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A Phragmen Lindelof Theorem at Infinity for Intermittent Domains

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Abstract

A Phragmèn-Lindelöf Theorem which yields the behavior at infinity of solutions of Dirichlet problems for elliptic quasilinear second-order partial differential equations in "Intermittent Domains" is established. This addresses a question raised by Jin and Lancaster in 2003.

1 Introduction

Suppose $n \geq 2$, $\upsilon \in S^{n-1}$ and Ω is an open region in $\mathbb{R}^n$ which "in the direction υ" consists of approximately identical connected components as one goes to infinity, say $\Omega = \cup_{i=1}^{\infty} \Omega_i$ with each component being roughly the translation in the direction υ through a distance d_i of a prototypical region; such a domain might be called "intermittent". Suppose further that Q is a second-order, elliptic partial differential operator on $\mathbb{R}^n$, $\phi \in C^0(\partial\Omega)$ and $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ is the solution of the Dirichlet problem $Qu = 0$ in Ω and $u = \phi$ on $\partial\Omega$. If Q has a "limiting form at ∞" $Q^{(\infty)}$ in the direction υ and ϕ has limiting values at infinity in the direction υ, then does u converge at infinity in the direction υ to translations of a function k which is determined by $Q^{(\infty)}$ and the limiting values of ϕ?

This is the type of question which arises in Example 5.3 of [5]. In this note, a special "Phragmèn-Lindelöf theorem at infinity" will be established which answers this question affirmatively under certain conditions. Related Phragmèn-Lindelöf theorems at infinity in which the structure of Ω is less important have been obtained by various authors; the reader is referred to the papers [4] and [5] and their references for some examples. The author obtained a less general version of this Phragmèn-Lindelöf theorem at infinity in his dissertation ([1]), which appears in [2]. Examples at the end of this note illustrate the results obtained here.

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2 Statement of the Theorem

Let $n \geq 2$ be an integer. For $M > 0$, let S_M denote the set

$$\{(x_1, \ldots, x_n) \in \mathbf{R}^n \mid |x_n| < M\} = \{(x_1, \ldots, x_n) \in \mathbf{R}^n \mid x_n \in I_M\},$$

where $I_M = (-M, M)$. Let Ω be an unbounded, open set in $\mathbf{R}^n$ contained in S_M. Let $T(\Omega)$ denote the set of directions $\omega \in S^{n-2}$ at infinity of the set $\pi(\Omega) = \{\vec{x} \in \mathbf{R}^{n-1} \ : \ (\vec{x}, y) \in \Omega \quad \text{for some} \quad y \in I_M\}$; that is

$$T(\Omega) = \cap_{N=1}^{\infty} \overline{\cup_{r \geq N} \{\omega \in S^{n-2} \ : \ r\omega \in \pi(\Omega)\}}. \tag{1}$$

Notice that $\omega \in T(\Omega)$ if and only if there exists a sequence $\{(\mathbf{x_j}, y_j)\}$ in Ω with $|\mathbf{x_j}| \to \infty$ and $\frac{\mathbf{x_j}}{|\mathbf{x_j}|} \to \omega$ as $j \to \infty$. Also $\omega \in T(\Omega)$ if and only if $\upsilon = (\omega, 0) \in S^{n-1}$ is a direction at infinity of $\Omega \subset S_M$.

Suppose (a_{ij}) is a $n \times n$ (symmetric) positive semidefinite matrix of functions in $C^0(\Omega \times \mathbf{R} \times \mathbf{R}^n)$ with trace one (e.g. [7], [3]), so that

$$\sum_{i=1}^{n} a_{ii}(X, z, P) = 1 \quad \text{for} \quad (X, z, P) \in \Omega \times \mathbf{R} \times \mathbf{R}^n. \tag{2}$$

Let b be a function in $C^0(\Omega \times \mathbf{R} \times \mathbf{R}^n)$ such that $b(X, z, P)$ is nonincreasing in z and let Q be the non-hyperbolic quasilinear second-order operator defined by

$$Qu(X) = \sum_{i,j=1}^{n} a_{ij}(X, u(X), Du(X)) D_{ij}u(X) + b(X, u(X), Du(X)). \tag{3}$$

Consider the Dirichlet problem of finding a function $f \in C^2(\Omega) \cap C^0(\overline{\Omega})$ which satisfies

$$Qf = 0 \quad \text{in} \ \Omega \tag{4}$$

and

$$f = \phi \quad \text{on} \ \partial\Omega. \tag{5}$$

Let $\omega, \nu \in S^{n-2}$ with $\omega \cdot \nu > 0$; for notational simplicity, we will assume Ω has been rotated so that $\omega = (1, 0, \ldots, 0)$ (and hence $\nu_1 > 0$). Set $m_j = \nu_j / \nu_1$, $1 \leq j \leq n-1$. Fix a bounded open subset Λ of $\mathbf{R} \times I_M$. For $d > 0$, let

$$\Lambda_d = \{(x_1, \mathrm{x}', y) \in \mathbf{R}^n : (\sum_{j=1}^{n-1} m_j x_j - d, y) \in \Lambda\}.$$

An element $X = (x_1, \ldots, x_n)$ of $\mathbf{R}^n$ may be represented in either of two ways:
(i) $X = (\vec{x}, y)$, where $\vec{x} = (x_1, \ldots, x_{n-1})$ and $y = x_n$.
(ii) $X = (x_1, \mathrm{x}', y)$, where $\mathrm{x}' = (x_2, \ldots, x_{n-1})$ and $y = x_n$.
If $h : \mathbf{R} \to (0, \infty)$, set

$$L_h = \{(x_1, \mathrm{x}', y) \in \mathbf{R} \times \mathbf{R}^{n-2} \times \mathbf{R} \ : \ |\mathrm{x}'| < h(x_1)\}.$$

Assumption 1 There exist $R_0 > 0$, a positive, increasing function $h : \mathbf{R} \to (0, \infty)$, and sequences (d_l), (e_l) in $(0, \infty)$ such that $\lim_{t\to\infty} h(t) = \infty$, $\lim_{l\to\infty} d_l = \infty$, $\lim_{l\to\infty} e_l = 0$,

$$\Omega^h = \cup_{l=1}^{\infty} \Omega_l,$$

$\Omega_i \cap \Omega_j = \emptyset$ if $i \neq j$ and $\operatorname{dist}(\Omega_l, \Lambda_{d_l}) \le e_l$ for $l = 1, 2, \ldots$, where $\Omega^h = \{(\vec{x}, y) \in \Omega \cap L_h \ : \ |\vec{x}| > R_0\}$.

Remark 2.1 *For $\delta \in (0, \sqrt{2})$ and $\omega \in S^{n-2}$, consider the "cone"*

$$S_\omega^\delta = \{(\vec{x}, y) \in \mathbf{R}^{n-1} \times \mathbf{R} \ : \ |\frac{\vec{x}}{|\vec{x}|} - \omega| < \delta\}.$$

If $\omega = (1, 0, \ldots, 0)$ and $h(t) = mt$, where $m = \delta\sqrt{4 - \delta^2}/(2 - \delta^2)$, then $S_\omega^\delta = L_h$.

Assumption 2 There exist $E, A, B \in C^0(I_M \times \mathbf{R}^3)$ such that $E(y, z, p_1, q)$ is nonincreasing in z, $A(y, z, p_1, q)$ and $B(y, z, p_1, q)$ are independent of z and

$$\frac{b(\vec{x}, y, z, \vec{p}, q)}{a_{nn}(\vec{x}, y, z, \vec{p}, q)} \to E(y, z, p_1, q), \tag{6}$$

$$\frac{\sum_{i,j=1}^{n-1} m_i m_j a_{ij}(\vec{x}, y, z, \vec{p}, q)}{a_{nn}(\vec{x}, y, z, \vec{p}, q)} \to A(y, z, p_1, q), \tag{7}$$

and

$$\frac{\sum_{j=1}^{n-1} m_j a_{jn}(\vec{x}, y, z, \vec{p}, q)}{a_{nn}(\vec{x}, y, z, \vec{p}, q)} \to B(y, z, p_1, q) \tag{8}$$

as $|\vec{x}| \to \infty$ with $\frac{\vec{x}}{|\vec{x}|} \to \omega$ and $|\mathrm{p}'| = |(p_2, \ldots, p_{n-1})| \to 0$ uniformly for $|y| < M$ and $z, p_1, q \in \mathbf{R}$.

Let $Q^{(\infty)}$ be the operator on $C^2(\mathbf{R} \times I_M)$ defined by

$$Q^{(\infty)}v(t, y) = A(t, y, v, Dv)\frac{\partial^2 v}{\partial t^2} + 2B(t, y, v, Dv)\frac{\partial^2 v}{\partial t \partial y} + \frac{\partial^2 v}{\partial y^2} + E(t, y, v, Dv)$$

Assumption 3 There exist functions $H \in C^0(\overline{I_M})$ and $k \in C^0(\overline{\Lambda}) \cap C^2(\Lambda)$ such that

$$\phi(\vec{x}, y) \to H(y) \quad \text{uniformly as} \quad |\vec{x}| \to \infty \quad \text{and} \quad \frac{\vec{x}}{|\vec{x}|} \to \omega \quad \text{for} \quad (\vec{x}, y) \in \partial\Omega, \tag{9}$$

$$Q^{(\infty)}k = 0 \qquad \text{in} \quad \Lambda \tag{10}$$

and

$$k(t, y) = H(y) \qquad \text{for} \quad (t, y) \in \partial\Lambda. \tag{11}$$

Remark 2.2 *One can observe that the regularity of k (i.e. $k \in C^2(\Lambda)$) and equation (10) play no role in the proof and are, in fact, unnecessary. They will hold in "typical" situations and are included for the edification of the reader.*

Assumption 4 For each $\epsilon > 0$, there exist positive constants B_0, B_1, δ, σ (depending only on $Q^{(\infty)}$, Λ, k and ϵ) and $k_1,\ k_2 \in C^1(\overline{\Lambda^\sigma}) \cap C^2(\Lambda^\sigma)$ such that

$$|k_j(t,y) - k(t,y)| \le \epsilon, \qquad (t,y) \in \Lambda, \quad j = 1,2, \tag{12}$$

$$Q^{(\infty)} k_1 \ge \delta, \qquad (t,y) \in \Lambda^\sigma, \tag{13}$$

$$Q^{(\infty)} k_2 \le -\delta, \qquad (t,y) \in \Lambda^\sigma \tag{14}$$

$$|Dk_j(t,y)| \le B_0, \qquad (t,y) \in \Lambda^\sigma,\ j = 1,2, \tag{15}$$

and

$$|D^2 k_j(t,y)| \le B_1, \qquad (t,y) \in \Lambda^\sigma,\ j = 1,2, \tag{16}$$

where

$$\Lambda^\sigma = \{(x_1, y) \in \mathbf{R} \times I_M \ :\ |x_1 - t| < \sigma \quad \text{for some} \quad (t,y) \in \Lambda\}.$$

In addition, assume $\sigma(\epsilon) \to 0$ as $\epsilon \to 0$.

Theorem 2.3 Let $\Omega \subset S_M$ and $\omega = (1, 0, \ldots, 0) \in \mathbf{R}^{n-1}$ be as above. Suppose

1) Q satisfies (2);
2) $f \in C^2(\Omega) \cap C^0(\overline{\Omega}) \cap L^\infty(\Omega)$ satisfies (4) & (5);
3) Assumptions 1-4 are satisfied;
4) there exist $L \ge 0$ and a positive continuous function σ_1 on $[1, \infty)$ such that

$$a_{nn}(\vec{x}, y, z, \vec{p}, q) \ge \sigma_1(|\vec{p}|^2 + q^2) \tag{17}$$

whenever $\vec{x}, \vec{p} \in \mathbf{R}^{n-1}$, $y, z, q \in \mathbf{R}$ with $|\vec{x}| \ge L$ and $|y| \le M$.

Let $(t,y) \in \overline{\Lambda}$. Suppose $(\vec{x}_{(i)})$ is a sequence in $\mathbf{R}^{n-1}$ and $(y_{(i)})$ is a sequence in $\overline{I_M}$ such that

(i) $(\vec{x}_{(i)}, y_{(i)}) = (x_{(i),1}, x_{(i),2}, \ldots, x_{(i),n-1}, y_{(i)}) \in \overline{\Omega_i}$ for $i \in \mathbf{N}$,
(ii) $h(x_{(i),1}) - |\mathrm{x}'_{(i)}| \to \infty$ as $i \to \infty$ and

$$\lim_{l \to \infty} \frac{\vec{x}_{(l)}}{|\vec{x}_{(l)}|} = \omega$$

(iii) $(\sum_{l=1}^{n-1} m_l x_{(i),l} - d_i, y_{(i)}) \to (t,y)$ as $i \to \infty$.

Then

$$\lim_{i \to \infty} f(\vec{x}_{(i)}, y_{(i)}) = k(t,y). \tag{18}$$

3 Proof

Let $\epsilon > 0$. Let $B_0, B_1, \delta, \sigma \in (0, \infty)$ and $k_1, k_2 \in C^1(\overline{\Lambda^\sigma}) \cap C^2(\Lambda^\sigma)$ be as given in Assumption 4 so that (12)-(16) hold. In particular,

$$AD_{11}k_2 + 2BD_{1n}k_2 + D_{nn}k_2 + E(k_2, D_1k_2, D_nk_2) = Q^{(\infty)}k_2 \leq -\delta. \tag{19}$$

Assumption 1 implies that there exists a natural number N_ϵ such that

$$e_i < \frac{\epsilon}{B_0} \qquad \text{whenever} \quad i \geq N_\epsilon. \tag{20}$$

Assumption 3 implies that there exist $\delta_1 > 0$ and R_1 such that if $(\vec{x}, y) \in \partial\Omega$, $|\vec{x}| \geq R_1$, $|y| \leq M$, and $|\frac{\vec{x}}{|\vec{x}|} - \omega| < \delta_1$, then

$$|\phi(\vec{x}, y) - H(y)| < \epsilon. \tag{21}$$

Assumption 2 implies there exist $\delta_2 > 0$ and R_2 such that

$$|\frac{b(\vec{x}, y, z, \vec{p}, q)}{a_{nn}(\vec{x}, y, z, \vec{p}, q)} - E(y, z, p_1, q)| \leq \frac{\delta}{16} \tag{22}$$

$$|\frac{a_{11}(\vec{x}, y, z, \vec{p}, q)}{a_{nn}(\vec{x}, y, z, \vec{p}, q)} - a^*_{11}(y, z, p_1, q)| \leq \frac{\delta}{16B_1} \tag{23}$$

$$|\frac{a_{1n}(\vec{x}, y, z, \vec{p}, q)}{a_{nn}(\vec{x}, y, z, \vec{p}, q)} - a^*_{1n}(y, z, p_1, q)| \leq \frac{\delta}{16B_1} \tag{24}$$

if $|\vec{x}| \geq R_2$, $|\mathrm{p}'| = |(p_2, \ldots, p_{n-1})| \leq \delta_2$, and $|\frac{\vec{x}}{|\vec{x}|} - \omega| \leq 2\delta_2$. Consider the compact set

$$K = \{(\vec{p}, q) \in \mathbf{R}^n : |\vec{p}|^2 + q^2 \leq 2(1 + \|\frac{\partial k_2}{\partial y}\|^2_\infty + \|\frac{\partial k_2}{\partial t}\|^2_\infty \sum_{i=1}^{n-1} m_i^2)\}.$$

Now (17) implies that there exists $\mu = \mu(K) > 0$ such that

$$a_{nn}(\vec{x}, y, z, \vec{p}, q) \geq \mu(K) \qquad \text{if} \quad (\vec{p}, q) \in K, \vec{x} \in \mathbf{R}^{n-1} \quad \text{and} \quad y, z \in \mathbf{R}.$$

Set $T_2(f) = \sup\{|f(\vec{x}, y) - k_2(t, y)| : (\vec{x}, y) \in \Omega, (t, y) \in \Lambda^\sigma\}$. There exists $\delta_3 > 0$ such that

$$|E(y, z, p_1 + s, q + t) - E(y, z, p_1, q)| \leq \frac{\delta}{16}, \tag{25}$$

$$|A(y, z, p_1 + s, q + t) - A(y, z, p_1, q)| \leq \frac{\delta}{16B_1}, \tag{26}$$

$$|B(y, z, p_1 + s, q + t) - B(y, z, p_1, q)| \leq \frac{\delta}{16B_1} \tag{27}$$

for $|s|, |t| \leq \delta_3$, $|y| < M$, $|z| < T_2(f)$, and $(p_1, q) \in K$.

There exists $\delta_4 > 0$ such that $2|\sum_{i=1}^{n-1} m_i x_i| \geq |\vec{x}|$ whenever $|\frac{\vec{x}}{|\vec{x}|} - \omega| \leq \delta_4$. Set $\delta_0 = \min\{1, \delta_1, \delta_2, \delta_3, \delta_4\}$. Now choose $H \geq 2$ such that $\frac{2M}{H} < \epsilon$, $\chi(H) \leq \ln(2)$,

$$A(H) \geq 16T_2(f), \qquad A(H) \geq \frac{5}{\mu(K)\delta}, \tag{28}$$

and

$$\frac{2\sqrt{2T_2(f)A(H)e^{\chi(H)}}}{A(H)} + \frac{2}{H} < \delta_0, \tag{29}$$

where $A(H)$ is given in (7.8) of [4] with

$$\eta(\beta) = \begin{cases} \frac{1}{\sqrt{2\beta}} & \text{if } 0 < \beta < \frac{1}{2} \\ e^{\frac{1}{2}-\beta} & \text{if } \frac{1}{2} \leq \beta < \infty. \end{cases} \tag{30}$$

There exists $R_3 > 0$ such that if $|\vec{x}_0| \geq R_3$, $|\vec{x} - \vec{x}_0| \leq A(H)e^{\chi(H)}$, and $|\frac{\vec{x}_0}{|\vec{x}_0|} - \omega| < \delta_0$, then $|\frac{\vec{x}}{|\vec{x}|} - \omega| < 2\delta_0$. There exists $R_4 > 0$ such that if $(\vec{x}, y) \in \overline{\Omega}$, $|\vec{x}| \geq R_4$ and $|\frac{\vec{x}}{|\vec{x}|} - \omega| < \delta_4$, then $(\vec{x}, y) \in \overline{\Omega_i}$ for some $i \geq N_\epsilon$. Set $R_0 = \max\{R_1, R_2, R_3, R_4\} + A(H)e^{\chi(H)}$ and define

$$W = \{\vec{x} \mid |\vec{x}| > R_0, \ \ |\frac{\vec{x}}{|\vec{x}|} - \omega| < \delta_0, \ \ h(x_1) - \sqrt{\sum_{j=2}^{n-1} x_j^2} \geq A(H)e^{\chi(H)}\}.$$

Claim: If $(\vec{x}_0, y_0) \in \overline{\Omega}$ and $\vec{x}_0 = (x_1^{(0)}, x_2^{(0)}, \ldots, x_{n-1}^{(0)}) \in W$, then $(\vec{x}_0, y_0) \in \overline{\Omega_l}$ for some $l = l(\vec{x}_0) \in \mathrm{N}$ with $l \geq N_\epsilon$ and

$$k_1(\sum_{i=1}^{n-1} m_i x_i^{(0)} - d_l, y_0) - 4\epsilon < f(\vec{x}_0, y_0) < k_2(\sum_{i=1}^{n-1} m_i x_i^{(0)} - d_l, y_0) + 4\epsilon. \tag{31}$$

Let $w(\vec{x}, y) = w_{a,\vec{x}_0,\gamma,H}(\vec{x}, y)$ be the upper barrier given in §7 of [4], specifically (7.14) of [4], with $\gamma = 3\epsilon$, $a = A(H)$ and η as in (30). Set $l = l(\vec{x}_0)$. Notice then that $w \geq \gamma = 3\epsilon$ on $\Omega_{a,\vec{x}_0,H}$. Now set

$$V_l = \{(\vec{x}, y) \in \Omega_{a,\vec{x}_0,H} \cap \Omega_l \ : \ |\vec{x} - \vec{x}_0| < \sqrt{2T_2(f)A(H)e^{\chi(H)} - T_2(f)^2}\ \}. \tag{32}$$

Define $u_2, v_2 \in C^1(\overline{V_l}) \cap C^2(V_l)$ by $v_2(\vec{x}, y) = k_2(\sum_{i=1}^{n-1} m_i x_i - d_l, y)$ and $u_2 = w + v_2$. Notice that

$$D_1 v_2(\vec{x}, y) = \frac{\partial v_2}{\partial x_1}(\vec{x}, y) = \frac{\partial k_2}{\partial t}(t, y) \quad \text{with } \ t = \sum_{i=1}^{n-1} m_i x_i - d_l, \tag{33}$$

since $m_1 = 1$, and so $D_j v_2 = m_j D_1 v_2$, $1 \leq j \leq n-1$. Notice also if $(\vec{x}, y) \in V_l$, then $|\vec{x}| \geq \max\{R_1, R_2, R_3, R_4\}$, $h_a'(h_a^{-1}(y + M)) \geq H \geq 2$, $A(H) < h_a^{-1}(y + M) < A(H)e^{\chi(H)}$, and $|\frac{\vec{x}}{|\vec{x}|} - \omega| < 2\delta_0$.

Let $\zeta \geq 0$. We claim that

$$Q(u_2 + \zeta) < 0 \quad \text{in} \quad \Omega_{1,l}. \tag{34}$$

From §7, [4], it follows that

$$\frac{\partial^2 w}{\partial x_i \partial x_j}(\vec{x}, y) = \frac{\delta_{ij} S^2 + (x_i - x_i^{(0)})(x_j - x_j^{(0)})}{S^3} \quad \text{for} \quad 1 \leq i, j \leq n-1,$$

$$\frac{\partial^2 w}{\partial x_i \partial y}(\vec{x}, y) = \frac{-(x_i - x_i^{(0)})Z}{S^3 h_a'(Z)} \quad \text{for} \quad 1 \leq i \leq n-1,$$

$$\frac{\partial^2 w}{\partial y^2}(\vec{x}, y) = \frac{S^2\big(Z\ h_a''(Z) - h_a'(Z)\big) + Z^2 h_a'(Z)}{S^3 (h_a'(Z))^3}$$

where $\vec{x}_0 = (x_1^{(0)}, \ldots, x_{n-1}^{(0)})$,

$$Z = h_a^{-1}(y+M), \quad \text{and} \quad S = \sqrt{(h_a^{-1}(y+M))^2 - |\vec{x} - \vec{x}_0|^2} = \sqrt{Z^2 - |\vec{x} - \vec{x}_0|^2}.$$

Since $A(H) < Z < 2A(H)$, $|\vec{x} - \vec{x}_0|^2 < 2T_2(f)A(H)e^{\chi(H)} - T_2(f)^2 \leq 4T_2(f)A(H) \cdot T_2(f)^2$, and $A(H) \geq 16T_2(f)$, it is easy to see that $2S^2 \geq (A(H))^2$. Notice then that

$$|Dw(\vec{x}, y)| \leq \frac{|\vec{x} - \vec{x}_0|}{S} + \frac{Z}{S|h_a'(Z)|} \leq \frac{2\sqrt{2T_2(f)A(H)e^{\chi(H)} - T_2(f)^2}}{A(H)} + \frac{2}{H}$$

and so (29) implies $|Dw(\vec{x}, y)| < \delta_0$.

If one sets $\xi_i = \frac{x_i - x_i^{(0)}}{S}$ for $1 \leq i \leq n-1$, $\xi_n = \frac{-Z}{S h_a'(Z)}$, and $\vec{\xi} = (\xi_1, \ldots, \xi_n)$, then $|\vec{\xi}| \leq 1$ and $\sum_{i,j=1}^n a_{ij}(\vec{x}, y, u_2 + \zeta, Du_2)\xi_i \xi_j \leq 1$. Since

$$\sum_{i,j=1}^{n-1} a_{ij}(\vec{x}, y, u_2 + \zeta, Du_2)\frac{\delta_{ij}}{S} = \frac{1}{S}(1 - a_{nn}(\vec{x}, y, u_2 + \zeta, Du_2))$$

and

$$\frac{Z h_a''(Z)}{S(h_a'(Z))^3} = -\frac{1}{S},$$

then

$$\sum_{i,j=1}^{n} a_{ij}(\vec{x}, y, u_2 + \zeta, Du_2) D_{ij} w(\vec{x}, y)$$

$$\leq \frac{2}{S} - \frac{1}{S}\big(2 + \frac{1}{(h_a'(Z))^2}\big) a_{nn}(\vec{x}, y, u_2 + \zeta, Du_2) < \frac{2}{S}.$$

Since $|Dw(\vec{x}, y)| < \delta_0 \leq 1$ when $(\vec{x}, y) \in \Omega_1$, $Du_2(\vec{x}, y) \in K$ and so $a_{nn}(\vec{x}, y, u_2 + \zeta, Du_2) \geq \mu(K)$ if $(\vec{x}, y) \in V_l$. Set $\mu = \mu(K)$. From (28), one obtains $\frac{2}{S} \leq \frac{\mu\delta}{2}$.

Notice that

$$E(y, u_2 + \zeta, p_1, q) \leq E(y, u_2, p_1, q) \leq E(y, k_2, p_1, q) \tag{35}$$

for all $y \in I_M$ and $p_1, q \in \mathrm{R}$ since $\zeta \geq 0$ and $u_2 = w + k_2 \geq 2\epsilon + k_2 > k_2$. Using (22), (25) and (35), we see that

$$\left|\frac{b(\vec{x}, y, u_2(\vec{x}, y) + \zeta, Du_2(\vec{x}, y))}{a_{nn}(\vec{x}, y, u_2(\vec{x}, y) + \zeta, Du_2(\vec{x}, y))} - E(y, k_2(t, y), Dk_2(t, y))\right| < \frac{\delta}{8}, \tag{36}$$

where $t = \sum_{i=1}^{n-1} m_i x_i - d_l$ and $Dk_2 = (\frac{\partial k_2}{\partial t}, \frac{\partial k_2}{\partial y})$. From (23), (26) and (35), we have

$$\left|\frac{\sum_{i,j=1}^{n-1} m_i m_j a_{ij}(\vec{x}, y, u_2 + \zeta, Du_2)}{a_{nn}(\vec{x}, y, u_2 + \zeta, Du_2)} - A(y, k_2(t, y), Dk_2(t, y))\right| < \frac{\delta}{8B_1} \tag{37}$$

and from (24), (27) and (35), we obtain

$$\left|\frac{\sum_{i=1}^{n-1} m_i a_{in}(\vec{x}, y, u_2 + \zeta, Du_2)}{a_{nn}(\vec{x}, y, u_2 + \zeta, Du_2)} - B(y, k_2(t, y), Dk_2(t, y))\right| < \frac{\delta}{8B_1}, \tag{38}$$

where t is as above. Recalling (19), we obtain

$$Qu_2(\vec{x}, y) = Q(w + k_2 + \zeta)(\vec{x}, y)$$

$$= \sum_{i,j=1}^{n} a_{ij} D_{ij} w + a_{11} D_{11} k_2 + 2a_{1n} D_{1n} k_2 + a_{nn} D_{nn} k_2 + b$$

$$< \frac{\mu\delta}{2} + (\frac{a_{11}}{a_{nn}} D_{11} k_2 + 2\frac{a_{1n}}{a_{nn}} D_{1n} k_2 + D_{nn} k_2 + \frac{b}{a_{nn}}) a_{nn}$$

$$< \frac{\mu\delta}{2} + a_{nn}\big[|D_{11}k_2|\frac{\delta}{8B_1} + 2|D_{1n}k_2|\frac{\delta}{8B_1} + \frac{\delta}{8}\big]$$

$$+ a_{nn}\big[AD_{11}k_2 + 2BD_{1n}k_2 + D_{nn}k_2 + E(k_2, D_1k_2, D_nk_2)\big]$$

$$\leq \frac{\mu\delta}{2} + \big[\frac{\delta}{8} + \frac{|D_{11}k_2|\delta}{8B_1} + \frac{2|D_{1n}k_2|\delta}{8B_1} - \delta\big] a_{nn}(Du_2)$$

$$\leq \frac{\mu\delta}{2} - \frac{\mu\delta}{2} = 0,$$

where $a_{ij} = a_{ij}(\vec{x}, y, u_2 + \zeta, Du_2)$, $A(D_1u_2, D_nu_2) = A(y, D_1u_2, D_nu_2)$, $b = b(u_2 + \zeta, Du_2) = b(\vec{x}, y, u_2 + \zeta, Du_2)$, etc.

Suppose $(\vec{x}, y) \in \partial V_l \cap \partial\Omega_l$. Then there exists $(t, y) \in \partial\Lambda$ such that

$$|t - (\sum_{i=1}^{n-1} m_i x_i - d_l)| < e_l < \frac{\epsilon}{B_0}.$$

Using (11), (12) and (21), we obtain

$$f(\vec{x}, y) = \phi(\vec{x}, y) < H(y) + \epsilon = k(t, y) + \epsilon \leq k_2(t, y) + 2\epsilon$$

$$< k_2(\sum_{i=1}^{n-1} m_i x_i - d_l, y) + B_0 e_l + 2\epsilon \le k_2(t,y) + w(\vec{x}, y).$$

Thus

$$f(\vec{x},y) - u_2(\vec{x},y) < 0 \quad \text{on} \quad \partial V_l \cap \partial\Omega_l.$$

Suppose $(\vec{x},y) \in \Omega_l \cap \partial V_l$. Then $|\vec{x}-\vec{x}_0| = \sqrt{2T_2(f)A(H)e^{\chi(H)} - T_2(f)^2}$ and so

$$w(\vec{x},y) = 3\epsilon + A(H)e^{\chi(H)} - \sqrt{\left(h_a^{-1}(y+M)\right)^2 - |\vec{x}-\vec{x}_0|^2}$$

$$\ge 3\epsilon + A(H)e^{\chi(H)} - \sqrt{\left(A(H)e^{\chi(H)}\right)^2 - 2T_2(f)A(H)e^{\chi(H)} + T_2(f)^2} = 3\epsilon + T_2(f).$$

Hence

$$f(\vec{x},y) - k_2(\sum_{i=1}^{n-1} m_i x_i - d_l, y) \le T_2(f) \le w(\vec{x},y) - 3\epsilon < w(\vec{x},y)$$

and so $f(\vec{x},y) < u_2(\vec{x},y)$ for $(\vec{x},y) \in \Omega_l \cap \partial V_l$.

Let $U_0 = \{(\vec{x},y) \in V_l \;:\; f(\vec{x},y) > u_2(\vec{x},y)\}$. Since $f < u_2$ on ∂V_l, U_0 is a relatively compact subset of V_l and $f = u_2$ on ∂U_0. A standard argument (e.g. §3 of [6]) then implies $U_0 = \emptyset$ and so $f \le u_2$ on V_l. Therefore,

$$f(\vec{x}_{(0)}, y) \le w(\vec{x}_0, y) + k_2(\sum_{i=1}^{n-1} m_i x_i^{(0)} - d_l, y) < 4\epsilon + k_2(\sum_{i=1}^{n-1} m_i x_i^{(0)} - d_l, y)$$

since $w(\vec{x}_0, y) \le 3\epsilon + \frac{2M}{H} < 4\epsilon$ for $|y| \le M$. Repeating this argument with the lower barrier $u_1(\vec{x},y) = -w(\vec{x},y) + k_1(\sum_{i=1}^{n-1} m_i x_i^0 - d_l, y)$ in place of $u_2(\vec{x},y)$, one obtains

$$k_1(\sum_{i=1}^{n-1} m_i x_i^{(0)} - d_l, y) - 4\epsilon < f(\vec{x}_0, y), \qquad |y| \le M.$$

Thus the claim (31) is valid.

Now suppose $(t,y) \in \overline{\Lambda}$, $(\vec{x}_0, y_0) \in \overline{\Omega}$ and $\vec{x}_0 \in W$. Then $(\vec{x}_0, y_0) \in \Omega_l$ for some $l = l(\vec{x}_0) \ge N_\epsilon$. Suppose $|(\sum_{i=1}^{n-1} m_i x_i^{(0)} - d_l, y_0) - (t,y)| \le \lambda$ for some $\lambda \ge 0$. Then (15) implies $|k_j(\sum_{i=1}^{n-1} m_i x_i^{(0)} - d_l, y_0) - k_j(t,y)| \le B_0\lambda$ for $j = 1, 2$. Since $k(t,y) - \epsilon \le k_1(t,y)$ and $k_2(t,y) \le k(t,y) + \epsilon$, one sees that

$$k(t,y) - 5\epsilon - B_0\lambda \le f(\vec{x}_0, y_0) \le k(t,y) + 5\epsilon + B_0\lambda.$$

The conclusion of the theorem follows from this.

4 Examples

To illustrate the results in this note, let $M = 1$ and $n = 3$ and consider the following elliptic operators:

$$Q_1 u = \frac{1}{3}\triangle u \tag{39}$$

and

$$Q_2u = \frac{1}{3+2|\nabla u|^2}\mathrm{div}(Tu), \qquad \text{where} \quad Tu = \frac{\nabla u}{\sqrt{1+|\nabla u|^2}}. \tag{40}$$

Notice that Q_1 is the Laplace operator and Q_2 is the minimal surface operator in $\mathbf{R}^3$.

For $-\frac{\pi}{2} < \beta < \frac{\pi}{2}$, let $\nu = (\cos(\beta), -\sin(\beta)) \in S^{n-2}$. Notice that $m_1 = 1$ and $m_2 = -\tan(\beta)$; hence

$$Q_1^{(\infty)}v(t,y) = \sec^2(\beta)v_{tt} \ + \ v_{yy} \tag{41}$$

and

$$Q_2^{(\infty)}v(t,y) = \frac{\left(\sec^2(\beta)(1+|\nabla v|^2) - v_t^2\right)v_{tt} - 2v_tv_yv_{ty} + v_{yy}}{1+v_t^2}. \tag{42}$$

Notice that the results of [1] and [2] do not apply unless $\beta = 0$ and $e_l = 0$ in Assumption 1 for each $l \in \mathbf{N}$.

Example 4.1 *Let* $\Lambda = \{(x_1, y) \in \mathbf{R}\times I_M : -1 < x_1 < 1, \ -1 < y < 1\}$, $d_l = 3^l$, $\Omega_l = \Lambda_{d_l}$ *and* $e_l = 0$ *for* $l \in \mathbf{N}$, $\Omega = \cup_{l=1}^{\infty}\Omega_l$, $H(y) = \cos(\frac{\pi}{2}y)\cosh(\frac{\pi}{2}\cos(\beta))$ *and* $\phi(x_1, x_2, y) \to H(y)$ *as* $x_1 \to \infty$ *with* $\frac{x_2}{x_1} \to 0$. *If* $(t, y) \in \overline{\Lambda}$, $\{(x_{(l),1}, x_{(l),2}, y_{(l)})\}$ *is a sequence in* $\overline{\Omega}$ *with* $(x_{(l),1}, x_{(l),2}, y_{(l)}) \in \overline{\Omega_l}$ *for each* $l = 1, 2, \ldots,$

$$(x_{(l),1} - \tan(\beta)x_{(l),2} - d_l, y_{(l)}) \to (t, y) \quad \text{as} \ \ l \to \infty,$$

and

$$\lim_{l\to\infty} \frac{(x_{(l),1}, x_{(l),2})}{\sqrt{x_{(l),1}^2 + x_{(l),2}^2}} = (1, 0),$$

then

$$\lim_{l\to\infty} f(x_{(l),1}, x_{(l),2}, y_{(l)}) = k(t, y),$$

where $f \in C^2(\Omega) \cap C^0(\overline{\Omega}) \cap L^{\infty}(\Omega)$ *satisfies (4) & (5) with* $Q = Q_1$ *and*

$$k(t,y) = \cos(\frac{\pi}{2}y)\cosh(\frac{\pi}{2}\cos(\beta)t).$$

In particular, $\lim_{l\to\infty} f(3^l + t, 0, y) = k(t, y)$ *for all* $|t| \le 1$ *and* $|y| \le 1$.
(Here we don't need $h(t)$ *and* $k_1(t, y) = \cos(\frac{\pi}{2}y)\cosh(\frac{\pi}{2}\cos(\beta)(1+\eta)t)$ *and* $k_2(t, y) = \cos(\frac{\pi}{2}y)\cosh(\frac{\pi}{2}\cos(\beta)(1-\eta)t)$ *for sufficiently small* $\eta > 0$.*) If Assumptions 3 & 4 hold for* $Q = Q_2$, *then the conclusion of the theorem holds when* $Q = Q_2$.

Example 4.2 *Fix* $\beta \in [0, \frac{\pi}{2})$. *Let* $\{d_l\}$ *be any sequence in* $\mathbf{R}$ *satisfying* $d_1 \ge 2$ *and* $d_{l+1} > d_l + 1$. *For each* $l \in \mathbf{N}$, *let*

$$\Omega_l = \{(x_1, x_2, y) : |y| < 1, \sec^2(\beta)d_l^2 - 2d_l + 1 < x_1^2 + (x_2 - \tan(\beta)d_l)^2 < \sec^2(\beta)d_l^2\}.$$

The boundary of Ω_l *consists of two concentric circles with center* $(0, \tan(\beta)d_l)$; *the inner circle passes through* $(d_l - 1, 0)$, *the outer circle passes through* $(d_l, 0)$ *and the radius through* $(d_l, 0)$ *meets the* x_1*−axis in an angle of* β.

Set $\Omega = \cup_{l=1}^{\infty}\Omega_l$ *and* $\Lambda = (-1,0) \times (-1,1)$. *Select* h *to be an increasing function which satisfies* $f(t) = o(\sqrt{t})$ *at* $t \to \infty$ *and* $h(t) \to \infty$ *as* $t \to \infty$. *The reader can verify that Assumption 1 is satisfied with the appropriate values of* $e_l > 0$ *which depend on* h. *For example, if* $h(t) = Ct^{\alpha}$ *with* $0 < \alpha < 1/2$, *then* $e_l = O(d_l^{2\alpha-1})$ *as* $d_l \to \infty$. *If Assumptions 3 & 4 hold for* $Q = Q_1$ *or* $Q = Q_2$, *the conclusion of the theorem holds when* $Q = Q_1$ *or* $Q = Q_2$ *respectively.*

Example 4.3 *In Example 5.3 of [5],* $n = 3$, $M = 1$, $Qu = \frac{1}{3}(\triangle u - u)$ *and* $\phi \equiv \cosh(1)$. *When* $\theta \in (-\frac{\pi}{2}, 0)$ *and* $\omega = (\cos(\theta), \sin(\theta))$, *an analysis similar to that in the previous example with* $\beta = 0$ *applies, where*

$$\Lambda = (-\arcsin(1 + \frac{4\theta}{\pi}), \pi + \arcsin(1 + \frac{4\theta}{\pi})) \times (-1, 1)$$

and $k \in C^2(\Lambda) \cap C^0(\overline{\Lambda})$ *is the solution of* $k_{tt} + k_{yy} - k = 0$ *in* Λ *and* $k = \cosh(1)$ *on* $\partial\Lambda$.

REFERENCES

[1] H. Almefleh, Asymptotic Behavior of Solutions of Elliptic Partial Differential Equations, PhD Dissertation, Wichita State University, 2003.

[2] H. Almefleh, Asymptotic Behavior of Solutions of Elliptic Partial Differential Equations, submitted for publication.

[3] Z. Jin & K. Lancaster, Theorrms of Phragmén-Lindelöf type for quasilinear elliptic equations, J. reine angew. Math. **514** (1999), 165-197.

[4] Z. Jin & K. Lancaster, Phragmén-Lindelöf Theorrms and the Asymptotic Behavior of solutions of quasilinear elliptic equations in slabs, Proc. Roy. Soc. Edinburgh **130A** (2000), 335-373.

[5] Z. Jin & K. Lancaster, A Phragmén-Lindelöf Theorem and the Behavior at Infinity of solutions of non-hyperbolic equations, Pacific Journal of Mathrmatics **211** (2003), 101-121.

[6] K. Lancaster, Phragmén-Lindelöf theorrms in slabs for some systems of non-hyperbolic second-order quasilinear equations, Proc. Roy. Soc. Edinburgh **133A** (2003), 1155-1173.

[7] J. Serrin, *The problem of Dirichlet for quasilinear equations with many independent variables*, Phil. Trans. Royal Soc. London Ser. A **264** (1969), 413–496.

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