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A CLASS OF RATIO-CUM-RATIO-TYPE EXPONENTIAL ESTIMATORS FOR POPULATION MEAN WITH SUB SAMPLING THE NON-RESPONDENTS

Surya K. Pal [(1)] AND Housila P. Singh[(2)]

ABSTRACT: This paper addresses the problem of estimating population mean using auxiliary information in presence of non-response in sample surveys. We have suggested a class of ratio-cum-ratio-type exponential estimators for population mean along with its properties. Range of the scalar involved in the proposed class of estimators is found in which the suggested class of estimators is more efficient than usual unbiased estimator, ratio estimator and ratio-type exponential estimator. Double sampling version of the suggested class of estimators is also given along with its properties. Numerical illustrations are given in support of present study.

1. INTRODUCTION

Usually almost all surveys suffer from the problem of non-response. Lack of information, absence at the time of survey, and refusal of the respondents are main reason of the non-response. However an extensive description of the different types of non-response and their effects on surveys could be found in Cochran (1977). Hansen and Hurwitz (1946) considered the problem of non-response while estimating the population mean by taking a subsample from the non-respondents group with the help of some extra efforts and an estimator was suggested by combining the information available from response and non-response groups. In estimating population parameters like the mean or total survey statisticians make the use of auxiliary information to improve the precision of the estimates. Cochran (1977), Rao (1986, 1987), Okafor and Lee (2000), Olufadi and Kumar (2014), Kumar (2015) and Pal and Singh (2016) among others have used the auxiliary information in presence of non-response and

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incorporating the Hansen and Hurwitz (1946) technique suggested some estimators along with their properties.

The aim of this paper is to develop a class of estimators for estimating the population mean along with its properties in presence of non-response. Double sampling version of the suggested class of estimators is also given along with its properties. The estimator based on estimated optimum value has been investigated along with its mean squared error (*MSE*) formula. The optimum values of sample size(s) and the sub sampling fraction of the non-responding group have been determined for the fixed cost in case of double sampling. Empirical studies are carried in support of the present study.

2. The suggested class of ratio-cum-ratio-type exponential estimators

Consider a finite population $\Omega = (\Omega_1, \Omega_2, ..., \Omega_N)$ of size N from which a simple random sample of size n is drawn without replacement. Let the characteristic of interest 'say' y and the auxiliary characteristic x take the values (y_i, x_i), on units $\Omega_i (i = 1,2,..., N)$. It is assumed that the study variate y is positively correlated with the auxiliary variate x.

In surveys on human populations, frequently n_1 units respond on the items under examination in the first attempt while remaining $n_2 (= n - n_1)$ units do not provide any response. When non-response occurs in the initial attempt, Hansen and Hurwitz (1946) considered a double sampling scheme for estimating population mean comprising the following steps:

(i) a simple random sample of size n is selected and the questionnaire is mailed to the sample units;

(ii) a sub sample of size $r = n_2 / k, (k \geq 1)$ from the n_2 non-responding units in the initial attempt is contacted through personal interviews.

In this procedure the population is assumed to be composed of two strata of sizes N_1 and $N_2 = N - N_1$ of 'respondents' and 'non-respondents'.

Let $(\overline{Y}, \overline{X})$ be the population means of the study variate y and the auxiliary variate x respectively. The population means $(\overline{Y}, \overline{X})$ can be expressed as

$$\overline{Y} = W_1\overline{Y}_1 + W_2\overline{Y}_2 \text{ and } \overline{X} = W_1\overline{X}_1 + W_2\overline{X}_2,$$

where $W_1 = (N_1 / N)$ and $W_2 = (N_2 / N)$ are population proportions of the 'response' and 'non-response' groups respectively.

Using Hansen and Hurwitz (1946) procedure, the estimator for the population mean $\bar{Y}$ using $(n_1 + r)$ observations on study variate y is given by

$$\bar{y}^* = w_1\bar{y}_1 + w_2\bar{y}_{2r}, \tag{2.1}$$

where $w_1 = n_1 / n$ and $w_2 = n_2 / n$ are 'responding' and 'non-responding' proportions in the sample. The variance of $\bar{y}^*$ is given by

$$V(\bar{y}^*) = \frac{(1-f)}{n} S_y^2 + \frac{W_2(k-1)}{n} S_{y(2)}^2, \tag{2.2}$$

where $f = \frac{n}{N}$, S_y^2 and $S_{y(2)}^2$ are the mean square of y for the entire population and for the non-responding group of the population. Similarly an unbiased estimator $\bar{x}^*$ for the population mean $\bar{X}$ of the auxiliary variate x is given by

$$\bar{x}^* = w_1\bar{x}_1 + w_2\bar{x}_{2r} \tag{2.3}$$

where ($\bar{x}_1$, $\bar{x}_{2r}$) are the sample means based on n_1 and r observations.

The variance of $\bar{x}^*$ is given by

$$V(\bar{x}^*) = \frac{(1-f)}{n} S_x^2 + \frac{W_2(k-1)}{n} S_{x(2)}^2, \tag{2.4}$$

where S_x^2 and $S_{x(2)}^2$ are the population mean square of x for the entire population and 'non-responding' group of the population.

Let the population mean $\bar{X}$ of the auxiliary variate x be known and incomplete information on the study variate y and the auxiliary variate x is available. In this case, we observe n_1 units respond for study variate y and the auxiliary variate x in the initial attempt and r units respond at the second stage.

Thus we have $(n_1 + r)$ observations on the characters (y, x) and the population mean $\bar{X}$ is known. Thus the usual ratio estimator and the ratio-type exponential estimator for population mean $\bar{Y}$ are respectively defined by

$$t_R = \bar{y}^* \frac{\bar{X}}{\bar{x}^*}, \tag{2.5}$$

$$t_{\mathrm{Re}} = \bar{y}^* \exp\left(\frac{\bar{X} - \bar{x}^*}{\bar{X} + \bar{x}^*}\right). \tag{2.6}$$

Taking the linear combination of the two estimators t_R and t_{Re}, we suggest a class of ratio-cum-ratio-type exponential estimators for population mean $\bar{Y}$ as

$$t = \alpha\, \bar{y}^*\left(\frac{\bar{X}}{\bar{x}^*}\right) + (1-\alpha)\bar{y}^* \exp\left(\frac{\bar{X} - \bar{x}^*}{\bar{X} + \bar{x}^*}\right), \tag{2.7}$$

where α is suitable chosen constant.

For $\alpha = 1$, 't' reduces to the ratio estimator t_R while for $\alpha = 1$ boils down to ratio-type exponential estimator t_{Re}. Thus the proposed class of ratio-cum-ratio-type exponential estimators 't' is the new class of estimators which generalizes the ratio estimator t_R and ratio-type exponential estimator t_{Re}.

To the first degree of approximation, the bias of the proposed class of ratio-cum-ratio-type exponential estimators 't' is given by

$$B(t) = \left(\frac{\bar{Y}V_x}{8}\right)[(5-4V)(\alpha+1)-2] \tag{2.8}$$

Taking the limit as $n = \infty$ (i.e. sample size n is sufficiently large) of both sides of (2.8) we have

$$\begin{aligned}\lim_{n=\infty} B(t) &= \lim_{n=\infty}\left(\frac{\bar{Y}V_x}{8}\right)[(5-4V)(\alpha+1)-2] \\ &= \left(\frac{\bar{Y}}{8}\right)[(\alpha+1)\{5\lim_{n=\infty} V_x - 4\lim_{n=\infty} V_{xy}\} - 2\lim_{n=\infty} V_x].\end{aligned}$$

Since $\lim_{n=\infty} V_x = 0$ and $\lim_{n=\infty} V_{xy} = 0$. Therefore $\lim_{n=\infty} B(t) = 0$; this proves the consistency of the proposed class of ratio-cum-ratio-type exponential estimators t.

The *MSE* of the proposed class of ratio-cum-ratio-type exponential estimator 't' to the first degree of approximation is given by

$$MSE(t) = \bar{Y}^2\left[V_y + \frac{(\alpha+1)}{4}V_x(\alpha - 4V + 1)\right], \tag{2.9}$$

where

$$V_y = \left[\frac{(1-f)}{n}C_y^2 + \frac{W_2(k-1)}{n}C_{y(2)}^2\right], V_x = \left[\frac{(1-f)}{n}C_x^2 + \frac{W_2(k-1)}{n}C_{x(2)}^2\right],$$

$$V_{xy} = \left[\frac{(1-f)}{n}\rho C_y C_x + \frac{W_2(k-1)}{n}\rho_{(2)} C_{y(2)} C_{x(2)}\right], V = \frac{V_{xy}}{V_x} \text{ and } (\rho, C_y, C_x);$$

are the population correlation coefficient between y and x and coefficients of variation y and x respectively for the entire population; and $\rho_{(2)}$ is the correlation coefficient between the variates y and x in the 'non-responding' group of the population, $C_{y(2)} = (S_{y(2)} / \bar{Y})$ and $C_{x(2)} = (S_{x(2)} / \bar{X})$.

The $MSE(t)$ at (2.9) is minimized for

$$\alpha = (2V - 1). \tag{2.10}$$

Thus the resulting minimum *MSE* of t is given by

$$\min .MSE(t) = \bar{Y}^2 (V_y - V^2 V_x) = \bar{Y}^2 V_y (1 - \rho^{*2}), \tag{2.11}$$

where $\rho^* = (V_{xy} / \sqrt{V_x V_y})$.

Now we state the following theorem.

Theorem 2.1: To the first degree of approximation,

$$MSE(t) \geq \bar{Y}^2 V_y (1 - \rho^{*2})$$

with equality holding if $\alpha = (2V - 1)$.

Putting $\alpha = (2V - 1)$ in (2.8) we get the resulting bias of t as

$$B_0(t) = \left(\frac{\bar{Y} V_x}{4}\right)[(5 - 4V)V - 1]. \tag{2.12}$$

3. Bias comparison

Inserting $\alpha = 1$, 0 in (2.8) we get the bias of the estimators t_R and t_{Re} to the first degree of approximation, respectively as:

$$B(t_R) = \bar{Y}V_x(1-V)\,, \tag{3.1}$$

$$B(t_{\text{Re}}) = \frac{\bar{Y}V_x}{8}(3-4V)\,. \tag{3.2}$$

From (2.8) and (3.1) we have that

$|B(t)| < |B(t_R)|$ if

$$|(5-4V)(\alpha+1)-2| < 8|(1-V)| \tag{3.3}$$

Further from (2.8) and (3.2) we have that

$|B(t)| < |B(t_{\text{Re}})|$ if

$$|(5-4V)(\alpha+1)-2| < |(3-4V)| \tag{3.4}$$

or, equivalently

$$\min.\left\{0, \frac{2(4V-3)}{(5-4V)}\right\} < \alpha < \max.\left\{0, \frac{2(4V-3)}{(5-4V)}\right\} \tag{3.5}$$

Thus the proposed class of ratio-cum-ratio-type exponential estimators 't' is less biased than the conventional ratio estimator t_R and ratio-type exponential estimator t_{Re} as long as the condition (3.3) and (3.5) respectively hold well.

4. Efficiency Comparison

4.1. When the scalar α in (2.7) exactly coincide with its optimum value $\alpha = (2V-1)$

Putting $\alpha = 1$, 0 in (2.9) we get the *MSE*s of the ratio estimator t_R and ratio-type exponential estimator t_{Re} to the first degree of approximation, respectively as

$$MSE(t_R) = \bar{Y}^2[V_y + V_x(1-2V)]\,, \tag{4.1}$$

$$MSE(t_{\text{Re}}) = \bar{Y}^2\left[V_y + \frac{V_x}{4}(1-4V)\right]. \tag{4.2}$$

The expressions in (2.2) can be expressed in term of V_y as:

$$V(\bar{y}^*) = \bar{Y}^2\left[\frac{(1-f)}{n}C_y^2 + \frac{W_2(k-1)}{n}C_{y(2)}^2\right] = \bar{Y}^2 V_y . \tag{4.3}$$

From (2.11), (4.1), (4.2) and (4.3) it can be easily shown that the proposed class of ratio-cum-ratio-type exponential estimators ‘ t ’ is more efficient than usual unbiased estimator $\bar{y}^*$, ratio estimator t_R and ratio-type exponential estimator t_{Re} at its optimum condition.

4.2. When the scalar α in (2.7) does not coincide with its optimum value

- From (2.9) and (4.3) we have

$$V(\bar{y}^*) - MSE(t) = \frac{(\alpha+1)V_x}{4}\bar{Y}^2(4V-\alpha-1) > 0 \text{ if}$$

$$\min.\{-1,(4V-1)\} < \alpha < \max.\{-1,(4V-1)\} \tag{4.4}$$

- From (2.9) and (4.1) we have

$$MSE(t_R) - MSE(t) = \frac{\bar{Y}^2 V_x}{4}(1-\alpha)(3-4V+\alpha) > 0 \text{ if}$$

$$\min.\{1,(4V-3)\} < \alpha < \max.\{1,(4V-3)\} \tag{4.5}$$

- From (2.9) and (4.2) we have

$$MSE(t_{\mathrm{Re}}) - MSE(t) = \frac{\bar{Y}^2 V_x}{4}\alpha(4V-\alpha-2) > 0 \text{ if}$$

$$\min.\{0,2(2V-1)\} < \alpha < \max.\{0,2(2V-1)\} \tag{4.6}$$

Thus the proposed class of ratio-cum-ratio-type exponential estimator ‘ t ’ is more efficient than the usual unbiased estimator $\bar{y}^*$, ratio estimator t_R and ratio-type exponential estimator t_{Re} as long as the condition (4.4), (4.5) and (4.6) are respectively holds good.

5. Estimator Based on Estimated Optimum Values

Substitution of optimum value of $\alpha = (2V-1)$ in the class of estimators t at (2.7) we get the asymptotically optimum estimator (*AOE*) for the population mean $\bar{Y}$ as

$$t_0 = (2V-1)\bar{y}^*\left(\frac{\bar{X}}{\bar{x}^*}\right) + 2(1-V)\bar{y}^* \exp\left(\frac{\bar{X}-\bar{x}^*}{\bar{X}+\bar{x}^*}\right) \tag{5.1}$$

It is observed from (5.1) that the *AOE* depends on unknown parameter $V = \beta^*(\bar{X}/\bar{Y})$ which lacks the practical utility of the *AOE* t_0 in (5.1), $\beta^* = Cov(\bar{y}^*, \bar{x}^*)/V(\bar{x}^*)$ being the population regression coefficient of $\bar{y}^*$ on $\bar{x}^*$. Hence the alternative is to replace V by its consistent estimated optimum value $\hat{V} = \hat{\beta}^*(\bar{X}/\bar{y}^*)$, $\hat{\beta}^* = \hat{C}ov(\bar{y}^*, \bar{x}^*)/\hat{V}(\bar{x}^*)$ being the estimate of β^* based on the data available at hand. Thus the substitution of $\hat{V}$ in the place of V in (5.1) yields an estimator (based on estimated optimum value) as

$$\hat{t}_0 = \left[(2\hat{V}-1)\bar{y}^*\left(\frac{\bar{X}}{\bar{x}^*}\right) - 2(1-\hat{V})\bar{y}^* \exp\left(\frac{\bar{X}-\bar{x}^*}{\bar{X}+\bar{x}^*}\right)\right] \tag{5.2}$$

Following the same procedure as adopted by Singh and Ruiz- Espejo (2003) it can be easily shown to the first degree of approximation as

$$MSE(\hat{t}_0) = \bar{Y}^2(V_y - V^2 V_x) = \bar{Y}^2 V_y (1-\rho^{*2}), \tag{5.3}$$

which is same as the minimum *MSE* of the class of estimators t [or the *MSE* of the optimum estimator t_0 in (5.1)].

Thus the estimator $\hat{t}_0$ based on estimated optimum value can be treated as an alternative to the usual regression estimator $t_{lr} = \bar{y}^* + \hat{\beta}^*(\bar{X} - \bar{x}^*)$.

Let non-response be occur only on the study variable y and the information x be available for complete sample size n. Then in such a situation, we define a ratio-cum-ratio-type exponential estimator for the population mean $\bar{Y}$ as

$$t^* = \alpha^* \bar{y}^*\left(\frac{\bar{X}}{\bar{x}}\right) + (1-\alpha^*)\bar{y}^* \exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right), \tag{5.4}$$

where α^* is a suitable chosen constant. For $\alpha^* = 1$, it reduces to the ratio estimator $t_R^* = \bar{y}^*(\bar{X}/\bar{x})$;

while for $\alpha^* = 0$, it boils to the ratio-type exponential estimator

$$t_{\mathrm{Re}}^* = \bar{y}^* \exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right).$$

To first degree of approximation, the bias of the proposed class of ratio-cum-ratio-type exponential estimators t^* is given by

$$B(t^*) = \frac{\bar{Y}(1-f)}{8n} C_x^2[(5-4C)(\alpha+1)-2] \tag{5.5}$$

which is negligible, if sample size n is sufficiently large.

To the first degree of approximation the *MSE* of the proposed class of ratio-cum-ratio-type exponential estimators t^* is given by

$$MSE\ (t^*) = \bar{Y}^2\left[V_y + \frac{(1-f)}{4n} C_x^2(\alpha - 4C + 1)\right], \tag{5.6}$$

where $C = \rho(C_y / C_x)$.

The $MSE(t^*)$ at (5.6) is minimized for $\alpha^* = (2C-1)$.

Thus the resulting minimum *MSE* of t^* is given by

$$\min .MSE(t^*) = \bar{Y}^2\left[V_y - \frac{(1-f)}{n}\rho^2 C_y^2\right]$$

$$= \frac{(1-f)}{n} S_y^2(1-\rho^2) + \frac{W_2(k-1)}{n} S_{y(2)}^2 . \tag{5.7}$$

Putting $\alpha^* = (2C-1)$ in (5.5) we get the resulting bias of t^* as

$$B_0(t^*) = \frac{\bar{Y}(1-f)}{4n} C_x^2[(5-4C)C-1]. \tag{5.8}$$

This is same as the variance of linear regression estimator $t_{lr}^* = \bar{y}^* + \hat{\beta}(\bar{X} - \bar{x}^*)$, where $\hat{\beta}$ is the estimate of the population regression coefficient β of y on x. It can be easily shown that the proposed class of estimators t^* is more efficient than:

(i) the usual unbiased estimator $\bar{y}^*$ if

$$\min .\{-1, (4C-1)\} < \alpha^* < \max .\{-1, (4C-1)\} \tag{5.9}$$

(ii) the usual ratio estimator t_R^* if

$$\min .\{1, (4C-3)\} < \alpha^* < \max .\{1, (4C-3)\} \tag{5.10}$$

(iii) the ratio-type exponential estimator t^*_{Re} if

$$\min.\{0,2(2C-1)\} < \alpha^* < \max.\{1,2(2C-1)\} \tag{5.11}$$

Remark 5.2: Let non-response be occur on both the variables y and x. It is assumed that the correlation coefficient between y and x is negative and the population mean $\bar{X}$ of x is known. In such a situation one can define a class of product-cum-product-type exponential estimator for population mean $\bar{Y}$ as

$$t_1 = \delta\bar{y}^*\left(\frac{\bar{x}^*}{\bar{X}}\right) + (1-\delta)\bar{y}^* \exp\left(\frac{\bar{x}^* - \bar{X}}{\bar{x}^* + \bar{X}}\right), \tag{5.12}$$

where δ is a suitable chosen constant.

Further if the non-response occurs only on study variable y and the population mean $\bar{X}$ of x is known, then one can define another class of product-cum-product-type exponential estimators for population mean $\bar{Y}$ as

$$t_1^* = \delta^* \bar{y}^*\left(\frac{\bar{x}}{\bar{X}}\right) + (1-\delta^*)\bar{y}^* \exp\left(\frac{\bar{x} - \bar{X}}{\bar{x} + \bar{X}}\right), \tag{5.13}$$

where δ^* is a suitable chosen constant. The properties of (t_1, t_1^*) can be easily studied under large sample approximation.

3. Empirical Study

In this section we have used the data sets earlier considered by Kumar and Vishwanathaiah (2013).

Population I: Source: **Khare and Sinha (2004, p.53)**

The data belongs to the data on physical growth of upper-socio-economic group of 95 school children of Varanasi under an *ICMR* study, Deportment of Pediatrics; *BHU* during 1983-1984 has under taken in this study. The first 25% (i.e. 24 children) units have been considered as non-response units. The values of the parameters related to the study variable y (the weight in Kg.) and the auxiliary variable x (the chest circumferences in cm.) has been given below:

$N = 95$ $n = 35$ $\bar{Y} = 19.4968$ $\bar{X} = 55.8611$ $S_y = 3.0435$ $S_x = 3.2735$
$S_{y(2)} = 2.3552$ $S_{x(2)} = 3.5137$ $\rho = 0.8460$ $\rho_2 = 0.7290$ $W_2 = 0.25$

Population II: Source: **Khare and Srivastava (1993, p.50)**

A list of seventy villagers in a Tehsil of India along with their population in 1981 and cultivated area (in acres) in the same year is taken into consideration. Here the cultivated area (in acres) is taken as main study character and the population of village is taken as auxiliary character. The performance of the population is as follows

$$N = 70, n = 25, \bar{Y} = 981.29, \bar{X} = 1755.53, S_y = 613.66, S_x = 1406.13,$$

$$S_{y(2)} = 244.11, S_{x(2)} = 631.51, \rho = 0.778, \rho_2 = 0.445, W_2 = 0.20.$$

We have computed the range of α in which the proposed class of estimators t is more efficient than $\bar{y}^*$, t_R and t_{Re}. Findings are given in Table 6.1.

We have also computed the percent relative efficiency (*PRE*) of the proposed class of estimators t with respect to the usual unbiased estimator $\bar{y}^*$ for both the population data sets I and II for different values of α and k by using the following formula:

$$PRE(t, \bar{y}^*) = \frac{V_y}{\left[V_y + \frac{(\alpha+1)}{4} V_x (\alpha - 4V + 1) \right]} \times 100 \tag{6.1}$$

Findings are shown in Tables 6.2 and 6.3.

Table 6.1: Range of α

Range of α in which the proposed class of estimators t is better than:				
Population I				
Estimator	1/5	1/4	1/3	1/2
$\bar{y}^*$	(-1.0000, 7.4415)	(-1.0000, 7.5258)	(-1.0000, 7.6368)	(-1.0000, 7.7899)
t_R	(1.0000, 5.4415)	(1.0000, 5.5258)	(1.0000, 5.6368)	(1.0000, 5.7899)
t_{Re}	(0.0000, 6.4415)	(0.0000, 6.5258)	(0.0000, 6.6368)	(0.0000, 6.7899)
α_{opt}	3.2208	3.2629	3.3184	3.3949
Common Range	(1.0000, 5.4415)	(1.0000, 5.5258)	(1.0000, 5.6368)	(1.0000, 5.7899)
Population II				
$\bar{y}^*$	(-1.0000, 1.1434)	(-1.0000, 1.2013)	(-1.0000, 1.2671)	(-1.0000, 1.3425)
t_R	(-0.8566, 1.0000)	(-0.7987, 1.0000)	(-0.7329, 1.0000)	(-0.6575, 1.0000)
t_{Re}	(0.0000, 0.1434)	(0.0000, 0.2013)	(0.0000, 0.2671)	(0.0000, 0.3425)
α_{opt}	0.0717	0.1007	0.1336	0.1713
Common Range	(0.0000, 0.1434)	(0.0000, 0.2013)	(0.0000, 0.2671)	(0.0000, 0.3425)

Table 6.2: PRE of the proposed class of ratio-cum-ratio-type exponential estimators t with respect to the usual unbiased estimator $\bar{y}^*$ for population I

α \ $1/k$	$1/5$	$1/4$	$1/3$	$1/2$
-1.0000	100.0000	100.0000	100.0000	100.0000
-0.7500	107.7135	107.8082	107.9337	108.1077
-0.5000	116.1239	116.3437	116.6355	117.0415
-0.2500	125.2694	125.6523	126.1621	126.8740
0.0000	135.1750	135.7686	136.5611	137.6728
0.2500	145.8460	146.7085	147.8642	149.4931
0.5000	157.2577	158.4595	160.0763	162.3679
0.7500	169.3440	170.9682	173.1631	176.2939
1.0000	181.9835	184.1254	187.0347	191.2138
1.2500	194.9859	197.7504	201.5265	206.9939
1.5000	208.0794	211.5759	216.3813	223.3998
1.7500	220.9042	225.2379	231.2332	240.0724
2.0000	233.0159	238.2755	245.6021	256.5122
Table 6.2 Continued…				
2.2500	243.9044	250.1467	258.9039	272.0793
2.5000	253.0306	260.2646	270.4845	286.0201
2.7500	259.8803	268.0540	279.6793	297.5281
3.0000	**264.0285**	**273.0227**	**285.8940**	**305.8380**
3.2500	**265.2010**	**274.8337**	**288.6922**	**310.3381**
3.5000	**263.3179**	**273.3595**	**287.8678**	**310.6758**
3.7500	258.5070	268.7039	283.4822	306.8242
4.0000	251.0826	261.1847	275.8529	299.0867
4.2500	241.4963	251.2811	265.4981	288.0402
4.5000	230.2738	239.5633	253.0549	274.4333
4.7500	217.9516	226.6198	239.1921	259.0730
5.0000	205.0272	212.9993	224.5379	242.7261
5.2500	191.9261	199.1721	209.6320	226.0534
5.5000	178.9873	185.5125	194.9030	209.5774
5.7500	166.4628	172.2973	180.6661	193.6787
6.0000	154.5257	159.7156	167.1344	178.6103
6.2500	143.2830	147.8832	154.4361	164.5201
6.5000	132.7900	136.8585	142.6340	151.4763
6.7500	123.0635	126.6576	131.7427	139.4892
7.0000	114.0928	117.2671	121.7437	128.5306
7.2500	105.8490	108.6539	112.5971	118.5482
7.3500	102.7463	105.4164	109.1656	114.8140
7.4100	100.9363	103.5289	107.1667	112.6416

Table 6.3: *PRE* of the proposed class of ratio- cum-ratio-type exponential estimators t with respect to the usual unbiased estimator $\bar{y}^*$ for population II

α \ $1/k$	$1/5$	$1/4$	$1/3$	$1/2$
-1.0000	100.0000	100.0000	100.0000	100.0000
-0.7500	125.7331	126.3642	127.0719	127.8711
-0.5000	155.1066	157.1866	159.5608	162.2963
-0.2500	182.4329	187.1015	192.5694	199.0614
0.0000	**197.7762**	**205.6717**	**215.2044**	**226.9426**
0.0500	**198.5692**	**207.0499**	**217.3499**	**230.1251**
0.2500	193.3697	203.4945	216.0462	232.0165
0.5000	171.6111	181.7932	194.6049	211.2158
0.7500	142.3836	151.0125	161.8993	176.0637
1.0000	114.1566	120.8053	129.1518	139.9414
1.0500	109.0265	115.2991	123.1614	133.3055
1.1400	100.3162	105.9485	112.9875	122.0353

Table 6.1 exhibits that there is enough scope of choosing the value of scalar α to obtain estimators better than $\bar{y}^*$, t_R and t_{Re}. The range of α (in which the proposed class of ratio-cum-ratio-type exponential estimators 't' is better than $\bar{y}^*$, t_R and t_{Re}) becoming wider for decreasing value of k . However, it is to be mentioned that the range of α is, wider in population I as compared to population II. So the flexibility of choosing the appropriate value of α to get better estimators than $\bar{y}^*$, t_R and t_{Re} heavily depends on the constituents of data. It is observed, from Tables 6.2 and 6.3 that $PRE(t,\bar{y}^*)$ is larger than 100% for the suitable values of (α,k) . Thus the estimators t_R , t_{Re} and the proposed class of ratio-cum-ratio-type exponential estimators 't'are better than the usual unbiased estimator $\bar{y}^*$ which does not utilize the auxiliary information. The proposed class of ratio-cum-ratio-type exponential estimators't' is better than the usual unbiased estimator $\bar{y}^*$, ratio estimator t_R and ratio-type exponential estimator t_{Re} in the common range of α [see Table 6.1]. The larger gain in efficiency by using proposed class of ratio-cum-ratio-type exponential estimators 't' over $\bar{y}^*$, t_R and t_{Re} are seen in the neighborhood of the optimum value of scalar α . However, the maximum gain in efficiency is observed at optimum value of scalar α . This fact is true for both the population data sets I and II. We further note that the ratio estimator t_R is better than usual unbiased estimator $\bar{y}^*$ and the ratio-type exponential estimator t_{Re} in population I[see Table

6.2] while the ratio-type exponential estimator t_{Re} is better than usual unbiased estimator $\bar{y}^*$ and ratio estimator t_R in population II [see Table 6.3]. The *PRE* of the estimators t_R, t_{Re} and the proposed class of ratio-cum-ratio-type exponential estimators 't' increase for decreasing values of k [see Tables 6.2 and 6.3].

From above discussion our recommendation is in the favour of present study regarding in search of estimators better than conventional unbiased estimator $\bar{y}^*$, ratio estimator t_R and ratio-type exponential estimator t_{Re} .

7. Estimation of population mean under two phase sampling with sub sampling the non-respondents

In certain practical situations the population mean $\bar{X}$ is not known a priori in which case the technique of double sampling can be employed successfully. When the population mean $\bar{X}$ of the auxiliary variable x is not known then it is suggested to select a first phase sample of size n' from the population of size N by using simple random sampling without replacement (*SRSWOR*) procedure of sampling. We assume that at the first phase all the n' units supplied information on the auxiliary character x . A smaller second phase sample of size $n(< n')$ is selected by the *SRSWOR* sampling scheme, and the character y is measured on it. At the second phase, let n_1 units respond on the study variable y and then n_2 units do not respond in the sample of size n .Using Hansen and Hurwitz (1946) procedure to sub sampling, from the n_2 non-respondents a sub sample, of size r units is selected using the *SRSWOR* sampling scheme, and enumerated by direct interview, such that $r = (n_2 / k)$, $k > 1$, where k is the inverse sampling rate.

Here we assume that response is obtained for all r units. This procedure of double sampling can be applied in a household survey, where the household size is used as an auxiliary variate x for the estimation of family expenditure. Information can be obtained completely on the family size, while there may be some non-response on the household expenditure, see Kumar et al. (2011, p.290).

Let the sample mean of the auxiliary variate x based on n' units be denoted by $\bar{x}' = (1/n')\sum_{i=1}^{n'} x_i$.

Then we define a class of ratio-cum-ratio-type exponential estimators for population mean $\bar{Y}$ as:

$$t_d = \alpha_1 \bar{y}^* \left(\frac{\bar{x}'}{\bar{x}^*} \right) + (1-\alpha_1)\bar{y}^* \exp\left(\frac{\bar{x}' - \bar{x}^*}{\bar{x}' + \bar{x}^*} \right), \tag{7.1}$$

where α_1 is suitable chosen constant. For $\alpha_1 = 1$ in (7.1), the class of estimators t_d reduces to the ratio estimator

$$t_{Rd} = \bar{y}^* \left(\frac{\bar{x}'}{\bar{x}^*} \right), \tag{7.2}$$

due to Khare and Srivastava (1993), Okafor and Lee (2000) and Tabasum and Khan (2004). Setting $\alpha_1 = 0$ in (7.1) we get the ratio-type exponential estimator

$$t_{Rde} = \bar{y}^* \exp\left(\frac{\bar{x}' - \bar{x}^*}{\bar{x}' + \bar{x}^*} \right). \tag{7.3}$$

7.1. Bias and *MSE* of the proposed class of ratio-cum-ratio-type exponential estimators t_d

To the first degree of approximation, the bias and *MSE* of the proposed class of estimators t_d are respectively given by

$$B(t_d) = \bar{Y}\left[\left(\frac{1}{n} - \frac{1}{n'} \right) C_x^2 \{(5-4C)(\alpha_1+1)-2\} + \frac{W_2(k-1)}{n} C_{x(2)}^2 \{(5-4C_{(2)})(\alpha_1+1)-2\} \right], \tag{7.4}$$

$$MSE(t_d) = \bar{Y}^2 \left[V_y + \frac{(\alpha+1)^2}{4} V_x^* - (\alpha_1+1) V_{xy}^* \right], \tag{7.5}$$

where

$$V_x^* = \left[\left(\frac{1}{n} - \frac{1}{n'} \right) C_x^2 + \frac{W_2(k-1)}{n} C_{x(2)}^2 \right],$$

$$V_{xy}^* = \left[\left(\frac{1}{n} - \frac{1}{n'} \right) C C_x^2 + \frac{W_2(k-1)}{n} C_{(2)} C_{x(2)}^2 \right].$$

For large N, (i.e. $\frac{1}{N} \cong 0$), $MSE(t_d)$ in (7.4) reduces to:

$$MSE(t_d) = \bar{Y}^2\left[\frac{1}{n}C_y^2 + \frac{W_2(k-1)}{n}C_{y(2)}^2 + \frac{(\alpha+1)^2}{4}V_x^* - (\alpha_1+1)V_{xy}^*\right] \quad (7.6)$$

which is minimized for $\alpha_1 = (2V^* - 1)$ with $V^* = (V_{xy}^* / V_x^*)$.

Thus the resulting minimum *MSE* of t_d is given by

$$\min .MSE(t_d) = \bar{Y}^2\left[\frac{1}{n}C_y^2 + \frac{W_2(k-1)}{n}C_{y(2)}^2 - V^{*2}V_x^*\right] \quad (7.7)$$

Now, we state the following theorem.

Theorem 7.1: To the first degree of approximation,

$$MSE(t_d) \geq \bar{Y}^2\left[\frac{1}{n}C_y^2 + \frac{W_2(k-1)}{n}C_{y(2)}^2 - V^{*2}V_x^*\right]$$

with equality holding if $\alpha_1 = (2V^* - 1)$.

The biases and *MSE*s of the estimators t_{Rd} and t_{Rde} to the first degree of approximation (for large N), are respectively given by

$$B(t_{Rd}) = \bar{Y}V_x^*(1 - V^*), \quad (7.8)$$

$$B(t_{Rde}) = \bar{Y}V_x^*(3 - 4V^*), \quad (7.9)$$

$$MSE(t_{Rd}) = \bar{Y}^2\left[\frac{1}{n}C_y^2 + \frac{w_2(k-1)}{n}C_{y(2)}^2 + V_x^*(1 - 2V^*)\right], \quad (7.10)$$

$$MSE(t_{Rde}) = \bar{Y}^2\left[\frac{1}{n}C_y^2 + \frac{w_2(k-1)}{n}C_{y(2)}^2 + \frac{V_x^*}{4}(1 - 4V^*)\right]. \quad (7.11)$$

Further, for large N , the variance/*MSE* of the usual unbiased estimator $\bar{y}^*$ is given by

$$MSE(\bar{y}^*) = V(\bar{y}^*) = \bar{Y}^2\left[\frac{1}{n}C_y^2 + \frac{w_2(k-1)}{n}C_{y(2)}^2\right] = \bar{Y}^2V_y^*. \quad (7.12)$$

The expressions for $\min . MSE(t_d)$ in (7.7) implies that the proposed class of estimators t_d will always be better than the estimators $\bar{y}^*, t_{Rd}, t_{Rde}$ as these estimators are members of the suggested class of estimators t_d. Nevertheless, we shall further determined a range of α-values for which the estimator t_d will always have smaller *MSE* than that of the estimators $\bar{y}^*, t_{Rd}$ and t_{Rde}.

8. Efficiency Comparison

The range of α - values are established in this Section for which the *MSE* of the suggested class of estimators t_d is always less than that of the estimators $\bar{y}^*$, t_{Rd} and t_{Rde}.

It can be easily shown that the proposed class of ratio-cum-ratio-type exponential estimators t_d is more efficient than

(i) the usual unbiased estimator $\bar{y}^*$ if

$$\min .\{-1,(4V^*-1)\} < \alpha < \max .\{-1,(4V^*-1)\} \tag{8.1}$$

(ii) the double sampling ratio estimator t_{Rd} if

$$\min .\{1,(4V^*-3)\} < \alpha < \max .\{1,(4V^*-3)\} \tag{8.2}$$

(iii) the double sampling ratio-type exponential estimator t_{Rde} if

$$\min .\{0,2(2V^*-1)\} < \alpha < \max .\{0,2(2V^*-1)\} \tag{8.3}$$

9. Determination of n', n and k

Let C^* be the total cost (fixed) of the survey apart from over head cost. The expected total of the survey apart from overhead cost is given by

$$C = C_1'n' + n\left(C_1 + C_2W_1 + C_3\frac{W_2}{k}\right), \tag{9.1}$$

where

C_1: the cost per unit of mailing questioners /visiting the unit at the second phase ,

C_2 : the cost per unit of collecting and processing data obtained from n_1 responding units,

C_3 : the cost per unit of obtaining and processing data (after extra effort) from the sub sampled units,

C_1' : the cost per unit of identifying and observing auxiliary character, and $W_1 = \frac{N_1}{N}$, $W_2 = \frac{N_2}{N}$ denote the response and non response rate in the population respectively.

The expressions of variances of estimators $t_0 = t_d, t_1 = t_{Rd}, t_2 = t_{Rde}$ and $(t_3 = \bar{y}^*)$ respectively given by (7.6), (7.13), (7.14) and (7.15) can be express as

$$MSE(t_i) = \left(\frac{V_{0i}}{n} + \frac{V_{1i}}{n'} + \frac{kV_{2i}}{n} \right) + \text{(terms independent of } n\text{, } n' \text{ and } k\text{)}, \tag{9.2}$$

for $i = 0,1,2,3$; where V_{0i}, V_{1i} and V_{2i} are respectively the coefficients of the terms of $\frac{1}{n}$, $\frac{1}{n'}$ and $\frac{k}{n}$ in the expressions for $MSE(t_0 = t_d), MSE(t_1 = t_{Rd}), MSE(t_2 = t_{Rde})$ and $Var(t_3 = \bar{y}^*)$. Let us define a function of L given by

$$L = MSE(t_i) + \mu_i \left\{ C_1'n' + n\left(C_1 + C_2W_1 + C_3 \frac{W_2}{k} \right) \right\}. \tag{9.3}$$

for obtaining n, n' and k, where μ_i is a lagrangian multiplier.

We have adopted the same procedure as discussed in Kumar et al. (2011).

Differentiating L in (9.3) with respect to n, n' and k and equating to zero, we have

$$n' = \sqrt{\frac{V_{1i}}{\mu_i C_1'}} , \tag{9.4}$$

$$n = \sqrt{\frac{V_{0i} + kV_{2i}}{\mu_i (C_1 + C_2W_1 + C_3(W_2 / k))}} \tag{9.5}$$

and

$$\frac{n}{k}=\sqrt{\frac{V_{2i}}{\mu_i C_3 W_2}}\,. \tag{9.6}$$

Now putting the value of n from (9.5) in (9.6) we get

$$k_{opt}=\sqrt{\frac{C_3 W_2 V_{0i}}{(C_1+C_2 W_1)V_{2i}}}\,. \tag{9.7}$$

Using the value of k_{opt} from (9.7) while putting the values of n' and n from (9.4) and (9.5) into (9.1), we have

$$\sqrt{\mu_i}=\frac{1}{C}\left[\sqrt{C_1' V_{1i}}+\sqrt{(V_{0i}+k_{opt}V_{2i})\left(C_1+C_2 W_1+\frac{C_3 W_2}{k_{opt}}\right)}\right]. \tag{9.8}$$

Thus the minimum value of $MSE(t_i)$, $i=0,1,2,3$ for the optimum value of n', n and k is given by

$$\min.MSE(t_i)=\frac{1}{C}\left[\sqrt{C_1' V_{1i}}+\sqrt{(V_{0i}+k_{opt}V_{2i})\left(C_1+C_2 W_1+\frac{C_3 W_2}{k_{opt}}\right)}\right]^2, i=0,1,2,3 \tag{9.9}$$

Remark 9.1: Assume that complete information on the auxiliary information is available for both the first and second samples, and that incomplete information on the study variable y is available. Thus, in this case, we use information on the (n_1+r) responding units on the study variable y and complete information on the auxiliary variable x from the sample of size n.

With this background we propose a class of ratio-cum-ratio-type exponential t_d^* for the population mean $\bar{Y}$ of the study variable y in the presence of non-response, as

$$t_d^*=\alpha_2\bar{y}^*\left(\frac{\bar{x}'}{\bar{x}}\right)+(1-\alpha_2)\bar{y}^*\exp\left(\frac{\bar{x}'-\bar{x}}{\bar{x}'+\bar{x}}\right), \tag{9.10}$$

where α_2 is a suitable chosen constant.

For $\alpha_2 = 1$, t_d^* reduces to ratio estimator $t_{Rd}^* = \bar{y}^*(\bar{x}'/\bar{x})$.

While for $\alpha_2 = 0$, it reduces to ratio-type exponential estimator $t_{Rde}^* = \bar{y}^* \exp\left(\frac{\bar{x}' - \bar{x}}{\bar{x}' + \bar{x}}\right)$.

The bias and *MSE* of the proposed class of ratio-cum-ratio-type exponential estimators t_d^* to the first degree of approximation are respectively given by

$$B(t_d^*) = \left(\frac{\bar{Y}C_x^2}{8}\right)\left(\frac{1}{n} - \frac{1}{n'}\right)[(5-4C)(\alpha_2+1)-2], \tag{9.11}$$

$$MSE(t_d^*) = \bar{Y}^2\left[\frac{(1-f)}{n}C_y^2 + \frac{w_2(k-1)}{n}C_{y(2)}^2 + \left(\frac{1}{n} - \frac{1}{n'}\right)\frac{(\alpha_2+1)}{4}C_x^2(\alpha_2 - 4C + 1)\right]\cdot \tag{9.12}$$

The biases and *MSE*s of the estimators t_{Rd}^* and t_{Rde}^* can be easily obtained from the expressions (9.11) and (9.12) just by putting suitable values of α_2.

The $MSE(t_d^*)$ is minimum when $\alpha_2 = (2C-1)$.

Thus the resulting minimum *MSE* of t_d^* is given by

$$\min.MSE(t_d) = \bar{Y}^2\left[\left(\frac{1}{n} - \frac{1}{n'}\right)C_y^2(1-\rho^2) + \left(\frac{1}{n'} - \frac{1}{N}\right)C_y^2 + \frac{w_2(k-1)}{n}C_{y(2)}^2\right] \tag{9.13}$$

Now, we arrived at the following theorem.

Theorem 9.1: To the first degree of approximation,

$$MSE(t_d^*) \geq \bar{Y}^2\left[\left(\frac{1}{n} - \frac{1}{n'}\right)C_y^2(1-\rho^2) + \left(\frac{1}{n'} - \frac{1}{N}\right)C_y^2 + \frac{w_2(k-1)}{n}C_{y(2)}^2\right]$$

with equality holding if $\alpha_2 = (2C-1)$.

Adopting the procedure discussed in Section 9, cost aspects can be also provided.

Remark 9.2: When there is negative correlation between the study variable y and the auxiliary variable x, the two product-cum-product-type exponential estimators in two different situations can be proposed along with their properties under large sample approximation.

When non-response occurs on the both the variables (y, x) at the second phase (in double sampling) and the population mean $\bar{X}$ is unknown:

$$t_{1d} = \eta_1^* \bar{y}^* \left(\frac{\bar{x}^*}{\bar{x}'} \right) + (1 - \eta_1^*) \bar{y}^* \exp\left(\frac{\bar{x}^* - \bar{x}'}{\bar{x}^* + \bar{x}'} \right) \tag{9.14}$$

where η_1^* is a suitable chosen constant.

When non-response occurs only on the study variables y and the population mean $\bar{X}$ is unknown:

$$t_{1d}^* = \eta_1^{**} \bar{y}^* \left(\frac{\bar{x}}{\bar{x}'} \right) + (1 - \eta_1^{**}) \bar{y}^* \exp\left(\frac{\bar{x} - \bar{x}'}{\bar{x} + \bar{x}'} \right) \tag{9.15}$$

where η_1^{**} is suitable chosen constant.

The properties of the estimator (t_{1d} , t_{1d}^*) can be studied easily under large sample approximation.

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(1) SCHOOL OF STUDIES IN STATISTICS, VIKRAM UNIVERSITY, UJJAIN-456010, M. P, INDIA.

E-mail address: suryakantpal6676@gmail.com

(2) SCHOOL OF STUDIES IN STATISTICS, VIKRAM UNIVERSITY, UJJAIN-456010, M. P, INDIA.