THE PERIODS OF THE PELL P-ORBITS OF POLYHEDRAL AND CENTRO-POLYHEDRAL GROUPS

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Abstract. In this paper, we define the Pell p-orbit of a finitely generated group and then we obtain the lengths of the periods and the basic periods of the Pell p-orbits of the finite polyhedral groups and centro-polyhedral groups.

1. INTRODUCTION AND PRELIMINARIES

The study of recurrence sequences in groups began with the earlier work of Wall [3] where the ordinary Fibonacci sequences in cyclic groups were investigated. The concept extended to some special linear recurrence sequences by several authors; see for example, [1, 2, 5, 6, 8, 9, 10, 11, 13, 14, 15, 16]. In [12] extended the theory to the generalized Pell p-sequences. In this paper, we examine the behaviour of the periods and basic periods of the Pell p-orbits of the polyhedral groups (n, 2, 2), (2, n, 2), (2, 2, n), (2, 3, 3), (2, 3, 4), (2, 2, 5) and the centro-polyhedral groups (−2, n, 2), (2, n, −2), (n, −2, 2), (n, 2, −2), (2, −2, n), (−2, 2, n) for n > 2.

In [4], the generalized Pell (p, i) numbers was defined as follows:

for p (p = 1, 2, …), n > p + 1 and 0 ≤ i ≤ p,

\[ P_p^{(i)} (n) = 2P_p^{(i)} (n - 1) + P_p^{(i)} (n - p - 1) , \]

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with initial conditions $P_p^{(i)}(1) = \cdots = P_p^{(i)}(i) = 0$ and $P_p^{(i)}(i + 1) = \cdots = P_p^{(i)}(p + 1) = 1$.

Note that if $i = 0$, the initial conditions are $P_p^{(i)}(1) = P_p^{(i)}(2) = \cdots = P_p^{(i)}(p + 1) = 1$.

A sequence is periodic if, after a certain point, it consists only of repetitions of a fixed subsequence. The number of elements in the repeating subsequence is called the period of the sequence. For example, the sequence $a, b, c, d, b, c, d, \ldots$ is periodic after the initial element $a$ and has period 3. A sequence is simply periodic with period $k$ if the first $k$ elements in the sequence form a repeating subsequence. For example, the sequence $a, b, c, d, a, b, c, d, \ldots$ is simply periodic with period 4.

Reducing the generalized Pell $(p, p)$-sequence $\{P_p^{(p)}(n)\}$ by a modulus $m$, we can get repeating sequence, denoted by

$$\{P_p^{(p,m)}(n)\} = \{P_p^{(p,m)}(1), P_p^{(p,m)}(2), \ldots, P_p^{(p,m)}(p), P_p^{(p,m)}(p + 1), \ldots, P_p^{(p,m)}(i), \ldots\}$$

where $P_p^{(p,m)}(i) = P_p^{(p)}(i) \pmod{m}$. Also, it has the same recurrence relation as in (1.1) (Deveci et al.).([12, p.3])

**Theorem 1.1.** (Deveci et al.).([12, Theorem 2.1, p.3]) $\{P_p^{(p,m)}(n)\}$ is a simply periodic sequence.

The notation $h_p^p(m)$ is used for the smallest period of the sequence $\{P_p^{(p,m)}(n)\}$ (Deveci et al.).([12, p.3])

Let $G$ be a finite $j$-generator group and let

$$X = \{(x_0, x_1, \ldots, x_{j-1}) \in G \times G \times \cdots \times G \mid < \{x_0, x_1, \ldots, x_{j-1}\} > = G\}.$$  

We call $(x_0, x_1, \ldots, x_{j-1})$ a generating $j$-tuple for $G$.  

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**References:**

1. Deveci et al. ([12, p.3])
**Definition 1.1.** (Deveci et al. ([12, Definition 3.4, p.6])) A generalized Pell $p$-sequence $(p \geq 2)$ in a finite group is a sequence of group elements $x_0, x_1, \ldots, x_n, \ldots$ for which, given an initial (seed) set $x_0, \ldots, x_{j-1}, (p + 1 \geq j)$ each element is defined by

$$x_n = \begin{cases} x_0 (x_{n-1})^2 & \text{for } j \leq n < p + 1, \\ x_{n-p-1} (x_{n-1})^2 & \text{for } n \geq p + 1. \end{cases}$$

It is required that the initial (seed) set $x_0, \ldots, x_{j-1}$ of the group elements sequence generates the group, thus, forcing the generalized Pell $p$-sequence to reflect the structure of the group.

The generalized Pell $p$-sequence of a group generated by $x_0, \ldots, x_{j-1}$ is denoted by $Q(p) (G; x_0, x_1, \ldots, x_{j-1})$.

**Theorem 1.2.** (Deveci et al. ([12, Theorem 3.1, p.7])) A generalized Pell $p$-sequence in a finite group is simply periodic.

In (Deveci et al. ([12, p.7])), the period of the generalized Pell $p$-sequence $Q(p) (G; x_0, x_1, \ldots, x_{j-1})$ had been denoted by $\text{Per}Q(p) (G; x_0, x_1, \ldots, x_{j-1})$.

**Definition 1.2.** (Deveci et al. ([12, Definition 3.5, p.8])) Let $G$ be a finite $j$-generator groups. For a $j$-tuple $(x_0, x_1, \ldots, x_{j-1}) \in X$ the basic generalized Pell $p$-sequence $\overline{Q}(p) (G; x_0, x_1, \ldots, x_{j-1})$, $(p \geq 2, p + 1 \geq j)$ of the basic period $m$ is a sequence of group elements $a_0, a_1, a_2, \ldots, a_n, \ldots$ for which, given an initial (seed) set $a_0 = x_0$, $a_1 = x_1$, $a_2 = x_2, \ldots, a_{j-1} = x_{j-1}$, each element is defined by

$$a_n = \begin{cases} a_0 (a_{n-1})^2 & \text{for } j \leq n < p + 1, \\ a_{n-p-1} (a_{n-1})^2 & \text{for } n \geq p + 1. \end{cases}$$

where $m \geq 1$ is the least integer with

$$a_0 = a_m \theta, \ a_1 = a_{m+1} \theta, \ a_2 = a_{m+2} \theta, \ldots, \ a_p = a_{m+p} \theta,$$
for some $\theta \in \text{Aut}G$. Since $G$ is a finite $j$-generator group and $a_m, a_{m+1}, \ldots, a_{m+j-1}$ generate $G$, it follows that $\theta$ is uniquely determined. The basic generalized Pell $p$-sequence $Q^{(p)}(G; x_0, x_1, \ldots, x_{j-1})$ is finite containing $m$ element.

Also, in (Deveci et al.)([12, p.8]), the basic period of the basic generalized Pell $p$-sequence $Q^{(p)}(G; x_0, x_1, \ldots, x_{j-1})$ had been denoted by $BQ^{(p)}(G; x_0, x_1, \ldots, x_{j-1})$.

**Definition 1.3.** The polyhedral group $(l, m, n)$ for $l, m, n > 1$, is defined by the presentation

$$\langle x, y, z : x^l = y^m = z^n = xyz = 1 \rangle.$$

For the generating pair $(x, y)$, the polyhedral group $(l, m, n)$ have the presentations

$$\langle x, y : x^l = y^m = (xy)^n = 1 \rangle$$

and

$$\langle x, y : x^l = y^m = (xy)^{−n} = 1 \rangle,$$

where $l, m, n > 1$.

The polyhedral group $(l, m, n)$ is finite if and only if the number $k = lmn \left(\frac{1}{l} + \frac{1}{m} + \frac{1}{n} - 1\right) = mn+nl+lm−lmn$ is positive. Its order is $2lmn/k$ (Coxeter and Moser).([7, p. 67-68]).

In this paper, we consider polyhedral groups as 3-generator groups.

**Definition 1.4.** The centro-polyhedral group $h(l, m, n)$, for $l, m, n \in Z$ is defined by the presentation

$$\langle x, y, z : x^l = y^m = z^n = xyz \rangle.$$

For detailed information about these groups, see(Coxeter and Moser)([7, p. 70-71]).
2. MAIN RESULTS AND PROOFS

Definition 2.1. For a finitely generated group $G = \langle A \rangle$, where $A = \{a_1, a_2, \ldots, a_{p+1}\}$ such that $p \geq 2$, the Pell p-orbit of $G$ with respect to the generating set $A$, written $P^p_A(G)$ is the sequence $x_0 = a_1, x_1 = a_2, \ldots , x_p = a_{p+1}, x_{n+p} = (x_{n-1})(x_{n+p-1})^2$, $n \geq 1$. The length of the period of the sequence is called the Pell p-length of $G$ with respect to the generating set $A$, written $LP^p_A(G)$. Also, we denote the length of basic period of this sequence by $\overline{LP}^p_A(G)$, which is called the basic length of $G$ with respect to the generating set $A$.

Firstly, we consider the Pell p-lengths and the basic Pell p-lengths of the finite polyhedral groups by the following Theorems.

Theorem 2.1. Let $G$ be any of the polyhedral groups $(n, 2, 2)$, $(2, n, 2)$ and $(2, 2, n)$, where $n \geq 3$. Then

$$LP^2_{\{x,y,z\}}(G) = \begin{cases} \frac{3n}{2}, & n \text{ is even} \\ 3n, & n \text{ is odd} \end{cases}$$

and $\overline{LP}^2_{\{x,y,z\}}(G) = 3$.

Proof. Let us consider the group $(n, 2, 2)$. The orbit $P^2_{\{x,y,z\}}((n, 2, 2))$ is

$$x, y, z, x, yx^2, z, x, yx^4, z, x, yx^6, z, x, yx^8, z, x, yx^{10}, z, \ldots$$

This sequence can be said to form layers of length three. Using the above, the sequence becomes:

$$x_0 = x, x_1 = y, x_2 = z,$$
$$x_3 = x, x_4 = yx^2, x_5 = z,$$
$$x_6 = x, x_7 = yx^4, x_8 = z,$$
$$x_{3i} = x, x_{3i+1} = yx^{2i}, x_{3i+2} = z, \ldots$$

So, we need the smallest $i \in N$ such that $2i = nv_1$ for $v_1 \in N$. 

If $n$ is even, then $i = \frac{n}{2}$. Thus, $LP_{\{x,y,z\}}^2((n, 2, 2)) = \frac{3n}{2}$ and $LP_{\{x,y,z\}}^2((n, 2, 2)) = 3$ since $x\theta = x$, $y\theta = yx^{-2}$ and $z\theta = z$ where $\theta$ is an outer automorphism of order $\frac{n}{2}$.

If $n$ is odd, then $n = i$. Thus, $LP_{\{x,y,z\}}^2((n, 2, 2)) = 3n$ and $LP_{\{x,y,z\}}^2((n, 2, 2)) = 3$ since $x\theta = x$, $y\theta = yx^{-2}$ and $z\theta = z$ where $\theta$ is an outer automorphism of order $n$.

The proofs for the groups $(2, n, 2)$ and $(2, 2, n)$ are similar to the above and are omitted.

Theorem 2.2. i) $LP_{\{x,y,z\}}^2((2, 3, 3)) = \ell P_{\{x,y,z\}}^2((2, 3, 3)) = 65$.

ii) $LP_{\{x,y,z\}}^2((2, 3, 4)) = \ell P_{\{x,y,z\}}^2((2, 3, 4)) = 27$.

iii) $LP_{\{x,y,z\}}^2((2, 3, 5)) = \ell P_{\{x,y,z\}}^2((2, 3, 5)) = 175$.

Proof. The orbit $P_{\{x,y,z\}}^2((2, 3, 3))$ is

$x, y, z, y, 1, z, yxy, xy, x, xy, yxy^2, yx, y^2xy, yxy^2, yx, yxy, xy, y^2xy,$

$yxy, y^2xy, x, xy, x, yxy^2, x, xy, yxy^2, xyx, y^2xy, yxy^2, xyx, y^2xy, x, yxy, yx,$

$xyx, xyx, x, xyx, 1, x, xyx, yxy, x, xyx, yxy^2, x, yxy, z, y, xy, 1, x, yxy^2, 1, 1, z,$

$xy, y^2xy, z, z, yx, y, xyx, x, y, y^2xy, x, y, z, \ldots,$

which has period 65. Also, $\ell P_{\{x,y,z\}}^2((2, 3, 3)) = 65$ since $x\theta = x$, $y\theta = y$ and $z\theta = z$ where $\theta$ is identity automorphism.

The proofs of the cases ii and iii are similar to the above and are omitted.

Now we give the Pell p-lengths and the basic Pell p-lengths of some centro polyhedral groups by the following Theorem.

Theorem 2.3. Let $G$ be any of the centro-polyhedral groups $\langle -2, n, 2 \rangle$, $\langle 2, n, -2 \rangle$, $\langle n, -2, 2 \rangle$, $\langle 2, n, -2 \rangle$, $\langle 2, -2, n \rangle$ and $\langle -2, 2, n \rangle$, where $n \geq 3$. Then

$$LP_{\{x,y,z\}}^2(G) = \begin{cases} \frac{n}{2} \cdot h_2^2(4(n-1)), & n \text{ is even}, \\
\frac{n}{2} \cdot h_2^2(4(n-1)), & n \text{ is odd} \end{cases}$$

and $\ell P_{\{x,y,z\}}^2(G) = h_2^2(4(n-1))$, where $h_2^2(4(n-1))$ denotes the smallest period of the sequence $\left\{ P_2^{(2,4(n-1))}(n) \right\}$. 
Proof. Let us consider the group \((-2, n, 2)\). It is clear that the centro polyhedral group \((-2, n, 2)\) is defined the presentation
\[
\langle x, y, z : x^{-2} = y^n = z^2 = xyz \rangle.
\]
writing \(x^{-2} = y^n = z^2 = xyz = s\), we find that \(|s| = \frac{4n}{n^2 + 2} - 1 = n - 1\). Thus we obtain \(|\langle-2, n, 2\rangle| = 4n(n - 1), |x| = |z| = 4(n - 1)\) and \(|y| = 2n(n - 1)\). Also note that \(z^2\) is central element of the group \((-2, n, 2)\).

If \(n\) is a positive even integer, then the orbit \(P^2_{\langle x, y, z \rangle}((-2, n, 2))\) becomes:
\[
\begin{align*}
x_0 &= x, \ x_1 = y, \ x_2 = z, \\
x_{h_2^2(4(n-1))} &= x, \ x_{h_2^2(4(n-1)) + 1} = y, \ x_{h_2^2(4(n-1)) + 2} = zy^{k_1 \cdot 4(n-1)}, \\
x_{h_2^2(4(n-1))i} &= x, \ x_{h_2^2(4(n-1))i + 1} = y, \ x_{h_2^2(4(n-1))i + 2} = zy^{k_1 \cdot 4(n-1)i}, \ldots,
\end{align*}
\]
where \(k_1 \in N\) be such that \((k_1, \frac{n}{2}) = 1\). Since \(|y| = 2n(n - 1)\), we need the smallest \(i \in N\) such that \(k_1 \cdot 4(n-1)i = 2n(n - 1)v_2\) for \(v_2 \in N\). Then, we obtain \(i = \frac{n}{2}\) for \(v_2 = k_1\) since \(n\) is a positive even integer. Thus, \(LP^2_{\langle x, y, z \rangle}((-2, n, 2)) = \frac{n}{2} \cdot h_2^2(4(n-1))\) and \(LP^2_{\langle x, y, z \rangle}((-2, n, 2)) = h_2^2(4(n-1))\) since \(x \theta = x, y \theta = y\) and \(z \theta = zy^{k_1 \cdot 4(1-n)}\) where \(\theta\) is an outer automorphism of order \(\frac{n}{2}\) and \(t_1 \in N\) such that \((t_1, \frac{n}{2}) = 1\).

If \(n\) is a positive odd integer, then the orbit \(P^2_{\langle x, y, z \rangle}((-2, n, 2))\) becomes:
\[
\begin{align*}
x_0 &= x, \ x_1 = y, \ x_2 = z, \\
x_{h_2^2(4(n-1))} &= x, \ x_{h_2^2(4(n-1)) + 1} = y, \ x_{h_2^2(4(n-1)) + 2} = zy^{k_2 \cdot 4(n-1)}, \\
x_{h_2^2(4(n-1))i} &= x, \ x_{h_2^2(4(n-1))i + 1} = y, \ x_{h_2^2(4(n-1))i + 2} = zy^{k_2 \cdot 4(n-1)i}, \ldots,
\end{align*}
\]
where \(k_2 \in N\) be such that \((k_2, n) = 1\). Since \(|y| = 2n(n - 1)\), we need the smallest \(i \in N\) such that \(k_2 \cdot 4(n-1)i = 2n(n - 1)v_3\) for \(v_3 \in N\). Then, we obtain \(i = n\) for \(k_2 = v_3\) since \(n\) is a positive odd integer. Thus, \(LP^2_{\langle x, y, z \rangle}((-2, n, 2)) = n \cdot h_2^2(4(n-1))\) and \(LP^2_{\langle x, y, z \rangle}((-2, n, 2)) = h_2^2(4(n-1))\) since \(x \theta = x, y \theta = y\) and \(z \theta = zy^{k_2 \cdot 4(1-n)}\) where \(\theta\) is an outer automorphism of order \(n\) and \(t_2 \in N\) such that \((t_2, n) = 1\).
The proofs for the groups \(<2,n,-2>, \langle n,-2,2 \rangle, \langle n,2,-2 \rangle, \langle 2,-2,n \rangle\) and \(\langle -2,2,n \rangle\) are similar to the above and are omitted.

All necessary calculations were carried out on the computer using the GAP computational algebra system, see (The GAP group).[17]

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