Jordan Journal of Mathematics and Statistics (JJMS) 10(2), 2017, pp 95 - 112

MODIFIED GERAGHTY CONTRACTION INVOLVING FIXED POINT THEOREMS

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Abstract. In this paper, we introduced a $\alpha - \psi -$Geraghty contraction and established fixed point results for $\alpha -$admissible mappings with respect to η in complete metric spaces. In the comparison of Geraghty contraction our new modified $\alpha - \psi -$ Geraghty contraction is more stronger. The example is given to show the validity of our results. Our results generalize/improve several results existing in the literature.

1. Introduction and Preliminaries

The study of fixed point problems in nonlinear analysis has emerged as a powerful and very important tools in the last 60 years. Particularly, the techniques of fixed point have been applicable to the many diverse fields of sciences such as Economics, Engineering, Chemistry, Biology, Physics and Game Theory. Fixed point theorems are the major mathematical tools for solving fixed point problems. Over the years, fixed point theory has been generlized in multi-directions by several mathematicians (see [1-36]).

In 1973, Geraghty [14] studied different contractive conditions and proved some useful fixed point theorems.

2000 *Mathematics Subject Classification.* 46S40; 47H10; 54H25.

Key words and phrases. Unique fixed point; α admissible with respect to η; $\alpha - \psi -$Geraghty contraction; Geraghty contraction.

Received: Aug.1, 2016 Accepted: Aug.3, 2017 .

Recently Samet et al. [33], introduced a concept of $\alpha-\psi$- contractive type mappings and established various fixed point theorems for mappings in complete metric spaces. Afterwards Karapinar and Samet [10], refined the notions and obtain various fixed point results. Subsequently Karapinar et al. [11, 12] established various results in different aspect. Nawab et al. [18], enlarge the concept of α-admissible mappings and proved fixed point theorems. Subsequently Abdeljawad [4] introduced pairs of $\alpha-$admissible mappings satisfying new sufficient contractive conditions different from [18, 33], and obtained fixed point and common fixed point theorems. Lately, Salimi et al. [32], modified the concept of $\alpha - \psi-$ contractive mappings and established fixed point results.

We define Ω the family of nondecreasing functions $\psi : [0, +\infty) \to [0, +\infty)$ which satisfy the following conditions:

(i) ψ is nondecreasing;

(ii) ψ is continuous;

(iii) ψ is subadditive, that is $\psi(s + t) \leq \psi(s) + \psi(t)$;

(iii) $\psi(0) = 0 \Leftrightarrow t = 0$.

Here we introduce F is the class of all functions $\beta : [0, +\infty) \to [0, 1]$ such that, for any bounded sequence $\{t_n\}$ of positive reals, $\beta(t_n) \to 1$ implies $t_n \to 0$.

Theorem 1.1. [14] *Let (X, d) be a metric space and S be an operator. Suppose that there exist $\beta : [0, +\infty) \to [0, 1]$ satisfying condition for any bounded sequence $\{t_n\}$ of positive reals*

$$\beta(t_n) \to 1 \textit{ implies } t_n \to 0$$

If S satisfying the following inequality

$$d(Sx, Sy) \leq (\beta\left(d\left(x, y\right)\right) d(x, y)), \textit{ for any } x, y \in X,$$

then S has a unique fixed point.

Definition 1.1. Let (X,d) be a metric space and $S : X \to X$ be a given mapping. We say that S is a Geraghty contraction mapping if there exists $\beta \in \mathcal{F}$ such that

$$d(Sx,Sy) \leq \beta(d(x,y))d(x,y)$$

for all $x,y \in X$.

Definition 1.2. [33] Let (X,d) be a metric space and $T : X \to X$ be a given mapping. We say that T is an $\alpha-\psi-$ contractive mapping if there exist two functions $\alpha : X \times X \to [0,+\infty)$ and $\psi \in \Omega$ such that

$$\alpha(x,y)d(Tx,Ty) \leq \psi(d(x,y)),$$

for all $x,y \in X$.

Definition 1.3. [11] Let (X,d) be a metric space and $T : X \to X$ be a given mapping. We say that T is a $\psi-$Geraghty contraction mapping if there exists $\beta \in \mathcal{F}$ such that

$$\psi\left(d(Tx,Ty)\right) \leq \beta(\psi(d(x,y)))\psi(d(x,y)),$$

for all $x,y \in X$. Note that since $\beta \in \mathcal{F}$, we have

$$\psi\left(d(Tx,Ty)\right) \leq \beta(\psi(d(x,y)))\psi(d(x,y)) < \psi(d(x,y)) \text{ for any } x,y \text{ with } x \neq y.$$

Definition 1.4. [33] Let $T : X \to X$ and $\alpha : X \times X \to [0,+\infty)$. We say that T is α-admissible if $x,y \in X$, $\alpha(x,y) \geq 1 \Rightarrow \alpha(Tx,Ty) \geq 1$.

Definition 1.5. [4] Let $T,S : X \to X$ and $\alpha : X \times X \to [0,+\infty)$. A pair (S,T) are α-admissible if $x,y \in X$, $\alpha(x,y) \geq 1 \Rightarrow \alpha(Sx,Ty) \geq 1$ and $\alpha(Tx,Sy) \geq 1$.

Example 1.1. *Consider $X = (0,\infty)$. Define $S : X \to X$ and $\alpha : X \times X \to [0,\infty)$ by $Sx = 2x$, and*

$$\alpha\left(x,y\right) = \begin{cases} e^{\frac{y}{x}} & \text{if } x \geq y, \\ 0 & \text{if } \quad x < y. \end{cases}$$

Then S is $\alpha-$admissible.

Definition 1.6. [32] Let $T : X \to X$ and let $\alpha, \eta : X \times X \to [0, +\infty)$ be two functions. We say that T is α-admissible mapping with respect to η if $x, y \in X$, $\alpha(x, y) \geq \eta(x, y) \Rightarrow \alpha(Tx, Ty) \geq \eta(Tx, Ty)$. Note that if we take $\eta(x, y) = 1$, then this definition is reduces to definition [33]. Also if we take $\alpha(x, y) = 1$, then we says that T is an η-subadmissible mapping.

2. Main results

In this section, we prove a fixed point thorem for $\alpha-$admissible mappings for modified $\alpha - \psi-$Geraghty contraction in a complete metric space.

Theorem 2.1. *Let (X, d) be a complete metric space and let S be an $\alpha-$admissible mappings with respect to η and $\beta \in \digamma$, such that*

$$(\psi\left(d(Sx, Sy)\right) + l)^{\alpha(x,Sx)\alpha(y,Sy)} \leq \left(\beta\left(\psi(d(x, y)\right)\psi(d(x, y)) + l\right)^{\eta(x,Sx)\eta(y,Sy)} \tag{2.1}$$

where $\psi \in \Omega$, $l \geq 1$, for all $x, y \in X$ and suppose that one of the following holds:

(i) S is continuous;

(ii) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}\cup\{0\}$ and $x_n \to p \in X$ as $n \to +\infty$, then

$$\alpha(p, Sp) \geq \eta(p, Sp).$$

If there exists $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq \eta(x_0, Sx_0)$, then S has a fixed point.

Proof. Let $x_0 \in X$ and define

$$x_{n+1} = Sx_n, \text{ for all } n \geq 0. \tag{2.2}$$

We shall assume that $x_n \neq x_{n+1}$ for each n. Otherwise, there exists an n such that $x_n = x_{n+1}$. Then $x_n = Sx_n$ and x_n is a fixed point of S. Since $\alpha(x_0, x_1) = \alpha(x_0, Sx_0) \geq$

$\eta(x_0, Sx_0) = \eta(x_0, x_1)$ and S is $\alpha-$admissible mapping with respect to η, we have

$$\alpha(x_1, x_2) = \alpha(Sx_0, Sx_1) \geq \eta(Sx_0, Sx_1) = \eta(x_1, x_2).$$

By continuing in this way, we have

$$\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1}) \tag{2.3}$$

for all $n \in \mathbb{N} \cup \{0\}$. From (2.3), we have

$$\alpha(x_{n-1}, x_n)\alpha(x_n, x_{n+1}) \geq \eta(x_{n-1}, x_n)\eta(x_n, x_{n+1}). \tag{2.4}$$

Thus applying the inequality (2.1), with $x = x_{k-1}$ and $y = x_k$, we obtain

$$\begin{aligned}
& (\psi\left(d(x_k, x_{k+1})\right) + l)^{\eta(x_{k-1}, Sx_{k-1})\eta(x_k, Sx_k)} \\
= \;& (\psi\left(d(Sx_{k-1}, Sx_k)\right) + l)^{\eta(x_{k-1}, Sx_{k-1})\eta(x_k, Sx_k)} \\
\leq \;& (\psi\left(d(Sx_{k-1}, Sx_k)\right) + l)^{\alpha(x_{k-1}, Sx_{k-1})\alpha(x_k, Sx_k)} \\
\leq \;& (\beta\left(\psi\left(d(x_{k-1}, x_k)\right)\right)\psi(d(x_{k-1}, x_k)) + l)^{\eta(x_{k-1}, Sx_{k-1})\eta(x_k, Sx_k)}
\end{aligned}$$

which implies that

$$\psi\left(d(x_k, x_{k+1})\right) \leq \beta\left(\psi\left(d(x_{k-1}, x_k)\right)\right)\psi(d(x_{k-1}, x_k)) < \psi(d(x_{k-1}, x_k)) \tag{2.5}$$

Since ψ is non decreasing, so we get

$$\psi\left(d(x_k, x_{k+1})\right) < \psi(d(x_{k-1}, x_k)). \tag{2.6}$$

It is clear that $\{\psi\left(d(x_{k-1}, x_k)\right)\}$ is a decreasing sequence. Therefore, there exists some positive number ϱ such that $\lim_{n\to\infty} \psi\left(d(x_k, x_{k+1})\right) = \varrho$. Now we shall prove that $\varrho = 0$. From (2.6), we have

$$\frac{\psi\left(d(x_k, x_{k+1})\right)}{\psi\left(d(x_{k-1}, x_k)\right)} \leq \beta(\psi\left(d(x_{k-1}, x_k)\right)) \leq 1. \tag{2.7}$$

Now by taking limit $k \to \infty$, we have

$$1 = \frac{d}{d} = \frac{\lim_{k\to\infty} \psi\left(d(x_k, x_{k+1})\right)}{\lim_{k\to\infty} \psi\left(d(x_{k-1}, x_k)\right)} \leq \beta(\psi\left(d(x_{k-1}, x_k)\right)) \leq 1$$

$$\lim_{k\to\infty} \beta(\psi\left(d(x_{k-1}, x_k)\right)) = 1. \tag{2.8}$$

By using property of β function, we have $\lim_{k\to\infty} \psi\left(d(x_{k-1}, x_k)\right) = 0$. Thus

$$\lim_{k\to\infty} d(x_{k-1}, x_k) = 0. \tag{2.9}$$

Now we prove that $\{x_n\}$ is a Cauchy sequence. Suppose on contrary that $\{x_n\}$ is not a Cauchy sequence. Then there exists $\epsilon > 0$ and sequences $\{m_k\}$ and $\{n_k\}$ such that, for all positive integers k, we have $n_k > m_k > k$,

$$\psi\left(d(x_{m_k}, x_{n_k})\right) \geq \epsilon \tag{2.10}$$

and

$$\psi\left(d(x_{m_k}, x_{n_{k-1}})\right) < \epsilon. \tag{2.11}$$

By the triangle inequality, we have

$$\begin{aligned} \epsilon &\leq \psi\left(d(x_{m_k}, x_{n_k})\right) \\ &\leq \psi\left(d(x_{m_k}, x_{n_{k-1}})\right) + \psi\left(d(x_{n_{k-1}}, x_{n_k})\right) \\ &< \epsilon + \psi\left(d(x_{n_{k-1}}, x_{n_k})\right), \end{aligned} \tag{2.1}$$

for all $k \in \mathbb{N}$. Now taking limit as $k \to +\infty$ in (2.12) and using (2.9), we have

$$\lim_{k\to\infty} \psi\left(d(x_{m_k}, x_{n_k})\right) = \epsilon. \tag{2.13}$$

Again using triangle inequality, we have

$$\psi\left(d(x_{m_k}, x_{n_k})\right) \leq \psi\left(d(x_{m_k}, x_{m_{k+1}})\right) + \psi\left(d(x_{m_{k+1}}, x_{n_{k+1}})\right) + \psi\left(d(x_{n_{k+1}}, x_{n_k})\right)$$

and

$$\psi\left(d(x_{m_{k+1}}, x_{n_{k+1}})\right) \leq \psi\left(d(x_{m_{k+1}}, x_{m_k})\right) + \psi\left(d(x_{m_k}, x_{n_k})\right) + \psi\left(d(x_{n_k}, x_{n_{k+1}})\right).$$

Taking limit as $k \to +\infty$ and using (2.9) and (2.13), we obtain

$$\lim_{k \to +\infty} \psi\left(d(x_{m_{k+1}}, x_{n_{k+1}})\right) = \epsilon. \tag{2.14}$$

By using (2.1), (2.13) and (2.14), we have

$$\begin{aligned} & \left(\psi\left(d(x_{m_{k+1}}, x_{n_{k+1}})\right) + l\right)^{\eta(x_{m_k}, Sx_{m_k})\eta(x_{n_k}, Sx_{n_k})} \\ \leq\ & \left(\psi\left(d(x_{m_{k+1}}, x_{n_{k+1}})\right) + l\right)^{\alpha(x_{m_k}, Sx_{m_k})\alpha(x_{n_k}, Sx_{n_k})} \\ \leq\ & \left(\psi\left(d(Sx_{m_k}, Sx_{n_k})\right) + l\right)^{\alpha(x_{m_k}, Sx_{m_k})\alpha(x_{n_k}, Sx_{n_k})} \\ \leq\ & \beta(d(x_{m_k}, x_{n_k}))\psi(d(x_{m_k}, x_{n_k}))^{\eta(x_{m_k}, Sx_{m_k})\eta(x_{n_k}, Sx_{n_k})} \end{aligned}$$

which implies that

$$\psi\left(d(x_{m_{k+1}}, x_{n_{k+1}})\right) \leq \beta(\psi\left(d(x_{m_k}, x_{n_k})\right))\psi(d(x_{m_k}, x_{n_k})). \tag{2.15}$$

Since ψ is non decreasing, so we get

$$\psi\left(d(x_{m_{k+1}}, x_{n_{k+1}})\right) \leq \beta(\psi\left(d(x_{m_k}, x_{n_k})\right))\psi\left(d(x_{m_k}, x_{n_k})\right) < \psi\left(d(x_{m_k}, x_{n_k})\right). \tag{2.16}$$

Therfore, we have

$$\frac{\psi\left(d(x_{m_{k+1}}, x_{n_{k+1}})\right)}{\psi\left(d(x_{m_k}, x_{n_k})\right)} \leq \beta(\psi d(x_{m_k}, x_{n_k})) \leq 1. \tag{2.17}$$

Now taking limit as $k \to +\infty$ in (2.17), we get

$$\lim_{n \to \infty} \beta(\psi\left(d(x_{m_k}, x_{n_k})\right)) = 1. \tag{2.18}$$

Hence $\lim_{k\to\infty} \psi\left(d(x_{m_k}, x_{n_k})\right) = 0 < \epsilon$, which is a contradiction. Hence $\{x_n\}$ is a Cauchy sequence. Since X is complete so there exists $p \in X$ such that $x_n \to p$. Now we prove that $p = Sp$. Suppose (i) holds that is, S is continuous, so we get

$$Sp = S \lim_{n\to\infty} x_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} x_{n+1} = p \tag{2.19}$$

thus $p = Sp$. Now we suppose that (ii) holds. Since

$$\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$$

for all $n \in \mathbb{N} \cup \{0\}$. By the hypotheses of (ii), we have

$$\alpha(p, Sp)\alpha(x_k, Sx_k) \geq \eta(p, Sp)\eta(x_k, Sx_k). \tag{2.20}$$

Using the triangle inequality and (2.1), we have

$$\begin{aligned} & (\psi\left(d(Sp, x_{k+1})\right) + l)^{\eta(p,Sp)\eta(x_k,Sx_k)} \\ = \; & (\psi\left(d(Sp, Sx_k)\right) + l)^{\eta(p,Sp)\eta(x_k,Sx_k)} \\ \leq \; & (\psi\left(d(Sp, Sx_k)\right) + l)^{\alpha(p,Sp)\alpha(x_k,Sx_k)} \\ \leq \; & (\beta(\psi\left(d(p, x_k)\right))\psi(d(p, x_k)) + l)^{\eta(p,Sp)\eta(x_k,Sx_k)} \end{aligned}$$

which implies that

$$\psi\left(d(Sp, x_{k+1})\right) \leq \beta(\psi\left(d(p, x_k)\right))\psi(d(p, x_k)) \tag{2.21}$$

Letting $k \to \infty$ and using the fact that $\psi(0) = 0$, we have $d(p, Sp) = 0$. Thus $p = Sp$. To ensure the uniqueness of the fixed point in Theorem, let there exists q be another fixed point of S and T, $q \in X$, $s.t$ $q = Sq = Tq$.

$$\begin{aligned} \psi\left(d(p, q)\right) + l & \leq \psi\left(d(Sp, Sq)\right) + l \\ & \leq (\psi\left(d(Sp, Sq)\right) + l)^{\eta(p,Sp)\eta(q,Tq)} \\ & \leq (d(Sp, Sq) + l)^{\alpha(p,Sp)\alpha(q,Tq)} \\ & \leq \beta(\psi\left(d(p, q)\right))\psi(d(p, q)) + l. \end{aligned}$$

So,

$$\psi\left(d(p, q)\right) \leq \beta(\psi d(p, q))\psi(d(p, q)). \tag{2.22}$$

By the property of β function, we have $d(p,q) = 0$, implies $p = q$. Hence S has a unique fixed point. □

If $\eta(x,y) = 1$ in the Theorem 2.1, we get the following corollary.

Corollary 2.1. *Let (X,d) be a complete metric space and let S be an $\alpha-$admissible mappings and $\beta \in \mathcal{F}$, such that*

$$(\psi\left(d(Sx,Sy)\right) + l)^{\alpha(x,Sx)\alpha(y,Sy)} \leq \left(\beta\left(\psi(d(x,y))\,\psi(d(x,y)\right) + l\right) \tag{2.23}$$

where $\psi \in \Omega$, $l \geq 1$, for all $x, y \in X$ then suppose that one of the following holds:

(i) S is continuous;

(ii) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}\cup\{0\}$ and $x_n \to p \in X$ as $n \to +\infty$, and

$$\alpha(p, Sp) \geq \eta(p, Sp).$$

If there exists $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq \eta(x_0, Sx_0)$, then S has a fixed point.

Taking $\psi(t) = t$ in Theorem 2.1, we get the following Corollary.

Corollary 2.2. *Let (X,d) be a complete metric space and let S be an $\alpha-$admissible mappings with respect to η and $\beta \in \mathcal{F}$, such that*

$$(d(Sx,Sy) + l)^{\alpha(x,Sx)\alpha(y,Sy)} \leq \left(\beta\left(d(x,y)\right) d(x,y) + l\right)^{\eta(x,Sx)\eta(y,Sy)} \tag{2.24}$$

for all $x, y \in X$ where $l \geq 1$. Suppose that either

(i) S is continuous, or

(ii) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}\cup\{0\}$ and $x_n \to p \in X$ as $n \to +\infty$, then

$$\alpha(p, Sp) \geq \eta(p, Sp).$$

If there exists $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq \eta(x_0, Sx_0)$, then S has a fixed point.

Taking $\eta(x,y)=1$, in Theorem 2.1, we get the following Corollary.

Corollary 2.3. *Let (X,d) be a complete metric space and let S be an $\alpha-$admissible mapping and $\beta \in \mathcal{F}$, such that*

$$(\psi(d(Sx,Sy))+l)^{\alpha(x,Sx)\alpha(y,Sy)} \leq \beta(\psi(d(x,y)))(d(x,y))+l \tag{2.25}$$

for all $x,y \in X$, where $\psi \in \Omega$, $l \geq 1$. Suppose that either

(i) S is continuous, or

(ii) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n,x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to p \in X$ as $n \to +\infty$, then

$$\alpha(p,Sp) \geq 1.$$

If there exists $x_0 \in X$ such that $\alpha(x_0,Sx_0) \geq 1$, then S has a fixed point.

Taking $\psi(t)=t$ and $\eta(x,y)=1$ in Theorem 2.1, we get the following Corollary.

Corollary 2.4. [18] *Let (X,d) be a complete metric space and let S be an $\alpha-$admissible mapping and $\beta \in \mathcal{F}$, such that*

$$(d(Sx,Sy)+l)^{\alpha(x,Sx)\alpha(y,Sy)} \leq \beta(d(x,y))\,d(x,y)+l \tag{2.26}$$

for all $x,y \in X$, where $l \geq 1$. Suppose that either

(i) S is continuous, or

(ii) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n,x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to p \in X$ as $n \to +\infty$, then

$$\alpha(p,Sp) \geq 1.$$

If there exists $x_0 \in X$ such that $\alpha(x_0,Sx_0) \geq 1$, then S has a fixed point.

Example 2.1. *Let $X = [0, \infty)$ with usual metric $d(x, y) = |x - y|$ for all $x, y \in X$ and $S : X \to X$, $\alpha : X \times X \to [0, +\infty)$ and $\beta : [0, +\infty) \to [0, 1]$ for all $x, y \in X$ be defined by*

$$Sx = \left\{ \begin{array}{c} 0 \text{ if } x \in [0,1] \\ \sqrt{x} \text{ if } x \in (1,5] \end{array} \right\}$$

$$\alpha(x, y) = \left\{ \begin{array}{ll} 1 & \text{if } x \geq y, \\ 0 & \text{if } x < y. \end{array} \right\}$$

$$\beta(t) = \frac{1}{\sqrt{t}}, \beta(0) \in [0, 1] \text{ and } \psi(t) = \frac{2}{3}t.$$

Let $x, y \in X$, clearly $Sx \leq x$ and $Sy \leq y$, then S of $\alpha-$admissible mapping $\alpha(x, y) \geq 1$, and $\alpha(x, Sx) \geq 1$, $\alpha(y, Sy) \geq 1$ and $\alpha(x, Sx)\alpha(y, Sy) \geq 1$ implies that

$$\begin{aligned} (d(Sx, Sy) + l)^{\alpha(x,Sx)\alpha(y,Sy)} &= Sx - Sy + l = \sqrt{x} - \sqrt{y} + l \leq \frac{x - y}{\sqrt{x} + \sqrt{y}} + l \\ &\leq \frac{2(x - y)}{3\sqrt{\frac{2}{3}(x - y)}} + l = \beta(\psi(d(x, y)))\psi(d(x, y)) + l. \end{aligned}$$

If $\alpha(x, Sx)\alpha(y, Sy) = 0$, then we have

$$(\psi(d(Sx, Sy)) + l)^{\alpha(x,Sx)\alpha(y,Sy)} = 1 \leq \beta(\psi(d(x, y)))\psi(d(x, y)) + l.$$

Let $x = 5$ and $y = 2$ then

$$\psi(d(S5, S2))^{\alpha(5,S5)\alpha(3,S3)} = 0.5479 \leq \beta(\psi(d(5, 3)))\psi(d(5, 3)) = \beta(\psi(2))\psi(2) = .8660.$$

Theorem 2.2. *Let (X, d) be a complete metric space and let S be an $\alpha-$admissible mappings and $\beta \in F$, such that*

$$\alpha(x, Sx)\alpha(y, Sy)\psi(d(Sx, Sy)) \leq \beta(\psi(d(x, y)))\psi(d(x, y)), \tag{2.27}$$

where $\psi \in \Omega$, for all $x, y \in X$ and suppose that one of the following holds:

(i) S is continuous;

(ii) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to p \in X$ as $n \to +\infty$, then

$$\alpha(p, Sp) \geq 1.$$

If there exists $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq 1$, then S has a fixed point.

Proof. Let $x_0 \in X$ and define

$$x_{n+1} = Sx_n, \text{ for all } \ n \geq 0. \tag{2.28}$$

We shall assume that $x_n \neq x_{n+1}$ for each n. Otherwise, there exists an n such that $x_n = x_{n+1}$. Then $x_n = Sx_n$ and x_n is a fixed point of S. Since $\alpha(x_0, x_1) = \alpha(x_0, Sx_0) \geq 1$ and S is $\alpha-$admissible mapping, we have

$$\alpha(x_1, x_2) = \alpha(Sx_0, Sx_1) \geq 1.$$

By continuing in this way, we have

$$\alpha(x_n, x_{n+1}) \geq 1$$

for all $n \in \mathbb{N} \cup \{0\}$. From (2.29), we have

$$\alpha(x_{n-1}, x_n)\alpha(x_n, x_{n+1}) \geq 1 \tag{2.29}$$

Thus applying the inequality (2.27), with $x = x_{k-1}$ and $y = x_k$, we obtain

$$\begin{aligned} & \alpha(x_{k-1}, Sx_{k-1})\alpha(x_k, Sx_k)(\psi\left(d(x_k, x_{k+1})\right) \\ \leq \quad & (\beta\left(\psi\left(d(x_{k-1}, x_k)\right)\right)\psi(d(x_{k-1}, x_k)) \end{aligned}$$

then

$$\psi\left(d(x_{2k+1}, x_{2k+2})\right) \leq \beta\left(\psi\left(d(x_{2k}, x_{2k+1})\right)\right)\psi(d(x_{2k}, x_{2k+1})). \tag{2.30}$$

It yields that $\{\psi\left(d(x_{2k+1}, x_{2k+2})\right)\}$ is a decreasing sequence. From (2.30), there exists $\varrho \in \mathbb{R}^{+}$ such that $\lim_{n\to\infty} \psi\left(d(x_n, x_{n+1})\right) = \varrho$. Now we shall prove that $\varrho = 0$. From (2.30), we have

$$\frac{\psi\left(d(x_{2k+1}, x_{2k+2})\right)}{\psi(d(x_{2k}, x_{2k+1}))} \leq \beta\left(\psi\left(d(x_{2k}, x_{2k+1})\right)\right) \leq 1.$$

Now takes limit $n \to \infty$, we have

$$\lim_{k\to\infty} \beta\left(\psi\left(d(x_{2k}, x_{2k+1})\right)\right) = 1.$$

By using property of β function, we have

$$\lim_{k\to\infty} \psi\left(d(x_{2k+1}, x_{2k+2})\right) = 0. \tag{2.31}$$

Now we will prove that sequence $\{x_n\}$ is Cauchy. Suppose on contrary, then there exists $\epsilon > 0$ and $\{m_k\}, \{n_k\}$ be two sequences such that, for all positive integer k, we have

$$n_k > m_k > k, \quad \psi\left(d(x_{n_k}, x_{m_k})\right) \geq \epsilon, \psi\left(d(x_{n_{k-1}}, x_{m_k})\right) < \epsilon, \tag{2.32}$$

Follows the similar lines in the proof of Theorem 9, we have

$$\lim_{k\to\infty} \psi\left(d(x_{n_k}, x_{m_k})\right) = \epsilon. \tag{2.32}$$

and

$$\lim_{k\to+\infty} \psi\left(d(x_{n_{k+1}}, x_{m_{k+1}})\right) = \epsilon. \tag{2.33}$$

By using (2.27), (2.32) and (2.33), we have

$$\begin{aligned}\psi\left(d(x_{n_{k+1}}, x_{m_{k+1}})\right) &\leq \alpha(x_{n_k}, Sx_{n_k})\alpha(x_{m_k}, Tx_{m_k})\psi\left(d(Sx_{n_k}, Sx_{m_k})\right)\\ &\leq \beta(\psi d(x_{n_k}, x_{m_k}))\psi(d(x_{n_k}, x_{m_k})).\end{aligned}$$

Therfore,

$$\frac{\psi\left(d(x_{n_{k+1}}, x_{m_{k+1}})\right)}{\psi(d(x_{n_k}, x_{m_k}))} \leq \beta(\psi\left(d(x_{n_k}, x_{m_k})\right)) \leq 1. \tag{2.34}$$

Now taking limit as $k \to +\infty$ in (2.34), we get

$$\lim_{n\to\infty} \beta(\psi\left(d(x_{n_k}, x_{m_k})\right)) = 1. \tag{2.35}$$

Then $\lim_{k\to\infty} \psi\left(d(x_{m_k}, x_{n_k})\right) = \psi(0) = 0 < \epsilon$, which is a contradiction. Hence $\{x_n\}$ is a Cauchy sequence. Since S is continuous

$$Sp = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} x_{n+1} = p.$$

So p is a fixed point of S. By the hypotheses of (ii), we have

$$\alpha(p, Sp)\alpha(x_k, Sx_k) \geq 1.$$

Using the triangle inequality and (2.27), we have

$$\begin{aligned} & \alpha(p, Sp)\alpha(x_k, Sx_k)(\psi\left(d(Sp, x_{k+1})\right)) \\ \leq \quad & (\beta(\psi\left(d(p, x_k)\right))\psi(d(p, x_k)) \end{aligned}$$

which implies that

$$\psi\left(d(Sp, x_{k+1})\right) \leq \beta(\psi\left(d(p, x_k)\right))\psi(d(p, x_k))$$

Letting $k \to \infty$ and using the fact that $\psi(0) = 0$, we have $d(p, Sp) = 0$. Thus $p = Sp$.
Uniqueness: Let q be another fixed point of S. Then

$$\begin{aligned} \psi\left(d(p, q)\right) & \leq \alpha(p, Sp)\alpha(q, Sq)\psi\left(d(Sp, Sq)\right) \\ & \leq \beta(\psi\left(d(p, q)\right))\psi(d(p, q)). \end{aligned}$$

So,

$$\psi\left(d(p, q)\right) \leq \beta(\psi\left(d(p, q)\right))\psi(d(p, q)).$$

By the property of β function, we have $d(p, q) = 0$, implies $p = q$. Hence S has a unique fixed point. □

Corollary 2.5. *Let (X,d) be a complete metric space and let S be an $\alpha-$admissible mapping and $\beta \in \mathcal{F}$, such that*

$$(\alpha(x,Sx)\alpha(y,Sy)\psi\left(d(Sx,Sy)\right) \le \beta\left(d(x,y)\right)\psi(d(x,y)), \tag{2.38}$$

where $\psi \in \Omega$, $l \ge 1$, for all $x, y \in X$ and suppose that one of the following holds:

(i) S is continuous;

(ii) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \ge \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to p \in X$ as $n \to +\infty$, then $\alpha(x_n, p) \ge \eta(x_n, p)$, for all $n \in \mathbb{N} \cup \{0\}$.

If there exists $x_0 \in X$ such that $\alpha(x_0, Sx_0) \ge \eta(x_0, Sx_0)$, then S has a unique fixed point.

If $\psi(t) = t$ in Theorem 2.2. we obtain the following corollary.

Corollary 2.6. [18] *Let (X,d) be a complete metric space and let S be an $\alpha-$admissible mapping and $\beta \in \mathcal{F}$, then*

$$\alpha(x,Sx)\alpha(y,Sy)d(Sx,Sy) \le \beta\left(d(x,y)\right)d(x,y), \tag{2.39}$$

where $l \ge 1$, for all $x, y \in X$ and suppose that one of the following holds:

(ii) S is continuous;

(iii) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \ge 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to p \in X$ as $n \to +\infty$, then $\alpha(x_n, p) \ge 1$, for all $n \in \mathbb{N} \cup \{0\}$.

If there exists $x_0 \in X$ such that $\alpha(x_0, Sx_0) \ge \eta(x_0, Sx_0)$, then S has a unique fixed point.

Remark 1. By combining the technique of Samet et al. [33], Hussain et al [18], Salimi et al. [32], and Erdal Karapınar [11] we obtain Theorem 2.1 and Theorem 2.2.

3. Conclusion

Our resuls are more general than [16, 18, 32, 33] and improve several results existing in the literature.

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