Abstract. In this paper, we give characterizations of Riesz bases and near Riesz bases in Banach spaces. The notion of atomic system is defined and a characterization of atomic system has been given. Also results exhibiting relationship between frames, atomic systems and Riesz bases have been proved. Further, we show that every atomic system is a projection of a Riesz basis in Banach spaces. Finally, we give some duality results of an atomic system for Banach spaces.

1. INTRODUCTION AND PRELIMINARIES

Frames were introduced in 1952 by Duffin and Schaeffer [5]. They in fact abstracted Gabor’s [9] method to define frames for Hilbert space. Let $\mathcal{H}$ be a real (or complex) separable Hilbert space with inner product $(.,.)$. A countable sequence $\{f_n\} \subset \mathcal{H}$ is called a frame (or Hilbert frame) for $\mathcal{H}$, if there exist numbers $A, B > 0$ such that

$$A\|f\|^2_\mathcal{H} \leq \sum_{n=1}^{\infty} |\langle f, f_n \rangle|^2 \leq B\|f\|^2_\mathcal{H}, \text{ for all } f \in \mathcal{H}. \tag{1.1}$$

The scalars $A$ and $B$ are called the lower and upper frame bounds of the frame, respectively. They are not unique. The inequality in (1.1) is called the frame inequality of the frame. For more details related to frames and Riesz bases in Hilbert spaces, one may refer to [4, 11]. Feichtinger and Gröcheing [7] extended the notion of frames...
to Banach space and defined the notion of atomic decomposition. Gröcheing [10] introduced a more general concept for Banach spaces called Banach frame. Casazza, Christensen and Stoeva [2] studied $E_d$-frame and $E_d$-Bessel sequence. Recall that a BK-space is, by definition, a Banach (scalar) sequence space in which the coordinate functionals are continuous.

**Definition 1.1.** [2] Let $E$ be a Banach space and $E_d$ be a BK-space, a sequence $\{f_n\}_{n=1}^{\infty} \subseteq E^*$ is called an $E_d$-frame for $E$ if

1. $\{f_n(x)\} \in E_d$, for all $x \in E$,
2. there exist constants $A$ and $B$ with $0 < A \leq B < \infty$ such that

\[
(1.2) \quad A \| x \|_E \leq \| \{f_n(x)\} \|_{E_d} \leq B \| x \|_E, \quad \text{for all } x \in E.
\]

$A$ and $B$ are called $E_d$-frame bounds. If at least (1) and the upper bound condition in (1.2) are satisfied, then $\{f_n\}$ is called an $E_d$-Bessel sequence for $E$. If $\{f_n\}$ is an $E_d$-Bessel sequence for $E$, then $U : E \to E_d$ given by

\[
U(x) = \{f_n(x)\}, \quad \text{for } x \in E
\]

is a bounded linear operator and $U$ is called the analysis operator associated to $E_d$-Bessel sequence $\{f_n\}$. If $\{f_n\}$ is an $E_d$-frame and there exists a sequence $\{x_n\} \subseteq E$ such that $x = \sum_{n=1}^{\infty} f_n(x)x_n$, for all $x \in E$. Then, a pair $(x_n, f_n)$ is called an atomic decomposition for $E$ with respect $E_d$. Further, if $\{f_n\}$ is an $E_d$-frame for $E$ and there exists a bounded linear operator $S : E_d \to E$ such that $S(\{f_n(x)\}) = x$ for all $x \in E$, then $(\{f_n\}, S)$ is called a Banach frame for $E$ with respect to $E_d$.

In [15], Stoeva defined and studied $E_d$-Riesz basis.

**Definition 1.2.** [15, 3] For a Banach space $E$ and a BK-space $E_d$, an $E_d$-Riesz basis for $E$ is a sequence $\{x_n\}_{n=1}^{\infty} \subseteq E$, which is complete in $E$ and there exist constants
0 < A \leq B < \infty \text{ such that }
\begin{align*}
(1.3) \quad A \norm{\{c_n\}}_{E_d} \leq \norm{\sum_{n=1}^{\infty} c_n x_n}_E \leq B \norm{\{c_n\}}_{E_d} \text{ for every } \{c_n\}^{\infty}_{n=1} \in E_d.
\end{align*}

A number A (resp. B) in (1.3) is called a lower (resp. upper) $E_d$-Riesz basis bound.

We give few results in the form of lemmas which will be used in the subsequent work.

Lemma 1.1 (Cassaza–Christensen–Stoeva). ([2, Lemma 3.1]) Let $E_d$ be a BK-space for which the canonical unit vectors $\{e_n\}$ form a Schauder basis. Then the space $Y_d = \{\{h(e_n)\} : h \in E_d^*\}$ with norm $\norm{\{h(e_n)\}}_{Y_d} = \norm{h}_{E_d^*}$ is a BK-space isometrically isomorphic to $E_d^*$. Also, every continuous linear functional $\Phi$ on $E_d$ has the form $\Phi\{c_n\} = \sum_{n=1}^{\infty} c_n d_n$, where $\{d_n\} \in Y_d$ is uniquely determined by $d_n = \Phi(e_n)$, and $\norm{\Phi} = \norm{\{\Phi(e_n)\}}_{Y_d}$.

Lemma 1.2 (Taylor–Lay). ([16, Theorem 12.9, p. 251]) Let $X, Y$ be Banach spaces and $S : X \to Y$ be a bounded linear operator from $X$ into $Y$. Then the following are equivalent.

1. $S$ has a pseudoinverse operator $S^\dagger$.
2. There exist closed subspaces $W, Z$ of $X, Y$ such that $X = \ker S \oplus W, Y = S(X) \oplus Z$.

Lemma 1.3 (Stoeva). ([15, Propositions 3.3 and 3.4]) Let $E_d$ be BK-space which has a sequence of canonical unit vectors as Schauder basis and $\{x_n\}^{\infty}_{n=1} \subseteq E$ be a sequence. Then

1. every $E_d$-Riesz basis $\{x_n\}$ is a Schauder basis for $E$.
2. $\{x_n\}$ is a Riesz basis if and only if the operator $T$, given by $T\{\alpha_n\}^{\infty}_{n=1} = \sum_{n=1}^{\infty} \alpha_n x_n$ is an isomorphism of $E_d$ onto $E$. 
Throughout this paper, $E$ will denote a Banach space over the scalar field $\mathbb{K}$ (which is $\mathbb{R}$ or $\mathbb{C}$), $E^*$ the conjugate space of $E$, $[x_n]$ the closed linear span of $\{x_n\}$ in the norm topology of $E$, $[f_n]$ the closed linear span of $\{f_n\} \subseteq E^*$ in the $\sigma(E^*, E)$-topology. Further, $E_d$ denotes a BK-space which has a sequence of canonical unit vectors $\{e_n\}_{n=1}^\infty$ as Schauder basis, $E_d^*$ the conjugate space of $E_d$ and $Y_d = \{\{h(e_n)\} : h \in E_d^*\}$ denotes a BK-space which is defined in Lemma 1.1. $\pi : E \rightarrow E^{**}$ is the natural canonical projection from $E$ onto $E^{**}$. This paper is devoted to the study of frames, $E_d$-Riesz bases, near $E_d$-Riesz bases, atomic systems in Banach spaces and the extremality property of atomic systems. In Section 2, we give the necessary and sufficient conditions for the existence of frames in Banach spaces. We also study the relationship between $E_d$-Riesz bases and frames. A necessary and sufficient condition for a frame to be an $E_d$-Riesz basis is obtained. Also, we give a necessary and sufficient condition for a frame to be a near $E_d$-Riesz basis. Section 3 is devoted to atomic systems in Banach spaces. We prove that every atomic system is a frame in Banach spaces. We also show that $E_d$-Riesz bases and near $E_d$-Riesz bases are atomic systems in Banach spaces. Also, various properties of atomic systems which are similar to that of the properties of frames in Hilbert spaces have been given. Further, we show that atomic systems are compression of $E_d$-Riesz bases in Banach spaces. Finally, we discuss the extremality property of atomic systems in the conjugate of a Banach space.

2. Frames and $E_d$-Riesz bases

P. A. Terekhin [17, 18] introduced and studied the notion of frames for Banach spaces.

Definition 2.1. [17, 18] Let $E$ be a Banach space and $E_d$ be a BK-space which has a sequence of canonical unit vectors $\{e_n\}$ as Schauder basis. A sequence $\{x_n\}_{n=1}^\infty \setminus \{0\} \subseteq E$ is called a frame for $E$ with respect to $E_d$ if
(1) $\{f(x_n)\} \in Y_d$ for all $f \in E^*$.

(2) there exist constants $A$ and $B$ with $0 < A \leq B < \infty$ such that

\begin{equation}
A\|f\|_{E^*} \leq \|\{f(x_n)\}\|_{Y_d} \leq B\|f\|_{E^*}, \text{ for all } f \in E^*.
\end{equation}

We refer (2.1) as the frame inequalities. If at least (1) and the upper bound condition in (2.1) are satisfied, then $\{x_n\}$ is called Bessel sequence for $E$ with respect to $E_d$.

In the following result, we give the necessary and sufficient conditions for the existence of frames in Banach spaces.

**Theorem 2.1.** $\{x_n\}_{n=1}^{\infty} \subseteq E$ is a frame for $E$ with respect to $E_d$ if and only if there exists a bounded linear operator $T : E_d \rightarrow E$ from $E_d$ onto $E$ for which $T(e_n) = x_n$, for all $n \in \mathbb{N}$.

**Proof.** Let $f \in E^*$, $B$ be upper bound of the frame $\{x_n\}$. By Lemma 1.1, $\{f(x_n)\} = \{\Phi_f(e_n)\}$ for some $\Phi_f \in E_d^*$ and $\|\{f(x_n)\}\|_{Y_d} = \|\Phi_f\|_{E_d^*}$. Let $n, m \in \mathbb{N}$ with $n \leq m$ and $\{c_n\} \in E_d$, then

\[
\|\sum_{k=n}^{m} c_k x_k\|_E = \sup_{f \in E^*, \|f\|=1} \|\sum_{k=n}^{m} c_k f(x_k)\|_E
\]

\[
= \sup_{f \in E^*, \|f\|=1} \|\sum_{k=n}^{m} c_k \Phi_f(e_k)\|_E
\]

\[
= \sup_{f \in E^*, \|f\|=1} |\Phi_f(\sum_{k=n}^{m} c_k e_k)|
\]

\[
\leq \sup_{f \in E^*, \|f\|=1} \|\Phi_f\| \|\sum_{k=n}^{m} c_k e_k\|_{E_d}
\]

\[
= \sup_{f \in E^*, \|f\|=1} \|\{f(x_n)\}\|_{Y_d} \|\sum_{k=n}^{m} c_k e_k\|_{E_d}
\]

\[
\leq B \|\sum_{k=n}^{m} c_k e_k\|_{E_d^*}.
\]
Hence, \( T : E_d \to E \) given by \( T \{c_n\} = \sum_{n=1}^{\infty} c_n x_n, \{c_n\} \in E_d \) is well defined bounded linear operator from \( E_d \) into \( E \). Moreover, \( T(e_n) = x_n \), for all \( n \in \mathbb{N} \). By Lemma 1.1 and for \( f \in E^* \) we have

\[ \|\{f(x_n)\}\|_{Y_d} = \|\{f(T(e_n))\}\|_{Y_d} = \|\{T^*(f)(e_n)\}\|_{Y_d} = \|T^* f\|_{E_d^*} \]

and from the frame inequalities we have \( T^* \) is one-one and \( T^*(E^*) \) is closed. Thus by [13, Theorem 4.15, p. 103], \( T \) is onto.

Conversely, let \( T : E_d \to E \) be well defined bounded linear operator from \( E_d \) onto \( E \) with \( T(e_n) = x_n \), for all \( n \in \mathbb{N} \). So \( T^* \) is one-one and \( T^*(E^*) \) is closed by [13, Theorem 4.15, p. 103]. Again, by [6, Lemma 1, p. 487], there exists constant \( C > 0 \) such that \( \|f\| \leq C \|T^*(f)\| \) for all \( f \in E^* \). Let \( f \in E^* \). Also

\[ \{f(x_n)\} = \{f(T(e_n))\} = \{T^* f(e_n)\} \subseteq Y_d. \]

Then, by using Lemma 1.1 and for \( f \in E^* \) we have

\[ \|f\|_{E^*} \leq C \|T^*(f)\|_{E_d^*} = \|\{T^* f(e_n)\}\|_{Y_d} = \|\{f(T(e_n))\}\|_{Y_d} = \|\{f(x_n)\}\|_{Y_d}. \]

To show the upper inequality,

\[ \|\{f(x_n)\}\|_{Y_d} = \|\{T^* f(e_n)\}\|_{Y_d} = \|T^*(f)\|_{E_d^*} \leq \|T\| \|f\|_{E^*}, \text{ for all } f \in E^*. \]

Hence, \( \{x_n\} \) is a frame for \( E \).

Note that, \( \{x_n\} \) is a Bessel sequence for \( E \) if and only if \( T : E_d \to E \) is a well defined bounded linear operator from \( E_d \) into \( E \) for which \( T(e_n) = x_n \), for all \( n \in \mathbb{N} \).

The operator \( T \) is called the synthesis operator associated to Bessel sequence \( \{x_n\} \) and bounded linear operator \( R : E^* \to Y_d \) given by

\[ R(f) = \{f(x_n)\}, \text{ for } f \in E^*. \]
is called the analysis operator associated to Bessel sequence \( \{x_n\} \). As in Lemma 1.1, \( Y_d \) is isometrically isomorphic to \( E_d^* \). So, let \( J_d : Y_d \to E_d^* \) be isometrically isomorphism from \( Y_d \) onto \( E_d^* \). Therefore, \( T^* = J_d \circ \mathcal{R} \).

Remark 1. Every \( E_d \)-Riesz basis is a frame for \( E \) with respect to \( E_d \). This is very clear from Lemma 1.3 and Theorem 2.1.

Next, we give the equivalent conditions for a sequence \( \{x_n\}_{n=1}^{\infty} \subseteq E \) to be an \( E_d \)-Riesz basis in Banach spaces.

**Proposition 2.1.** Let \( \{x_n\}_{n=1}^{\infty} \subseteq E \). Then, the following conditions are equivalent.

1. \( \{x_n\} \) is an \( E_d \)-Riesz basis for \( E \).
2. There exists an isomorphism \( T \) from \( E_d \) onto \( E \) for which \( T(e_n) = x_n \), for all \( n \in \mathbb{N} \).
3. \( \{x_n\} \) is a Schauder basis for \( E \) and \( \sum a_n x_n \) converges if and only if \( \{a_n\} \subseteq E_d \).
4. \( \{x_n\} \) is complete in \( E \) and there exist constants \( 0 < A \leq B < \infty \) such that for every finite sequence of scalars \( c_1, c_2, ..., c_n \), we have
   \[
   A \|\{c_k\}_{k=1}^{n}\|_{E_d} \leq \left\| \sum_{k=1}^{n} c_k x_k \right\|_E \leq B \|\{c_k\}_{k=1}^{n}\|_{E_d} 
   \]
5. \( \{x_n\} \) is a complete Bessel sequence in \( E \) and possesses a biorthogonal system \( \{f_n\} \subseteq E^* \) which is also an \( E_d \)-Bessel sequence and total over \( E \) i.e. \( \{f_n\} = E^* \).

**Proof.** (1) \(\Leftrightarrow\) (2) Follows from Lemma 1.3.

(2) \(\Rightarrow\) (5) By given hypothesis, \( \{x_n\} \) is a frame for \( E \) and \( \{x_n\} \) is complete. Let \( \{l_n\} \) be sequence of coordinate functionals on \( E_d \) and take \( f_n = (T^{-1})^* l_n \), \( n \in \mathbb{N} \). Now for \( x \in E \), we have \( f_n(x) = l_n(T^{-1}(x)) \), \( n \in \mathbb{N} \). So, \( \{f_n(x)\} = T^{-1}(x) \in E_d \), for all \( x \in E \). Moreover,
   \[
   \|\{f_n(x)\}\|_{E_d} = \|T^{-1}(x)\|_{E_d} \leq \|T^{-1}\| \|x\|_E, \text{ for all } x \in E.
   \]
To show biorthogonality, let $i, j \in \mathbb{N}$ we have
\[ f_i(x_j) = l_i(T^{-1}(x_j)) = l_i(T^{-1}(e_j)) = l_i(e_j) = \delta_{ij}. \]

Finally, let $x \in E$ and $f_n(x) = 0$ for all $n \in \mathbb{N}$. Then, $T^{-1}(x) = 0$ which implies that $x = 0$. Hence, $[f_n] = E^*$.

(5) $\Rightarrow$ (2) Since, $\{x_n\}$ is a complete Bessel sequence and its synthesis operator $T$ given by $T(\{a_n\}) = \sum_{n=1}^{\infty} a_n x_n$ is well defined bounded linear operator from $E_d$ into $E$. Also $\{f_n(x)\} \in E_d$, for all $x \in E$. Let $x \in E$ and let $z = \sum_{n=1}^{\infty} f_n(x) x_n$. So, for any $j \in \mathbb{N}$ we have $f_j(z) = \sum_{n=1}^{\infty} f_n(x) f_j(x_n) = f_j(x)$. Therefore $f_j(z - x) = 0$ for all $j \in \mathbb{N}$. Since, $\{f_n\}$ is total over $E$, so $x = z$. Thus,
\[ x = \sum_{n=1}^{\infty} f_n(x) x_n = T(\{f_n(x)\}), \text{ for all } x \in E. \]

That shows $T$ is onto. To show one-one, let $T(\{a_n\}) = \sum_{n=1}^{\infty} a_n x_n = 0$. Take any $j \in \mathbb{N}$. Then, $f_j(\sum_{n=1}^{\infty} a_n x_n) = 0$. From here $a_j = 0$, for all $j \in \mathbb{N}$. Hence, $T$ is an isomorphism from $E_d$ onto $E$.

(4) $\Rightarrow$ (3) Take $c_n = 1$ and $c_j = 0$ if $j \neq n$. So, $0 < A \leq \|x_n\|$ which shows $x_n$ is non zero for all $n \in \mathbb{N}$. Take any finite sequence of scalars $c_1, c_2, ..., c_n$. Let $N, M \in \mathbb{N}$ with $N \leq M$, then
\[ \| \sum_{k=1}^{N} c_k x_k \|_E \leq B \| \sum_{k=1}^{N} c_k e_k \|_{E_d} \leq B \| \sum_{k=1}^{M} c_k e_k \|_{E_d} \leq \frac{B}{A} \| \sum_{k=1}^{M} c_k e_k \|_E. \]

Also, $\text{span}\{x_n\}_{n=1}^{\infty} = E$. Thus, $\{x_n\}$ is a Schauder basis for $E$. It remains to show that $\sum_{n} a_n x_n$ converges if and only if $\{a_n\} \in E_d$. For this, take $N, M \in \mathbb{N}$ with $N \leq M$
\[ A \| \sum_{k=N}^{M} a_k e_k \|_{E_d} \leq \| \sum_{k=N}^{M} a_k e_k \|_E \leq B \| \sum_{k=N}^{M} a_k e_k \|_{E_d}. \]
From here it clearly shows that $\sum\limits_n a_n x_n$ converges if and only if $\{a_n\} \in E_d$.

(3) $\Rightarrow$ (2) By the given conditions, Schauder basis $\{x_n\}$ is equivalent to Schauder basis $\{e_n\}$. So there exists an isomorphism $T$ from $E_d$ onto $E$ such that $T(e_n) = x_n$, for all $n \in \mathbb{N}$.

(2) $\Rightarrow$ (4) As $\{T(e_n) = x_n\}$ is a Schauder basis for $E$, so $\{x_n\}$ is complete in $E$. Let $c_1, c_2, ..., c_n$ be any $n$ scalars. Then

$$\|\{c_k\}_{k=1}^n\|_{E_d} = \|T^{-1}T(\{c_k\}_{k=1}^n)\|_{E_d} \leq \|T^{-1}\| \sum\limits_{k=1}^n c_k x_k \|_{E_d},$$

$$\|\sum\limits_{k=1}^n c_k x_k \|_{E} = \|T(\{c_k\}_{k=1}^n)\|_{E} \leq \|T\| \|c_k\|_{E_e} \|_E.$$

In the following theorem, we give the characterization of Riesz bases from frames in Banach spaces.

**Theorem 2.2.** Let $\{x_n\}$ be a frame for $E$ with respect to $E_d$ and $T$ be its synthesis operator. Then, the following conditions are equivalent.

1. $\{x_n\}$ is an $E_d$-Riesz basis for $E$.
2. $T$ is one-one.
3. $\{x_n\}$ is a Schauder basis for $E$.
4. $\{x_n\}$ has unique biorthogonal system $\{f_n\} \subseteq E^*$.
5. If $\sum\limits_{n=1}^\infty d_n x_n = 0$ for some $\{d_n\}_{n=1}^\infty \in E_d$, then $d_n = 0$ for all $n \in \mathbb{N}$.
6. $\{x_n\}$ is minimal, that is $x_j \notin \text{span}\{x_n\}_{n \neq j}$.

**Proof.** (1) $\Leftrightarrow$ (2) Obvious.

(2) $\Rightarrow$ (3) Since, $\{e_n\}$ is a Schauder basis for $E_d$ and $T$ is an isomorphism from $E_d$ onto $E$. So, $\{x_n\} = \{T(e_n)\}$ is a Schauder basis for $E$.
(2) \implies (4) Let \( \{ l_n \} \subseteq E_d^* \) be sequence of coordinate functionals on \( E_d \). Take \( f_n = (T^{-1})^* l_n \in E^* \), for \( n \in \mathbb{N} \). So that

\[
 f_n(x_j) = (T^{-1})^* l_n(x_j) = l_n T^{-1}(x_j) = l_n(T^{-1} e_j) = l_n(e_j) = \delta_{nj}.
\]

(4) \implies (3) Let \( \{ f_n \}_{n=1}^\infty \subseteq E^* \) be a sequence which is unique biorthogonal system to \( \{ x_n \} \). But for any \( x \in E \), \( x = \sum_{n=1}^\infty \alpha_n x_n \), \( \{ \alpha_n \} \in E_d \). Therefore, \( f_n(x) = \alpha_n \), for all \( n \in \mathbb{N} \). Thus, \( \{ x_n \} \) is a Schauder basis for \( E \).

(3) \implies (4) Obvious.

(2) \iff (5) Obvious.

(4) \iff (6). Follows from [11, Lemma 5.4, p. 155]. \( \square \)

**Definition 2.2.** A frame \( \{ x_n \} \subseteq E \) is called a *near \( E_d \)-Riesz basis* for \( E \) if there exists a finite subset \( \sigma \) of \( \mathbb{N} \) for which \( \{ x_n \}_{n \in \mathbb{N} \setminus \sigma} \) is an \( E_d \)-Riesz basis for \( E \).

Next, we give the following characterization of near \( E_d \)-Riesz bases in Banach spaces. This result generalizes the result due to Holub [12, Theorem 2.4], let \( \{ x_n \}_{n=1}^\infty \) be a frame in Hilbert space \( \mathcal{H} \) and \( Q : l^2 \to \mathcal{H} \) be the associated preframe operator. Then, ker\( Q \) is finite dimensional \( \iff \{ x_n \}_{n=1}^\infty \) is a near-Riesz basis for \( \mathcal{H} \).

**Theorem 2.3.** Let \( \{ x_n \} \) be a frame for \( E \) with respect to \( E_d \) and \( T \) be the associated synthesis operator. Then, ker\( T \) is finite dimensional if and only if \( \{ x_n \} \) is a near \( E_d \)-Riesz basis for \( E \).

**Proof.** Let ker\( T \) be finite dimensional subspace of \( E_d \). So, there exists a complemented subspace \( M \) of \( E_d \) such that \( E_d = M \oplus \ker T \). And let \( Q \) be projection from \( E_d \) onto \( M \) and \( I_{E_d} - Q \) is a projection from \( E_d \) onto ker\( T \). Define \( \Gamma : E_d \to E \oplus \ker T \) as

\[
 \Gamma(\alpha) = (T(\alpha), (I_{E_d} - Q)(\alpha)), \text{ for } \alpha \in E_d.
\]

Then, \( \Gamma \) is an isomorphism from \( E_d \) onto \( E \oplus \ker T \) by [18, Theorem 2]. Let \( \{ e_n \} \) be sequence of canonical unit vectors as Schauder basis of \( E_d \) and take \( y_n = \Gamma(e_n) \),
for all \( n \in \mathbb{N} \). Then, \( \{y_n\} \) is a Schauder basis of \( E \oplus \ker T \). Let \( P \) be projection from \( E \oplus \ker T \) onto \( E \). By the construction of \( \Gamma \), we have \( T = PT \). Thus \( x_n = T(e_n) = P(\Gamma(e_n)) = P(y_n) \), \( n \in \mathbb{N} \). Let \( \dim(\ker T) = N \), by [1, Theorem 2] there exists a finite subset \( \sigma = \{k_1, k_2, k_3, \ldots, k_N\} \) of \( \mathbb{N} \), where \( k_i \neq k_j \) at \( i \neq j \) for which \( \{P(y_n)\}_{n \in \mathbb{N} \setminus \sigma} \) is a Schauder basis for \( E \). Thus, by Theorem 2.2, \( \{x_n\}_{n \in \mathbb{N} \setminus \sigma} \) is a \( E_d \)-Riesz basis for \( E \). Hence, \( \{x_n\}_{n=1}^\infty \) is near \( E_d \)-Riesz basis for \( E \).

For the converse, let us assume that \( \ker T \) is infinite dimensional subspace of \( E_d \) and let \( \{u_n\}_{n=1}^\infty \) be the basis of \( \ker T \). By given hypothesis, there exists a finite subset \( \sigma \) of \( \mathbb{N} \) such that \( \{x_n\}_{n \in \mathbb{N} \setminus \sigma} \) is a Riesz basis. So, by Theorem 2.2, \( T|_{[e_n]_{n \in \mathbb{N} \setminus \sigma}} \) is an isomorphism from \( [e_n]_{n \in \mathbb{N} \setminus \sigma} \) onto \( [x_n]_{n \in \mathbb{N} \setminus \sigma} = E \). But \( \text{codim}(e_n|_{n \in \mathbb{N} \setminus \sigma}) = \text{card}(\sigma) = k < \infty \). By [14, Lemma 4.1, p. 268], there exists a non zero element \( u \in [e_n|_{n \in \mathbb{N} \setminus \sigma} \cap [u_n|_{n=1}^{k+1} \subset \ker T \). Thus, \( T(e_n|_{n \in \mathbb{N} \setminus \sigma}) = 0 \). That gives us \( u = 0 \), which is a contradiction. Hence, \( \ker T \) is finite dimensional subspace of \( E_d \).

**Corollary 2.1.** Let \( \{x_n\} \) be a frame for \( E \) with respect to \( E_d \) for which \( \{x_n\}_{n \in \mathbb{N} \setminus \sigma} \) is an \( E_d \)-Riesz basis for some finite subset \( \sigma \) of \( \mathbb{N} \) and \( T \) be the synthesis operator of \( \{x_n\} \). Then, \( \text{card}(\sigma) = \dim(\ker T) \).

**Proof.** From Theorem 2.3, \( \{y_n\} \) is a Schauder basis of \( E \oplus \ker T \) and \( \{x_n\}_{n \in \mathbb{N} \setminus \sigma} \) is a Schauder basis for \( E \). Thus, \( [y_n] = [x_n]_{n \in \mathbb{N} \setminus \sigma} \oplus \ker T \). Therefore, \( \text{card}(\sigma) = \dim(\ker T) \). \( \square \)

**Remark 2.** Let \( \{x_n\} \) be near \( E_d \)-Riesz basis and \( T \) be its associated synthesis operator, then \( T \) is a Fredholm operator. By Theorem 2.3, \( \ker T \) is finitely dimensional. But, \( \ker R = \ker T^* \) and from frame inequalities \( \ker T^* = \{0\} \). Also \( T \) is bounded linear operator from \( E_d \) onto \( E \). Hence, \( T \) is Fredholm operator.
3. Atomic systems

The notion of a family of local atoms for a closed subspace $\mathcal{H}_0$ of a Hilbert space $\mathcal{H}$ is introduced in [8]. We generalize this notion and define atomic system for Banach spaces.

**Definition 3.1.** Let $E$ be a Banach Space and $E_d$ be a BK-space which has a sequence of canonical unit vectors $\{e_n\}$ as Schauder basis. A Bessel sequence $\{x_n\}_{n=1}^{\infty}\setminus\{0\} \subseteq E$ for $E$ with respect to $E_d$ is called an atomic system for $E$ with respect to $E_d$ if there exists an $E_d$-Bessel sequence $\{f_n\} \subseteq E^*$ for $E$ such that for every $x \in E$, we have $x = \sum_{n=1}^{\infty} f_n(x)x_n$. We shall call $\{f_n\}$ as the associated $E_d$-Bessel sequence for atomic system $\{x_n\}$.

Remark 3. An atomic system for $E$ with respect to a BK-space $E_d$ is a frame for $E$ with respect to $E_d$. By definition, there exists an $E_d$-Bessel sequence $\{f_n\} \subseteq E^*$ with bound $D$ and satisfying $x = \sum_{n=1}^{\infty} f_n(x)x_n$, for all $x \in E$. So, for $f \in E^*$ we have

$$\|f\|_{E^*} = \sup_{x \in E, \|x\| = 1} |f(\sum_{n=1}^{\infty} f_n(x)x_n)| \leq \sup_{x \in E, \|x\| = 1} D \|x\|_E \|\{f(x_n)\}\|_{Y_d}$$

$$= D \|\{f(x_n)\}\|_{Y_d}.$$ 

Thus, $\{x_n\}$ is a frame for $E$ with respect to $E_d$.

Remark 4. It is observed that every frame $\{x_n\}$ for Hilbert space $\mathcal{H}$ is an atomic system. However, every frame for Banach need not be atomic system, that we will show later.

In the following, we shall show that every $E_d$-Riesz basis for $E$ is an atomic system for $E$. 
Theorem 3.1. Every $E_d$-Riesz basis $\{x_n\}$ for $E$ is an atomic system.

Proof. Let $A$, $B$ be bounds of an $E_d$-Riesz basis $\{x_n\}$. Obviously, $\{x_n\}$ is a frame and its synthesis operator $T$ is an isomorphism from $E_d$ onto $E$. Take $f_n = (T^{-1})^* l_n$, $n \in \mathbb{N}$ and $\{l_n\}_{n=1}^{\infty}$ be sequence of coordinate functionals on $E_d$. For $x \in E$ we have

$$f_n(x) = (T^{-1})^* l_n(x) = l_n(T^{-1}(x)), \text{ for all } n \in \mathbb{N}.$$  

So, $\{f_n(x)\} = T^{-1}(x) \in E_d$, for all $x \in E$. Moreover,

$$\|\{f_n(x)\}\|_{E_d} = \|T^{-1}(x)\|_{E_d} \leq \|T^{-1}\| \|x\|_E \leq A^{-1} \|x\|_E, \text{ for all } x \in E.$$  

Also, $x = TT^{-1}(x) = T(\{f_n(x)\}) = \sum_{n=1}^{\infty} f_n(x)x_n$, for all $x \in E$.  

Next, we give the following characterization of atomic systems.

Theorem 3.2. Let $\{x_n\}$ be frame for $E$ with respect to $E_d$ and $T$ be its associated synthesis operator, then the following are equivalent.

1. $\{x_n\}$ is an atomic system.
2. $T$ has pseudoinverse $T^\dagger$.
3. $\ker T$ is complemented subspace of $E_d$.
4. $T^*(E^*)$ is complemented subspace of $E_d^*$.
5. $R(E^*)$ is complemented subspace of $\mathcal{Y}_d$
6. $T^*$ has pseudoinverse $T^{*\dagger}$.
7. There exists a complemented subspace $M$ of $E_d$ with $TQ(E_d) = E$, where $Q$ is a projection from $E_d$ onto $M$ and positive constants $0 < A \leq B < \infty$ such that

$$A\|\{\alpha_n\}\|_{E_d} \leq \sum_{n} \alpha_n x_n \|E \leq B\|\{\alpha_n\}\|_{E_d}, \text{ for all } \{\alpha_n\} \in M.$$
Proof. (1) \( \Rightarrow \) (2) By given hypothesis, there exists an \( E_d \)-Bessel sequence \( \{f_n\} \subseteq E^* \) such that

\[
x = \sum_{n=1}^{\infty} f_n(x)x_n, \text{ for all } x \in E.
\]

Let \( U : E \to E_d \) be the associated analysis operator of \( E_d \)-Bessel sequence \( \{f_n\} \) given by \( U(x) = \{f_n(x)\}, x \in E \). Then, \( TU = I_E \) and \( TUT = T \). Hence, \( T \) has pseudoinverse.

(2) \( \Rightarrow \) (1) \( TT^\dagger \) is a projection from \( E \) onto \( T(E_d) = E \). So \( TT^\dagger = I_E \). Take \( f_n = (T^\dagger)^*(l_n), n \in \mathbb{N} \), where \( \{l_n\} \subseteq E^*_d \) is a sequence of coordinate functionals on \( E_d \). So, for \( x \in E \), we have

\[
f_n(x) = (T^\dagger)^*(l_n(x)) = l_n(T^\dagger(x)).
\]

This gives \( \{f_n(x)\} = T^\dagger(x) \in E_d \), for all \( x \in E \). Further

\[
\|\{f_n(x)\}\|_{E_d} \leq \|T^\dagger\|\|x\|_E, \text{ for all } x \in E.
\]

Thus, \( \{f_n\} \) is an \( E_d \)-Bessel sequence for \( E \). Also, for \( x \in E \), we have

\[
x = TT^\dagger(x) = T(\{f_n(x)\}) = \sum_{n=1}^{\infty} f_n(x)x_n.
\]

Hence, \( \{x_n\} \) is an atomic system for \( E \).

(2) \( \Leftrightarrow \) (6) Straight forward.

(4) \( \Leftrightarrow \) (5) Straight forward.

(2) \( \Leftrightarrow \) (3) Since \( T(E_d) = E \), by Lemma 1.2, the equivalence follows.

(4) \( \Leftrightarrow \) (6) Since \( \ker T^* = \{0\} \), by Lemma 1.2, the equivalence follows.

(3) \( \Rightarrow \) (7) By given hypothesis we have \( E_d = M \oplus \ker T \), where \( M \) is a closed subspace of \( E_d \). Let \( T_1 : M \to E \) is restriction of \( T \) to \( M \). By [16, Theorem 6.3, p. 29], \( T_1 \) is
an isomorphism from $M$ onto $E$. So, for $\{\alpha_n\} \in M$ we have

$$\|\{\alpha_n\}\|_{E_d} = \|T_1^{-1}T_1(\{\alpha_n\})\|_{E_d} \leq \|T_1^{-1}\| \|T_1(\{\alpha_n\})\|_E = \|T_1^{-1}\| \sum_n \alpha_n x_n \|_E.$$ 

Also,

$$\| \sum_n \alpha_n x_n \|_E = \|T_1(\{\alpha_n\})\|_E \leq \|T_1\| \|\{\alpha_n\}\|_{E_d}, \text{ for all } \{\alpha_n\} \in M.$$ 

(7) $\Rightarrow$ (2) Let $T_1 : M \to E$ as restriction of $T$ to $M$. Then, $T_1(\{\alpha_n\}) = \sum_n \alpha_n x_n$, for $\{\alpha_n\} \in M$. From the given condition, $T_1$ is invertible and $T = T_1Q$. Thus,

$$TT_1^{-1}T(\alpha) = T_1QT_1^{-1}T_1Q(\alpha) = T_1Q^2(\alpha) = T(\alpha), \text{ for all } \alpha \in E_d.$$ 

Hence, $T$ has pseudoinverse. \qed

Remark 5. A near $E_d$-Riesz basis for a Banach space $E$ is an atomic system.

A Banach space $X$ is said to be primary if each of its infinite-dimensional complemented subspace is isomorphic to $X$.

**Theorem 3.3.** Let $E_d$ be primary BK-space. Let $E$ be not isomorphic to $E_d$. Then, none of the frames $\{x_n\}$ for $E$ with respect to $E_d$ is an atomic system.

**Proof.** On the contrary, let $\{x_n\}$ be an atomic system. So by Theorem 3.2, $E_d = kerT \oplus M$, where $M$ is a closed subspace of $E_d$. Moreover, from the proof of Theorem 3.2, the restriction of $T$ on $M$, $T : M \to E$ is an isomorphism from $M$ onto $E$. But $M$ is isomorphic to $E_d$. Thus, $E$ is isomorphic to $E_d$ which is a contradiction. Hence, $\{x_n\}$ is not an atomic system. \qed

Now, we will show that a frame need not be an atomic system. We give the following example.
Example 3.1. Let $E = L[0, 1]$ be the Lebesgue space. We extend each function $\phi \in L[0, 1]$ beyond the unit interval by zero. Define

$$
\phi_n(t) = \phi_{j,k}(t) = 2^j \phi(2^j t - k), \quad n \in \mathbb{N},
$$

where $j=0,1,2,...$ and $k = 0, 1, ..., 2^j - 1$ are such that $n = 2^j + k$. In [18, Example 1], it is shown that system of functions $\{\phi_n(t) : t \in [0, 1]\}$ is a frame for $E$ with respect to $E_d = l_1$. But $l_1$ is a primary Banach space which is not isomorphic to $L[0, 1]$. Hence, by Theorem 3.3, this system of functions $\{\phi_n(t) : t \in [0, 1]\}$ is not an atomic system for $E$ with respect to $l_1$.

In the following, we shall show that an atomic system for $E$ is a projection of an $E_d$-Riesz basis of an ambient Banach space containing $E$.

Theorem 3.4. Let $\{x_n\}_{n=1}^\infty \subseteq E$. Then $\{x_n\}$ is an atomic system for $E$ with respect to $E_d$ if and only if there exist a Banach space $Z$ with $E$ as its complemented subspace and Riesz basis $\{y_n\}$ for $Z$ such that $P(y_n) = x_n$ for all $n \in \mathbb{N}$, where $P$ is a projection from $Z$ onto $E$.

Proof. Let $\{x_n\}$ be an atomic system for $E$ and $T$ be its synthesis operator. By Theorem 3.2, ker$T$ is a complemented subspace of $E_d$. So, $E_d = M \oplus$ ker$T$, where $M$ is a closed subspace of $E_d$ and let $Q$ be a projection from $E_d$ onto $M$. Take $Z = E \oplus$ ker$T$. From Theorem 2.3, a map $\Gamma : E_d \to E \oplus$ ker$T$ defined by $\Gamma(\alpha) = (T(\alpha), (I_{E_d} - Q)(\alpha))$ for $\alpha \in E_d$ is an isomorphism from $E_d$ onto $Z$. Take $y_n = \Gamma(e_n)$, for $n \in \mathbb{N}$. So, $\{y_n\}$ is a Schauder basis of $Z$ and $\text{span}\{y_n\}_{n=1}^\infty = Z$. Let $\alpha = \{\alpha_n\} \in E_d$ and we have

$$
\|\alpha\|_{E_d} = \|\Gamma^{-1}\Gamma(\alpha)\|_{E_d} \leq \|\Gamma^{-1}\|\|\Gamma(\alpha)\|_Z = \|\Gamma^{-1}\|\sum_{n=1}^\infty \alpha_n y_n\|_Z.
$$
Also,
\[ \left\| \sum_{n=1}^{\infty} \alpha_n y_n \right\|_Z = \left\| \Gamma(\alpha) \right\|_Z \leq \left\| \Gamma \right\| \left\| \alpha \right\|_{E_d}, \text{ for all } \alpha \in E_d. \]

Thus, \( \{y_n\} \) is a \( E_d \)-Riesz basis for \( E \). Let \( P \) be projection from \( Z \) onto \( E \). We know that \( T(e_n) = x_n \), for all \( n \in \mathbb{N} \). From the definition of \( \Gamma \), we have \( T = P \Gamma \). Hence, \( x_n = T(e_n) = P \Gamma(e_n) = P(y_n) \), for all \( n \in \mathbb{N} \).

Conversely, let \( Z = E \oplus W \), where \( W \) is a closed subspace of \( Z \). Let \( \{y_n\} \) be an \( E_d \)-Riesz basis for \( Z \) with bounds \( A, B \) and \( P(y_n) = x_n \), for all \( n \in \mathbb{N} \). Then, by Proposition 2.1 there is an isomorphism \( \Gamma : E_d \rightarrow Z \) from \( E_d \) onto \( Z \) such that \( \Gamma(e_n) = y_n \), for all \( n \in \mathbb{N} \). Define \( T : E_d \rightarrow E \) as \( T = P \Gamma \). So, \( T \) is surjective and \( T(e_n) = P \Gamma(e_n) = P(y_n) = x_n \), for all \( n \in \mathbb{N} \). By Theorem 2.1, \( \{x_n\} \) is a frame for \( E \) and \( T \) is the synthesis operator. Let \( S : E \rightarrow E_d \) be the restriction of \( \Gamma^{-1} \) to \( E \).

Then,
\[ TST = P \Gamma \Gamma^{-1} P \Gamma = P^2 \Gamma = P \Gamma = T. \]

Thus, \( S \) is the pseudoinverse of \( T \). By Theorem 3.2, \( \{x_n\} \) is an atomic system. \( \square \)

Next, we discuss dual atomic system of a given atomic system \( \{x_n\} \) analogue to that of the dual frame in Hilbert spaces. That is, if \( \{x_n\} \) is an atomic system for \( E \) with \( \{f_n\} \) as its associated \( E_d \)-Bessel sequence and \( E_d^* \) has a sequence of canonical unit vectors \( \{e_n^*\} \) as Schauder basis. Then, \( \{f_n\} \) is an atomic system for \( E^* \) with respect to \( Y_d \) and \( (f_n, \pi(x_n)) \) is an atomic decomposition for \( E^* \) with respect to \( Y_d \).

In the following, we give duality results of a given atomic system \( \{x_n\} \).

**Theorem 3.5.** Let \( E_d^* \) has a sequence of canonical unit vectors \( \{e_n^*\} \) as Schauder basis and \( \{x_n\} \subseteq E \). Then,

1. if \( \{x_n\} \) is an atomic system for \( E \) with respect to \( E_d \) and \( \{f_n\} \subseteq E^* \) be its associated \( E_d \)-Bessel sequence, then \( \{f_n\} \) is an atomic system for \( E^* \) with respect to \( Y_d \).
Thus, with respect to

(2) By Theorem 2.1, (3.1)

(2) Since, $U$ and $V$ have pseudoinverse $V^\dagger$ and by Theorem 3.2, $\{f_n\}$ is an atomic system for $E^*$ with respect to $Y_d$.

Proof. (1) By given hypotheses, $\{x_n\}$ is a Bessel sequence for $E$ with respect to $E_d$ and

\[ x = \sum_{n=1}^{\infty} f_n(x) x_n, \quad \text{for all } x \in E. \]

Let $T$ be the associated synthesis operator of $\{x_n\}$ and $U$ be the associated analysis operator of $\{f_n\}$. Then, $I = TU$. So $I^* = U^* T^*$ and $f = U^* T^*(f)$, for all $f \in E^*$.

Therefore, $U^* : E_d^* \to E^*$ is surjective. Define $V : Y_d \to E^*$ as $V = U^* J_d$. Obviously, $\{J_d^{-1}(e_n^*)\}_{n=1}^{\infty}$ is basis of $Y_d$ and $V$ is also surjective. Also, for $n \in \mathbb{N}$ we have

\[ U^* e_n^*(x) = e_n^*(U(x)) = e_n^*(\{f_n(x)\}) = f_n(x), \quad \text{for all } x \in E. \]

Thus, $f_n = U^* e_n^*$ and $V(J_d^{-1} e_n^*) = f_n$, for all $n \in \mathbb{N}$. Since $V$ is surjective, so for any $f \in E^*$ there exists $\alpha = \{\alpha_n\} \in Y_d$ such that $f = V(\alpha)$ and

\[ f = V(\alpha) = V(\sum_{n=1}^{\infty} \alpha_n J_d^{-1}(e_n^*)) = \sum_{n=1}^{\infty} \alpha_n U^* J_d J_d^{-1}(e_n^*) = \sum_{n=1}^{\infty} \alpha_n f_n. \]

By Theorem 2.1, $\{f_n\}$ is a frame for $E^*$ with $V$ as its synthesis operator. Moreover,

\[ V J_d^{-1} T^* V = U^* J_d J_d^{-1} T^* U^* J_d = U^* J_d = V \]

Thus, $V$ has pseudoinverse $V^\dagger$ and by Theorem 3.2, $\{f_n\}$ is an atomic system for $E^*$ with respect to $Y_d$.

(2) Since, $\{x_n\}$ is an atomic system, so $\{x_n\}$ is a frame for $E$. Let $\{f_n\}$ be its associated $E_d$-Bessel sequence with bound $C$. As in Proof (1), we have $T_{E^*} = U^* T^* = U^* J_d \mathcal{R}$ and $\{J_d^{-1}(e_n^*)\}$ is a basis of $Y_d$. Now, for any $f \in E^*$ we have

\[ f = U^* T^*(f) = U^* J_d \mathcal{R}(f) = U^* J_d(\{f(x_n)\})(f) = \sum_{n=1}^{\infty} f(x_n) U^* J_d J_d^{-1}(e_n^*). \]
Hence,

\[(3.2)\quad f = \sum_{n=1}^{\infty} f(x_n)f_n, \text{ for all } f \in E^*.\]

Conversely, since \((f_n, \pi(x_n))\) is an atomic decomposition of \(E^*\) with respect to \(Y_d\), so there exist constants \(0 < A \leq B < 0\) such that

\[A\|f\|_{E^*} \leq \|\{f(x_n)\}\|_{Y_d} \leq B\|f\|_{E^*}, \text{ for all } f \in E^*,\]

and

\[f = \sum_{n=1}^{\infty} f(x_n)f_n, \text{ for all } f \in E^*.\]

Also, \(\{f_n(x)\} \in E_d\), for all \(x \in E\). Let \(x \in E\) and \(N \in \mathbb{N}\). Then

\[
\|x - \sum_{k=1}^{N} f_k(x)x_k\|_E = \sup_{f \in E^*, \|f\|=1} \left| \sum_{k=N+1}^{\infty} f_k(x)f(x_k) \right|
\]

\[
\leq B\| \sum_{k=N+1}^{\infty} f_k(x)\|_{E_d} \to 0 \text{ as } N \to \infty
\]

Hence, \(x = \sum_{n=1}^{\infty} f_n(x)x_n\), for all \(x \in E\). \(\square\)

In [18], P. A. Terekhin discussed an analogue of the extremality property of frame expansion in \(E\). We shall discuss such similar property in the conjugate space \(E^*\) from a given atomic system. A closed subspace \(F\) in a Banach space \(E\) is said to be 1-complemented or constrained, if \(F\) is the range of a norm one projection on \(E\).

In the following result, we characterize 1-complemented subspace of \(Y_d\) in term of an atomic system for \(E\).

**Theorem 3.6.** Let \(E_d^*\) has a sequence of canonical unit vectors \(\{e_n^*\}\) as Schauder basis. Let \(\{x_n\}\) be frame for \(E\). Then, the following conditions are equivalent.

1. \(\mathcal{R}(E^*)\) is 1-complemented subspace of \(Y_d\).
(2) \( \{x_n\} \) is an atomic system for \( E \) such that among all \( \{\alpha_n\} \in \mathcal{Y}_d \) for which the representation (3.1) holds, the sequence \( \{f(x_n)\} \) of the coefficients of expansion (3.2) has minimum \( \mathcal{Y}_d \) norm,

\[
\|\{f(x_n)\}\|_{\mathcal{Y}_d} \leq \|\{\alpha_n\}\|_{\mathcal{Y}_d}.
\]

(3.3)

Proof. (2) \( \Rightarrow \) (1) By Theorem 3.2, \( \mathcal{R}(E^*) \) is a complemented subspace of \( \mathcal{Y}_d \). Let \( \{f_n\} \) be its associated \( E_d \)-Bessel sequence. As proved in Theorem 3.5, we have

\[
I_{E^*} = U^* T^* = U^* J_d J_d^{-1} T^* = VR
\]

So, \( V = VRV \) and \( \mathcal{R} = RVR \). Indeed, \( Q = RV \) is a projection from \( \mathcal{Y}_d \) onto \( \mathcal{R}(E^*) \) and \( \mathcal{Y}_d = \mathcal{R}(E^*) \oplus kerRV \). It is obvious that \( kerV \subseteq kerRV \). Let \( \alpha \in kerRV \), then \( RV(\alpha) = 0 \). So, \( VRV(\alpha) = 0 \) and \( V(\alpha) = 0 \). Thus, \( \mathcal{Y}_d = kerV \oplus \mathcal{R}(E^*) \). Let \( \alpha \in \mathcal{Y}_d \), then \( \alpha = \alpha_0 + Q(\alpha) \), where \( \alpha_0 \in kerV \) and \( Q(\alpha) \in \mathcal{R}(E^*) \). Moreover, \( Q(\alpha) = R(f) \) for some \( f \in E^* \) and

\[
f = VR(f) = VQ(\alpha) = V(\alpha - \alpha_0) = V(\alpha),
\]

which satisfies (3.1). Thus, \( \|Q(\alpha)\|_{\mathcal{Y}_d} = \|R(f)\|_{\mathcal{Y}_d} \leq \|\alpha\|_{\mathcal{Y}_d} \). Therefore, \( \|Q\| = 1 \). Hence \( \mathcal{R}(E^*) \) is 1-complemented subspace of \( \mathcal{Y}_d \).

(1) \( \Rightarrow \) (2) By Theorem 3.2, \( \{x_n\} \) is an atomic system for \( E \). Also, there exists a projection \( Q \) from \( \mathcal{Y}_d \) onto \( \mathcal{R}(E^*) \) and \( \|Q\| = 1 \). Take an arbitrary element \( \alpha = \{\alpha_n\} \in \mathcal{Y}_d \) which satisfies the representation (3.1). Then, \( f = V(\alpha) \), for \( f \in E^* \).

As shown above that \( E_d^* = kerV \oplus \mathcal{R}(E^*) \). Let \( \alpha = \alpha_0 + Q(\alpha) \), for \( \alpha_0 \in kerV \) and \( Q(\alpha) \in \mathcal{R}(E^*) \). Then, \( Q(\alpha) = f^1 \), for some \( f^1 \in E^* \). Also, as shown above

\[
f^1 = V\mathcal{R}(f^1) = V(Q(\alpha)) = V(\alpha) = f.
\]

Thus, \( Q(\alpha) = R(f) \). Finally, we have

\[
\|\{f(x_n)\}\|_{\mathcal{Y}_d} = \|R(f)\|_{\mathcal{Y}_d} = \|Q(\alpha)\| \leq \|Q\| \|\alpha\|_{\mathcal{Y}_d} = \|\alpha\|_{\mathcal{Y}_d}.
\]
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