WEAKLY C-NORMAL AND Cs-NORMAL SUBGROUPS OF FINITE GROUPS

MOHAMMAD TASHTOUSH

ABSTRACT

A subgroup $H$ of a finite group $G$ is weakly $c$-normal subgroup of $G$ if there exists a subnormal subgroup $N$ of $G$ such that $G = HN$, and $H \cap N \leq \text{core}_G(H)$, where $\text{core}_G(H)$ denotes the core of $H$ in $G$, which is the largest normal subgroup of $G$ contained in $H$. If $H \cap N \leq \text{core}_G(H)$, then $H$ is $c_s$-normal subgroup of $G$, where $\text{core}_G(H)$ denotes the higher core of $H$ in $G$, which is the largest subnormal subgroup of $G$ contained in $H$.

In this paper, we investigate some properties of weakly $c$-normal and $c_s$-normal subgroups of finite groups, and using the weakly $c$-normality and $c_s$-normality of some Sylow and maximal subgroups to determine the structure of finite groups.

1. INTRODUCTION

It is interesting to use some information on the subgroups of a finite group $G$ to determine the structure of the group $G$. The normality of subgroups of a finite group plays an important role in the study of finite groups. Wang, 1996 initiated the concepts of $c$-normal subgroups and used the $c$-normality of maximal subgroups to give some conditions for solvability and supersolvability of a finite group. Lujin Zhu and et al, 2002 have introduced the concepts of weakly $c$-normal subgroups and they have used the weakly $c$-normality of some maximal and Sylow subgroups to determine the structure of a finite group.

Definition 1.1 [10]: Let $H \leq G$. We say that $H$ is a subnormal subgroup of $G$ if there is a series from $H$ to $G$.

A subgroup $H$ of a group $G$ is called $c$-normal subgroup of $G$ if there exists a normal subgroup $N$ of $G$ such that $G = HN$ and $H \cap N \leq H_G$, where $H_G = \text{core}_G(H)$ is the largest normal subgroup of $G$ contained in $H$.


Keywords: Normality, Weakly $c$-normal, $c_s$-normal, Sylow and maximal subgroups.

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**Definition 1.2** [6]: Let $G$ be a group. We call a subgroup $H$ weakly $c$–normal subgroup of $G$ if there exists a subnormal subgroup $N$ of $G$ such that $G = HN$ and $H \cap N \leq H_G$, where $H_G = \text{core}_G(H)$ is the largest normal subgroup of $G$ contained in $H$.

**Example 1.3:** Let $H$ be a sylow $2$–subgroup of the symmetric group $S_3$. Then $H$ is weakly $c$–normal subgroup of $S_3$.

It is easy to see that, every $c$–normal subgroup of a group $G$ is weakly $c$–normal in $G$, however, the converse is not true, see [6].

### 2. PRELIMINARIES

In this section, we give some definitions and basic results which are essential in the sequel. Let $\pi$ be a nonempty set of primes, $\pi'$ the complement set of $\pi$ in the set of all prime numbers. Let $G$ be a group, we denote the set of all prime divisors of the order the group $G$ by $\pi(G)$; the maximal normal $p$–subgroup of $G$ by $O_p(G)$ and the Fitting subgroup of $G$ by $F(G)$. We introduce the following concept which is closely related with the subnormal subgroups of a group.

**Definition 2.1:** Let $G$ be a group. We call a subgroup $H$ $c_s$–normal subgroup of $G$ if there exists a subnormal subgroup $N$ of $G$ such that $G = HN$ and $H \cap N \leq H_G$, where $H_G$ denotes of the higher core of $H$ in $G$ which is the maximal subnormal subgroup of $G$ contained in $H$.

**Example 2.2:** Let $H$ be a sylow $2$–subgroup of the alternating group $A_4$. Then $H$ is $c_s$–normal subgroup of $A_4$.

It is easy to see that, every normal (subnormal) subgroup of a group $G$, is $c_s$–normal subgroup of $G$, but the converse is not true. To see this, let $G = S_3$, then the subgroup $H = \langle (12) \rangle$ of $G$ is $c_s$–normal in $G$, but $H$ is not normal subgroup of $G$.

Clearly, every $c$–normal subgroup of a group $G$ is $c_s$–normal subgroup of $G$, also, every weakly $c$–normal subgroup of a group $G$ is also $c_s$–normal subgroup of $G$. 
**Lemma 2.3 [10]:** Let $G$ be a group with subgroups $A$ and $B$ such that $A$ is an Abelian subgroup and $G = AB$. Then one of the following two conditions is satisfied:

(i) $A$ contains a normal subgroup $C$ of $G$ such that $C \neq 1$, or (ii) $B \cap (x^{-1}Ax) = 1$, for all $x \in G$.

The following example shows that the property of $c_s$-normality cannot imply weakly $c$-normality.

**Example 2.4:** Let $G = S_3 \times S_3$ be the direct product of $S_3$ by itself, and $K = \Delta(A_3 \times A_3) = \{1, ((123),(123)), ((132),(132))\}$ be the diagonal subgroup of $H = A_3 \times A_3$ with $K_G = 1$ and $K_{G_c} = K$.

Then $K \triangleleft \triangleleft G$ since $K \triangleleft H \triangleleft G$ and hence $K$ is $c_s$-normal subgroup of $G$. But there is no subnormal subgroup, say $N$ of $G$, such that $G = NK$ and $N \cap K \leq K_G$. Suppose not, then there exists a subnormal subgroup $N$ of $G$ with order having three cases:

**Case (i):** If $|N| < 12$, then $|NK| < 36$, and therefore there is no subnormal subgroup $N$ of $G$ such that $G = NK$.

**Case (ii):** If $|N| > 12$, then by Lagrange theorem there are two situations; either, (a) $|N| = 18$, then $|NK| > 36$, or (b) $|N| = 36$, then we have $G = N$, and thus $NK = GK = G$, but $N \cap K = G \cap K = K \triangleleft K_G = 1$. Therefore from the two situations there is no subnormal subgroup $N$ of $G$ such that $G = NK$.

**Case (iii):** Assume that $|N| = 12$, and suppose that $G = NK$. Since $K$ is an Abelian subgroup, then either $N$ is a normal subgroup of $G$, or one cannot find any normal subgroup $K$ such that $G = NK$. In this case $N$ is normal in $G$ if it is the maximal subgroup of $G$.

Assume that $N$ is normal in $G$, then (ii) of lemma 2.3 cannot be satisfied here, and hence we have $K \cap (x^{-1}Nx) = K \cap N \neq 1$, for all $x \in G$, hence $K \subset N$. Therefore $G \neq NK$. If $N$ is not normal of $G$ then $K$ cannot found any subnormal subgroup such that $G = NK$. Thus $K$ is not weakly $c$-normal subgroup of $G$. 

3. ELEMENTARY PROPERTIES

Lemma 3.1: Let $G$ be a group with subgroup $H$. Then (i) $H_{G_-} \lhd H$, and (ii) if $H \leq L_1 \cap L_2$ where $L_1$ and $L_2$ are two maximal subgroups of $G$ with $L_1 \neq L_2$, then $H_{G_-} \lhd H$.

Proof: (i) We know that $H_{G_-} \leq H$ and $H_{G_-} \lhd G$ (by definition), then $H_{G_-} \lhd (H \cap G)$. Thus $H_{G_-} \lhd H$. (ii) If $H$ is subnormal of $G$, then the result is obvious, so it is enough to show the case when $H$ is not subnormal of $G$. By the definition of the higher core and (i), there is a series of minimal length $n > 1$ that has the form $H_{G_-} = M_n \lhd \ldots \lhd M_n \lhd M_0 = H$, where $M_i$ is not subnormal of $G$ for all $i = 1, 2, \ldots, n$. Then there exists a unique maximal subgroup of $G$, say $M$, such that $M_i \leq M$ for all $i = 1, 2, \ldots, n$ which is a contradiction with $H \leq L_1 \cap L_2$. This impels that $n = 1$. Hence $H_{G_-} \lhd H$.

Lemma 3.2: Let $G$ be a group with a subgroup $H$. Then $H_G \leq H_{G_-}$.

Lemma 3.3 [6]: Let $G$ be a group. Then the following statements hold.

(i) If $H$ is weakly $c$-normal subgroup of $G$ with $H \leq K \leq G$, then $H$ is weakly $c$-normal subgroup of $K$.

(ii) Let $K$ be a normal subgroup of $G$ with $K \leq H$. Then $H$ is weakly $c$-normal of $G$ iff $H/K$ is weakly $c$-normal of $G/K$.

Lemma 3.4: Let $G$ be a group. Then the following statements hold.

(i) If $H$ is $c_s$-normal subgroup of $G$ with $H \leq K \leq G$, then $H$ is $c_s$-normal subgroup of $K$.

(ii) Let $K$ be a normal subgroup of $G$ with $K \leq H$. Then $H$ is $c_s$-normal subgroup of $G$ iff $H/K$ is $c_s$-normal subgroup of $G/K$.

Proof: (i) Suppose that $H$ is $c_s$-normal in $G$, then there exists a subnormal subgroup $N$ of $G$ such that $G = HN$ and $H \cap N \leq H_{G_-}$. Then...
\( K = K \cap G = K \cap H \cap N = H \left( K \cap N \right), \) and hence \((K \cap N)\) is subnormal of \( K \), and \( H \cap (N \cap K) = (H \cap N) \cap K \leq H_{G_{-}} \cap K \leq K_{G_{-}}. \) Thus \( H \) is \( c_{s} \) - normal in \( K \).

(ii) Suppose that \( \frac{H}{K} \) is \( c_{s} \) - normal subgroup in \( \frac{G}{K} \), then there exists a subnormal subgroup \( \frac{N}{K} \) of \( \frac{G}{K} \) such that \( \frac{G}{K} = \left( \frac{H}{K} \right) \left( \frac{N}{K} \right) \), and \( \left( \frac{H}{K} \right) \cap \left( \frac{N}{K} \right) \leq \left( \frac{H}{K} \right) \left( \frac{G}{K} \right). \) Then we have \( G = H \cap N \) and \( H \cap N \leq H_{G_{-}}. \)

Hence \( H \) is \( c_{s} \) - normal in \( G \).

Conversely, assume that \( H \) is \( c_{s} \) - normal subgroup in \( G \), then there exists a subnormal subgroup \( N \) of \( G \) such that \( G = H \cap N \) and \( H \cap N \leq H_{G_{-}}. \) Then we have that \( \frac{G}{K} = \left( \frac{H}{K} \right) \left( \frac{N}{K} \right) \), and then \( NK \) is a subnormal of \( G \), and

\[
\left( \frac{H}{K} \right) \cap \left( \frac{NK}{K} \right) = \left( \frac{H \cap NK}{K} \right) \leq \frac{KH_{G_{-}}}{K} \leq \left( \frac{H}{K} \right) \left( \frac{G}{K} \right).
\]

Hence \( \frac{H}{K} \) is \( c_{s} \) - normal subgroup of \( \frac{G}{K} \).

**Definition 3.5 [10]:** For any set \( \pi \) of prime numbers, we denote by \( \pi' \) the set of all primes which do not belong to \( \pi \). If \( H \leq G \), then \( H \) is said to be a Hall \( \pi \) - subgroup of \( G \) if \( |H| \) is a \( \pi \) - number and \( [G:H] \) is a \( \pi' \) - number.

**Definition 3.6 [2]:** A group \( G \) is called \( \pi \) - solvable if it has a subnormal series whose factors are \( \pi \) - groups or \( \pi' \) - groups and the \( \pi \) - factors are solvable.

**Lemma 3.7 [9]:** Let \( G \) be a \( \pi \) - solvable group. Then \( G \) has at least one solvable Hall \( \pi \) - subgroup \( G_{x} \), and for any \( \pi \) - subgroup \( A \) of \( G \), there is an element \( x \in G \) such that \( A^{x} \leq G_{x} \). In particular, any two Hall \( \pi \) - subgroups are conjugate in \( G \).

For the proof of this lemma, the reader can see [2] and [9].
Definition 3.8: Let $G$ be a group. We call a group $G$ weakly $p$–nilpotent if $G$ has a subnormal $p$–complement in $G$ i.e., if $H$ is a subnormal subgroup of $G$ and $P$ is a Sylow $p$–subgroup of $G$ such that $G = HP$ and $H \cap P = 1$ then $G$ is called weakly $p$–nilpotent.

Clearly, if $G$ is $p$–nilpotent, then $G$ is weakly $p$–nilpotent, however, the converse is not true. The following example shows that the property of weakly $p$–nilpotent cannot imply $p$–nilpotent.

4. THEOREMS

Theorem 4.1: If $H$ is weakly $c$–normal subgroup of a group $G$, then $H/H_G$ has a subnormal complement in $G/H_G$, i.e., there exists a subnormal subgroup $K/H_G$ of $G/H_G$ such that $G/H_G$ is the semidirect product of $K/H_G$ and $H/H_G$. Conversely, if $H$ is a subgroup of $G$ such that $H/H_G$ has a subnormal complement in $G/H_G$, then $H$ is weakly $c$–normal of $G$.

Proof: Let $H$ be a weakly $c$–normal subgroup of $G$, then there exists a subnormal subgroup $K$ of $G$ such that $G = HK$ and $H \cap K \leq H_G$. If $H_G = 1$, then $H \cap K = 1$. Hence $K$ is a subnormal complement of $H$ in $G$. Assume that $H_G \neq 1$, then we can construct the factor groups $H/H_G$ and $K/H_G$ and by Dedekind’s Identity, we have

$$\left( \frac{H}{H_G} \right) \cap \left( \frac{KH_G}{H_G} \right) = \frac{(H \cap KH_G)}{H_G} = \frac{H_G(H \cap K)}{H_G} \leq \frac{H}{H_G} = 1.$$ 

Hence $\frac{KH_G}{H_G}$ is a subnormal complement of $\frac{H}{H_G}$ in $\frac{G}{H_G}$.

Conversely, if $F$ is a subgroup of $G$ such that $\frac{F}{H_G}$ is a subnormal complement of $\frac{H}{H_G}$ in $\frac{G}{H_G}$, then we have that
\[
\frac{G}{H_G} = \left( \frac{H}{H_G} \right) \left( \frac{F}{H_G} \right), \quad \text{and} \quad \left( \frac{H}{H_G} \right) \cap \left( \frac{F}{H_G} \right) = 1.
\]

Then \( G = HF \), where \( F \) is a subnormal subgroup in \( G \), and \( H \cap F \leq H_G \). Therefore \( H \) is weakly \( c \)-normal subgroup in \( G \).

**Corollary 4.2:** Let \( H \) be a subgroup of a group \( G \) such that \( \frac{H}{H_G} \) has a subnormal complement in \( \frac{G}{H_G} \). Then \( H \) is \( c_s \)-normal in \( G \).

The following example shows that the converse of the above corollary is not necessarily true.

**Example 4.3:** Let \( G = H \rtimes Z_p \) be the semidirect product of a subgroup \( H \) and the cyclic group \( Z_p \) with; (i) \( Z_p \) is a maximal subgroup in \( G \), (ii) \( H \) is not normal in \( G \) with \( H_G \neq 1 \), (iii) \( \left( \frac{|H|}{Z_p} \right) = 1 \).

Then \( Z_p \triangleleft G \) since \( p \) does not divide \( |H| \) (by Sylow theorem), and hence \( Z_p \) is \( c_s \)-normal in \( G \), also \( H \) is \( c_s \)-normal in \( G \). We claim that \( \frac{H}{H_G} \) has no subnormal complement in \( \frac{G}{H_G} \), suppose not, i.e., \( \frac{H}{H_G} \) has a subnormal complement in \( \frac{G}{H_G} \), say \( \frac{K}{H_G} \). Then \( \frac{G}{H_G} = \left( \frac{H}{H_G} \right) \left( \frac{K}{H_G} \right) \), and

\[
\left( \frac{H}{H_G} \right) \cap \left( \frac{K}{H_G} \right) = \left( \frac{H \cap K}{H_G} \right) = 1.
\]

Then \( G = HK \), and \( H \cap K = H_G \). But we know that

\[
|G| = \frac{|H|}{H \cap K} = \frac{|H|}{|K|} = \frac{|G|}{|H|} = \frac{|K|}{|H_G|}.
\]

Since \( |G| = |H| \), \( |Z_p| \)

\[
|H| = \frac{|G|}{|Z_p|} = \frac{|G|}{p} \Rightarrow |G| = \frac{|G|}{|H_G|} = \frac{|K|}{|H_G|} = p \Rightarrow |K| = p |H_G|.
\]

If \( |K| = p \), we get a contradiction with \( 1 < H_G \leq K \). By using Cauchy Theorem we have \( K \) contains a subgroup of order \( p \), say \( L \), since \( K \leq G \), then \( L \)
will be a sylow $p$ – subgroup of $G$ since $\left( |H|, |Z_p| \right) = 1$. Since $Z_p$ is the unique sylow $p$ – subgroup of $G$, and its normal in $G$, then $L = Z_p$. Then $Z_p < K \neq G$. This contradicts the maximality of $Z_p$ in $G$.

**Theorem 4.4:** A group $G$ has a weakly $c$ – normal sylow $p$ – subgroup if and only if the factor group $G/O_p(G)$ is weakly $p$ – nilpotent.

**Proof:** Assume that $G$ has a weakly $c$ – normal sylow $p$ – subgroup, say $P$. Then by lemma 3.3 we have $P/P_G$ has a subnormal complement in $G/P_G$ (notice that $P_G = O_p(G)$). Hence $G/O_p(G)$ is weakly $p$ – nilpotent.

Conversely, assume that $G/O_p(G)$ is weakly $p$ – nilpotent. Then there exists a subnormal subgroup $K/O_p(G)$ of $G/O_p(G)$ such that

$$G/O_p(G) = \left( K/O_p(G) \right) \left( P/O_p(G) \right), \text{ and } \left( K/O_p(G) \right) \cap \left( P/O_p(G) \right) = 1.$$

Then we have $G = KP$ such that $K$ is a subnormal subgroup in $G$, and $K \cap P = O_p(G) = P_G$. Therefore $P$ is weakly $c$ – normal subgroup in a group $G$.

**Corollary 4.5:** If a factor group $G/O_p(G)$ is weakly $p$ – nilpotent, then $G$ has a $c_S$ – normal sylow $p$ – subgroup.

This corollary is obvious and we omit the proof. The converse of the Corollary 4.5 is not necessarily true; regarded with Example 4.3, we have seen that $Z_p$ is $c_S$ – normal sylow $p$ – subgroup of $G$ such that $O_p(G) = Z_p$. Hence $G/O_p(G) = H$ has no subnormal $p$ – complement.
Theorem 4.6: A group $G$ is metanilpotent if and only if every sylow subgroup of $G$ is weakly $c -$ normal in $G$.

Proof: Suppose that $G$ is metanilpotent. Let $p \in \pi(G)$ and $P$ be a sylow $p -$ subgroup of $G$. Since $G/F(G)$ and $F(G)$ are solvable. Then $G$ is solvable. Moreover $G$ is $p' -$ solvable. By lemma 3.5, we can replace $G_x$ by $K$, and $A^x$ by $O_p(F(G))$ to conclude that $G$ has a solvable Hall $p' -$ subgroup $K$. Since $O_p(F(G))$ is normal in $G$, $O_p(F(G)) \leq K$ and hence $O_p(G)K = F(G)K$.

Since $G/F(G)$ is nilpotent, then we have $KF(G)/F(G)$ is a normal Hall $p' -$ subgroup of $G/F(G)$. Hence $H = O_p(G)K = KF(G)$ is normal subgroup in $G$, and consequently, we deduce that $G = PH$ and $P \cap H = P \cap O_p(G)K = O_p(G)(P \cap K) = O_p(G) = P_G$. Therefore $P$ is weakly $c -$ normal in $G$.

Conversely, Suppose that every sylow subgroup of $G$ is weakly $c -$ normal in $G$. Then, by the third isomorphism theorem and Theorem 4.4 and the fact that $F(G) = O_p(G)$,

$$
\frac{G}{O_p(G)} \cong \left( \frac{G}{O_p(G)} \right) \left/ \left( \frac{F(G)}{O_p(G)} \right) \right. \cong \frac{G}{F(G)}
$$

is $p -$ nilpotent for all $p \in \pi(G)$. Therefore $G/F(G)$ is nilpotent, and hence $G$ is metanilpotent.

Corollary 4.7: A group $G$ is metanilpotent if and only if every sylow subgroup of $G$ is $c_s -$ normal in $G$.

This corollary is obvious and we omit the proof.
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Department of Mathematics and Statistics,Al-Hussein Bin Talal University, P.O.Box 20, Ma'an, 71110, Jordan

E-mail address: mhmtdtashtoush@ahu.edu.jo (Corresponing author)