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DETERMINING THE SUPPORT OF A RANDOM VARIABLE BASED ON ITS CHARACTERISTIC FUNCTION

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Abstract. In this article, we consider determining the support of a random variable X if only its corresponding characteristic function, $\phi_X(t) = E\left(e^{itX}\right)$, is known, where the support is the set of all its possible realizations.i.e. $S = \{x;\, f(x) > 0\}$. In other words, the amount of information contained in characteristic function about the support of a random variable is investigated. The two main components of the probability distribution on any random variable are: S and $f(x)$. Most of the proposed methods in the literature focused on determining only the density (mass) function, $f(x)$, assuming S is known. No one has yet considered retrieving the support, S, from the corresponding characteristic function before estimating $f(x)$. This paper is an attempt to complete the gap by retrieving S before estimating $f(x)$, especially when the underlying random variable is a mixture of both discrete and absolutely continuous random variables. It is found that the tail behavior of $\phi_X(t)$ reveals most of the information about S. The theorems relating the properties of $\phi_X(t)$ to S are utilized to formulate the proposed method. Several examples are listed for illustrating the usefulness of the studied method.

1. Introduction

Suppose X is a random variable with a distribution function $F(x)$ and a characteristic function $\phi_X(t) = E\left(e^{itX}\right) = \int_{-\infty}^{\infty} e^{itx} dF(x)$, where $i = \sqrt{-1}$. Let $f(x)$

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be a probability density (mass) function, then the support, S , of the random variable X is the set of all possible values that have positive probabilities (densities) i.e. $S = \{x; f(x) > 0\}$, where x is a realization of the random variable X . The uniqueness theorem in [7] states that there is a one-to-one correspondence between characteristic functions and distribution functions. Two distribution functions $F_1(x)$ and $F_2(x)$are equal if and only if their corresponding characteristic functions $\phi_1(t)$ and $\phi_2(t)$ are equal.

$\phi(t)$ has all the information about X than can be obtained from $f(x)$. Hypothetically, both the support and the corresponding probabilities (masses) of a random variable X can be obtained from $\phi(t)$. A significant amount of literature has been proposed on the topic of estimating $f(x)$ using the empirical characteristic functions assuming that the the support is known. See, [3] for properties and applications of empirical characteristic functions. [5], [2], and [10] developed methods of retrieving $f(x)$ by numerical inversion of characteristic functions.

[8] developed a method to estimate Skew-symmetric model parameters using empirical characteristic function. [11] used empirical characteristic functions to estimate parameters in distributions that are finite mixtures of normal distributions. For further details about this topic one may refer to the work done by [12], [4], and [6].

All of the previous research methods focused on estimating the distribution function values given its characteristic function, assuming the support is known. But no one yet has considered determining the support of the random variable given its characteristic function. The main aim of this paper is to effectively use the properties of the characteristic function to determine the support of the underlying random variable.

Here is the organization of the paper. In Section two we discuss and employ the properties of $\phi(t)$ to formulate techniques to find the support of X . This is followed by Section three, where the suggested techniques were applied to determine the support of four characteristic functions. Finally, we make concluding remarks in section four.

2. Characteristic Function Tail Behavior

2.1. **Characteristic Function Decomposition.** The decomposition theorem for the distribution functions, [7], says that every distribution function $F(x)$ can be decomposed uniquely to $F(x) = a_1 F_{ac}(x) + a_2 F_d(x) + a_3 F_s(x)$, where $a_i \geq 0$ for all $i = 1, 2, 3$ and $\sum_{i=1}^{3} a_i = 1$. $F_{ac}(x)$, $F_d(x)$, and $F_s(x)$are respectively absolutely continuous, discrete (step function), and singular distribution functions.

[7] stated

> *A singular distribution is not a discrete probability distribution because each discrete point has a zero probability. On the other hand, neither does it have a probability density function, since the Lebesgue integral of any such function would be zero.*

Therefore we will limit the scope of our work to cases where the distribution function $F(x)$ is decomposed uniquely to a mixture of absolutely continuous and discrete.

Since the characteristic function uniquely determines the distribution function $F(x)$, then any characteristic function $\phi(t)$ can be decomposed to $a_1\phi_{ac}(x) + a_2\phi_d(x) + a_3\phi_s(x)$, where $a_i \geq 0$ for all $i = 1, 2, 3$ and $\sum_{i=1}^{3} a_i = 1$. $\phi_{ac}(x)$, $\phi_d(x)$, and $\phi_s(x)$ respectively are characteristic functions for absolutely continuous, discrete, and singular distribution functions. The next two theorems are provided to show that different types (i.e. absolutely continuous and discrete) of characteristic functions behave differently as $t \to \infty$

2.2. **Limit of $\phi(t)$ at Infinity.**

Theorem 2.1. [1] *If $\phi(t)$ is a characteristic function for an absolutely continuous random variable, then $\lim_{t\to\infty} \phi(t) = 0$.*

The proof is based on the fact that the density function of the absolutely continuous distribution function is absolutely integrable, so it can be approximated by a step function on bounded intervals. The characteristic function of this step function approaches zero as $t \to \infty$.

Theorem 2.2. [7] *If $\phi(t)$ is a characteristic function for discrete random variable, then* $\underset{t\to\infty}{lim sup}\,|\phi(t)| = 1$.

The proof is motivated by the fact that if X is a discrete random variable, then its distribution function $F(x)$ has at most countable number of singularities, which produces a periodic characteristic function.

Theorem 2.3. [7] *If* $\underset{t\to\infty}{lim sup}\,|\phi(t)| = 0$*, then $\phi(t)$ is a characteristic function for continuous distribution function, which can be of absolutely continuous or singular type.*

Based on the previous three theorems, $\underset{t\to\infty}{lim sup}\,|\phi(t)|$ needs to be checked as follows:

(1) If the $\underset{t\to\infty}{lim sup}\,|\phi(t)|$is 0, then we are sure that the distribution function is not discrete, and it is of continuous type, further analysis will be given in section 2.3 to check if it is absolutely continuous.

(2) If the $\underset{t\to\infty}{lim sup}\,|\phi(t)|$is 1, then it is not of absolutely continuous type, further analysis will be given in section 2.4 to check if it is discrete, and in particular a test for a lattice distribution can be conducted.

(3) If the $\underset{t\to\infty}{lim sup}\,|\phi(t)|$is between 0 and 1, then we are sure that, it is neither purely discrete, nor absolutely continuous. In this case decomposing the given characteristic function will reveal extra information about the support using theorems in section 2.5.

2.3. **Absolutely Continuous Characteristic Functions.**

Theorem 2.4. [7] *If $\phi(t)$ is absolutely integrable over $\Re$(real line), then its distribution function is absolutely continuous.*

This result is based on the inversion formula, which says that

$$F(a) - F(b) = \lim_{T\to\infty} \frac{1}{2\pi} \int_{-T}^{T} \left(\frac{e^{-ita} - e^{-itb}}{it} \right) \phi(t)\, dt,$$

when $F(x)$ is continuous at both a and b.

Theorem 2.5. [7] *If $\phi(t)$ is a real-valued continuous function on $\Re$(real line) that satisfies the following, then the distribution function is absolutely continuous.*

(1) $\phi(0) = 1$.

(2) $\phi(-t) = \phi(t)$.

(3) ϕ *is convex for positive numbers.*

(4) $\lim_{t\to\infty} \phi(t) = 0$.

2.4. **Discrete Characteristic Functions.** Another important correspondence between characteristic functions and distribution functions is that, [7], If $\phi(t)$ is a characteristic function for a random variable X , then

$$P(X = x) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} e^{-itx} \phi(t)\, dt.$$

Notice that if X has a continuous distribution function, then for any real x

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} e^{-itx} \phi(t)\, dt = 0,$$

Theorem 2.6. [7] *A random variable X , with distribution function $F(x)$ is discrete, if and only if, the corresponding characteristic function $\phi(t)$ is almost periodic.*

One of the important discrete random variables is the Lattice distribution. Which is a discrete random variable, where the discontinuity points of the distribution function are a subset of the sequence $\{a+kd;\ k=0,\ \pm1,\ \pm2,\ \ldots\}$. It is of interest to check if the underlying distribution function of the given characteristic function is a lattice distribution.

Theorem 2.7. [1] *A distribution function $F(x)$ of a characteristic function $\phi(t)$ is lattice distribution, if and only if, $\phi(t_0)=1$, for some $t_0\neq 0$.*

The proof in, [1], is listed because some details of the proof will be used in section 3. If X is a lattice distribution, let $p_k = Pr(X=x_k)$, where $x_k\in\{a+kd;\ k=0,\ \pm1,\ \pm2,\ \ldots\}$. Then the characteristic function $\phi(t) = \sum\limits_{k=0} p_k e^{it(a+kd)} = e^{ita}\sum\limits_{k=0} p_k e^{itdk}$. So, when $t=t_0=\frac{2\pi}{d}$, then $|\phi(t_0)|=1$. On the other hand, if $|\phi(t_0)|=1$, for some $t_0\neq 0$, then $\phi(t_0)=e^{it_0 a}$ for some real a. At t_0, $\phi(t_0)=\int\limits_{-\infty}^{\infty} e^{it_0 x}dF(x)$. So,

$$e^{-it_0 a}\int\limits_{-\infty}^{\infty} e^{it_0 x}dF(x) = \int\limits_{-\infty}^{\infty} e^{it_0(x-a)}dF(x) = 1.$$

This implies that,

$$\int\limits_{-\infty}^{\infty}\left(1-e^{it_0(x-a)}\right)dF(x) = \int\limits_{-\infty}^{\infty}\left(1-cos\left(t_0\left(x-a\right)\right)\right)dF(x) - i\int\limits_{-\infty}^{\infty} sin\left(t_0\left(x-a\right)\right)dF(x) = 0.$$

Since, $\int\limits_{-\infty}^{\infty}\left(1-cos\left(t_0\left(x-a\right)\right)\right)dF(x)=0$, and $\left(1-cos\left(t_0\left(x-a\right)\right)\right)$is a non-negative function, then $\mu_F\left(\{t_0(x-a);\ k=0,\ \pm1,\ \pm2,\ \ldots\}\right)=1$, where μ_F is the Lebesgue-Stieltjes measure defined by F . Therefore F is supported by a discrete set

$$\{t_0(x-a);\ k=0,\ \pm1,\ \pm2,\ \ldots\} = \left\{x_k = a+\frac{2\pi}{t_0};\ k=0,\ \pm1,\ \pm2,\ \ldots\right\},$$

which is a support for lattice distribution.

This section is concluded by discussing some results about general characteristic function that corresponds to the convolution of distribution functions. These results are helpful in determining the support of the distribution function when decomposing the characteristic function.

2.5. **General Characteristic Functions.** [1]. If X_1 and X_2 are independent random variables, then the distribution function $F(x)$ for the sum $X = X_1 + X_2$ is the convolution of $F_1(x)$and $F_2(x)$ given by $F(x) = (F_1 * F_2)(x) = \int\limits_{-\infty}^{\infty} F_1(x - z)\, dF_2(z)\, dz$. The characteristic function $\phi(t)$ of $F(x)$is given by $\phi(t) = \int\limits_{-\infty}^{\infty} e^{itx} d(F_1 * F_2)(x) = E\left(e^{it(X_1+X_2)}\right) = E\left(e^{itX_1}\right) E\left(e^{itX_2}\right) = \phi_1(t)\,\phi_2(t)$.

Theorem 2.8. [7] *If $F(x) = (F_1 * F_2)(x)$, then the following are true:*

1. *If $F_i(x)$ is continuous, then $F(x)$ is continuous, for $i = 1, 2$.*
2. *If $F_i(x)$ is absolutely continuous, then $F(x)$ is continuous, for $i = 1, 2$.*
3. *$F(x)$ is discrete, if and only of, both $F_1(x)$and $F_2(x)$are discrete.*

Theorem 2.9. [1] *If $\phi(t)$ is the characteristic function for the distribution function $F(x)$, then $\lim\limits_{T\to\infty} \frac{1}{2T} \int\limits_{-T}^{T} |\phi(t)|^2\, dt = \sum\limits_k p_k^2$, where $\{p_k;\ k = 1,\ 2,\ \ldots\}$ are the jumps at the discontinuities of $F(x)$.*

This result is helpful in identifying mixture distributions.

Now we ready to use the above-mentioned theorems to determine the support of based on $\phi(t)$.

1. Check the behavior of the characteristic function at infinity, then use theorems 2.1, 2.2,and 2.3 to decide if it is purely discrete or absolutely continuous or a mixture.
2. Theorems 2.4, 2.5, 2.6, and 2.7 are helpful in determining the support, when the underlying distribution function is purely discrete or absolutely continuous.

(3) If it is neither purely discrete nor absolutely continuous then decomposing the given characteristic function into its natural components (absolutely continuous, purely discrete) together with theorems 2.8, 2.9 will be helpful. Now we are ready to start investigating the given characteristic functions.

3. Application

The proposed method is illustrated through different examples in this section. It is applied to find the support of the following four characteristic functions.

Example 3.1. *:* $\phi_1(t) = \frac{e^{iNt}-1}{N(e^{it}-1)}$*, where* N *is a positive integer.*

We need to observe the behavior of $|\phi_1(t)|$when $t \to \infty$, which is shown in Figure 1, where $N = 8$.

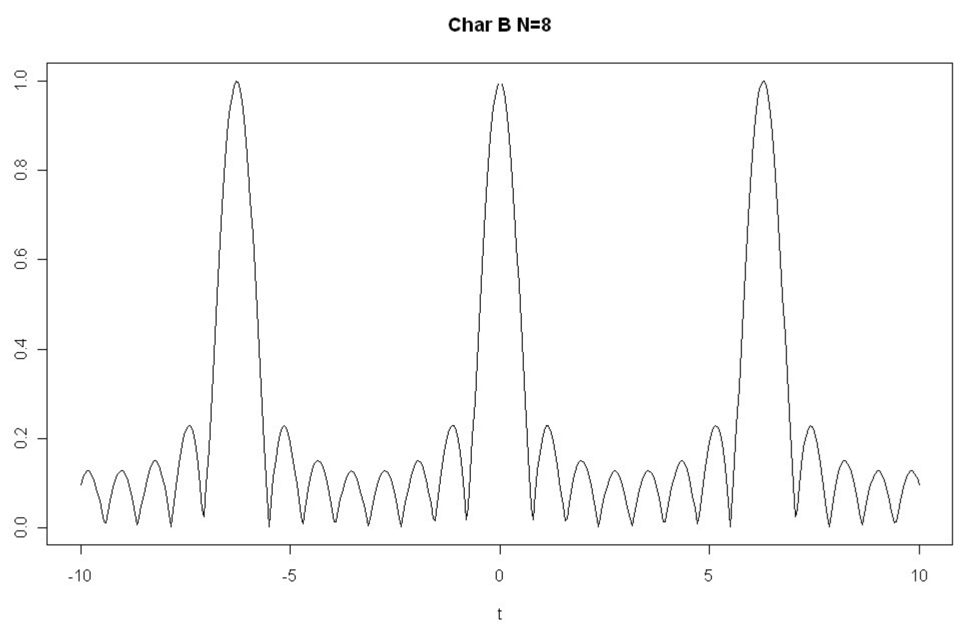

Figure 1. Plot for $|\phi_1(t)|$

Notice that it is a periodic function, based on Theorem 2.6 and Theorem 7 it is of discrete type, and particularly it is a lattice distribution. So, we know that its support is contained in $\{a+kd;\ k=0,\ \pm1,\ \pm2,\ \ldots\}$. To find d notice that $|\phi(2\pi k)|=1$, for any integer k, so it has a period of 2π, if we let $t_0=2\pi$, then based on the discussion by the end of Theorem 2.7, $d=\frac{2\pi}{t_0}=1$.

In general, the number of local peaks within any two peaks of 1 is $N-2$ for $\phi_1(t)$. It turns out that $\phi_1(t)$ is the characteristic function of the discrete random variable X with support of $\{x_k;\ k=0,\ 1,\ 2,\ \ldots,\ (N-1)\}$, where $Pr(X=x_k)=\frac{1}{N}$. Alternatively, direct application of the inversion formula to the given characteristic function produces the same support. Notice that $f(x)=\frac{1}{2\pi}\int\limits_{-\pi}^{\pi}\phi_1(t)\,e^{-itx}dt.$ $=$

$$\frac{1}{2\pi N}\sum_{l=0}^{N-1}\int\limits_{-\pi}^{\pi}e^{itl(-x)}dt=\begin{cases}\frac{1}{N} & \text{, if } x=0,\ 1,\ldots,(N-1)\\ 0 & \text{, Otherwise}\end{cases}.$$

Example 3.2. : $\phi_2(t)=\frac{\pi t}{sinh(\pi t)}$.

We need to observe the behavior of $|\phi_2(t)|$when $t\to\infty$, which is shown in Figure 2

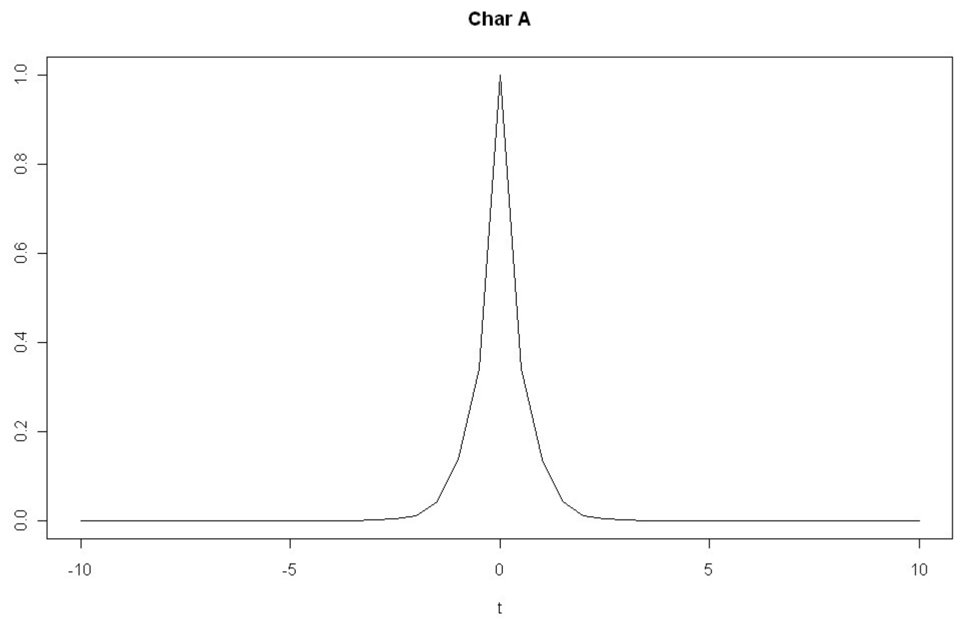

FIGURE 2. Plot for $|\phi_2(t)|$

Since $\limsup_{t\to\infty} |\phi_2(t)| = 0$, then based on Theorem 2.3 its distribution function is continuous. To check if the distribution function is absolutely continuous, notice that $\int_{-\infty}^{\infty} |\phi(t)|\, dt = \lim_{T\to\infty} \int_{-T}^{T} |\phi(t)|\, dt = \lim_{T\to\infty} \int_{-T}^{T} \phi(t)\, dt = \lim_{T\to\infty} \int_{-T}^{T} \frac{\pi t}{sinh(\pi t)} dt \approx 1.5708$. So, by Theorem 2.4 the distribution function is absolutely continuous.

By looking at the formula of this characteristic function, we notice that as a complex-valued function defined on complex numbers, $sinh(\pi z)$ is an entire function on the plane, which has zeros when $z_j = ij$, where $i = \sqrt{-1}$ and $j = 0, \pm 1, \pm 2, \ldots$. By Weierstrass Theorem, in [9], $sinh(\pi z)$ can be represented by $sinh(\pi z) = \pi z \prod_{j=1}^{\infty} \left(1 - \frac{z}{ij}\right)\left(1 + \frac{z}{ij}\right) = \pi z \prod_{j=1}^{\infty} \left(1 + \frac{z^2}{j^2}\right)$. Replacing z by t, we obtain $\phi_2(t) = \frac{\pi t}{sinh(\pi t)} = \prod_{j=1}^{\infty} \left(1 + \frac{z^2}{j^2}\right)^{-1}$.

But the characteristic function for Laplace random variable is m$\phi(t) = (1 + t^2)^{-1}$, with support $(-\infty, \infty)$. So, $\prod_{j=1}^{n} \left(1 + \frac{t^2}{j^2}\right)^{-1} = \prod_{j=1}^{n} \phi\left(\frac{t}{j}\right)$, which is the characteristic function for $\sum_{l=1}^{n} \frac{X_j}{j}$, where $X_1,\ X_2, \ldots, X_n$ are independent Laplace random variables. Notice that the support for $\sum_{l=1}^{n} \frac{X_j}{j}$ is $(-\infty,\ \infty)$ for every $n \geq 1$.

Since $\phi_2(t) = \frac{\pi t}{sinh(\pi t)}$ is the limit of $\prod_{j=1}^{n} \phi\left(\frac{t}{j}\right)$ as $n \to \infty$, then by the continuity theorem, in Casella [2], the distribution function $F_n(x)$ of the random variable $\sum_{l=1}^{n} \frac{X_j}{j}$ converges to the distribution function of interest $F(x)$, this convergence is for all continuity points x of $F(x)$. But, $F(x)$ is continuous, so the convergence is for all real numbers. The support in this case is all real numbers $(-\infty,\ \infty)$.

Example 3.3. : $\phi_3(t) = \frac{\frac{3}{4}}{1-\frac{1}{4}\frac{(e^{it}-1)}{it}}$.

Check the behavior of $|\phi_3(t)|$when $t \to \infty$, which is shown in Figure 3.

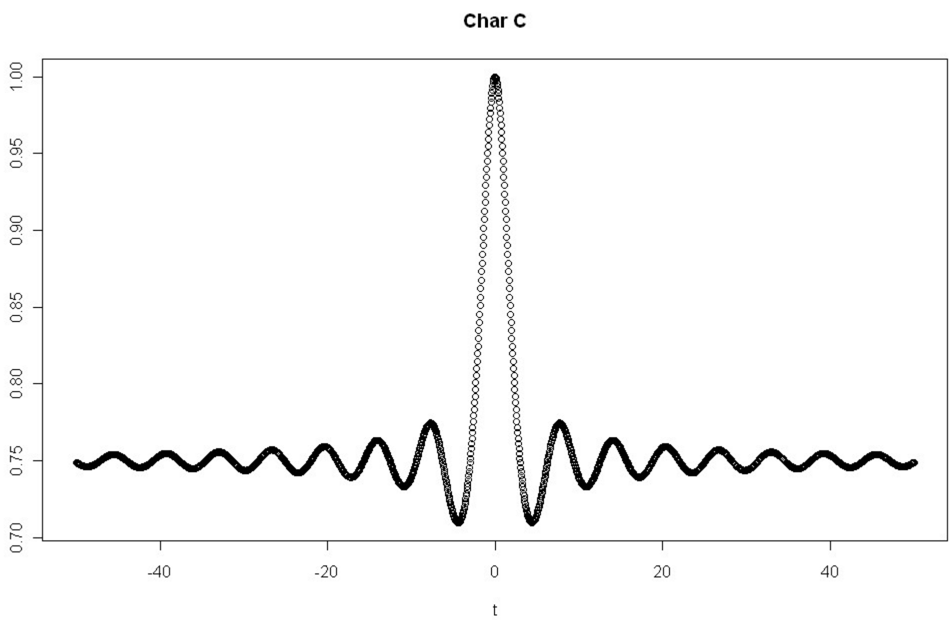

FIGURE 3. Plot for $|\phi_3(t)|$

Notice that,$\underset{t\to\infty}{lim\,sup}\left|\phi_3(t)\right| = 0.75$, which is neither 0 nor 1. So based on the discussion that follows Theorem 2.3, the corresponding distribution function is neither purely discrete, nor absolutely continuous. We suspect in this case that it is a mixture of distribution functions. To understand the nature of this characteristic function; firstly, notice that $\left|\phi_3(t)\right|$approaches 0.75 as $t \to \infty$. Based on Theorem 2.9, $\underset{T\to\infty}{lim}\frac{1}{2T}\int\limits_{-T}^{T}\left|\phi_3(t)\right|^2 dt = \sum\limits_k p_k^2$, where$\{p_k;\, k = 1,\, 2,\, \ldots\}$ are the jumps at the discontinuities of $F(x)$, where $F(x)$ is the distribution function of interest. By using (R package) we can approximately evaluate this integral, it turns out that $\frac{1}{2\pi}\int\limits_{-T}^{T}\left|\phi_3(t)\right|^2 dt \approx 0.5625 = 0.75^2$, when $T = 100000$.

Now based on the decomposition theorem of characteristic functions we would like to decompose $\phi_3(t)$to better understand its nature. This can be done by cross-multiplying $\phi_3(t) = \frac{\frac{3}{4}}{1-\frac{1}{4}\frac{(e^{it}-1)}{it}}$, which leads to

$$\phi_3(t) = \frac{1}{4}\underbrace{\left(\phi_3(t)\frac{(e^{it}-1)}{it}\right)}_{\phi_{ac}(t)} + \frac{3}{4}\phi_d(t),$$

where:

(1) $\left(\phi_3(t)\frac{(e^{it}-1)}{it}\right)$is the product between $\phi_3(t)$and $\frac{(e^{it}-1)}{it}$ which is the characteristic function of the uniform (0, 1). So, the corresponding distribution function is the convolution between the absolutely continuous uniform (0, 1)distribution function and $F(x)$, our distribution function of interest. Based on Theorem 2.8, this convolution is absolutely continuous since one of the components is absolutely continuous.

(2) $\phi_d(t) = 1$, is the characteristic function for the degenerate distribution function, which has a jump of one at its discontinuity point. Notice $\underset{T\to\infty}{lim}\frac{1}{2T}\int\limits_{-T}^{T}\left|\phi_3(t)\right|^2 dt = \sum\limits_k p_k^2 = 0.5625 = 0.75^2$, since our distribution of interest $F(x)$has only one discontinuity point with jump equals to 0.75. Our best guess for the

support is the closed unit interval $[0,1]$ with a mass of about 0.75 at $x=0$. This result is consistent with the convolution with uniform (0, 1) distribution.

Example 3.4. : $\phi_4(t) = \frac{\frac{1}{4}e^{-t^2}}{1-\frac{3}{4}(1+t^2)^{-1}}$.

Check the behavior of $|\phi_4(t)|$when $t \to \infty$, which is shown in Figure 4.

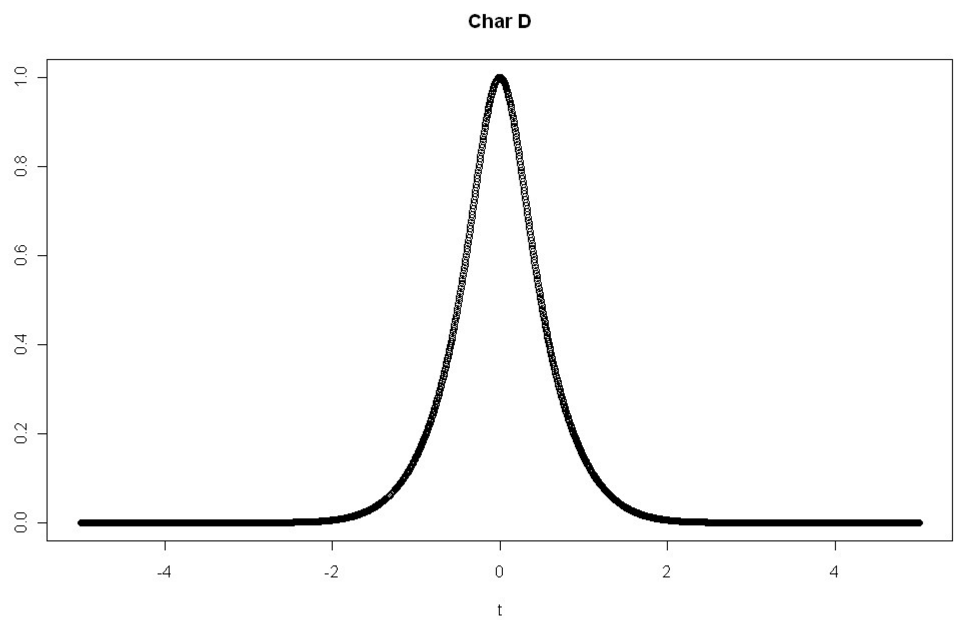

FIGURE 4. Plot for $|\phi_4(t)|$

It is clear from the formula that describes $\phi_4(t)$and the accompanying graph, that $\underset{t\to\infty}{lim\,sup}\,|\phi_4(t)| = \underset{t\to\infty}{lim}\phi_4(t) = 0$, which, based on Theorem 2.3 has a continuous distribution function. Notice that $|\phi_4(t)| \le e^{-t^2}$, for all t. Since, $\int\limits_{-\infty}^{\infty} e^{-t^2}dt = \sqrt{2\pi}$, then $\phi_4(t)$ is absolutely integrable. By Theorem 2.4, the corresponding distribution function is absolutely continuous. To further investigate the nature of the distribution function of interest $F(x)$, let's cross multiply the equation given to describe $\phi_4(t)$, which will lead to

$$\phi_4(t) = \frac{3}{4}\underbrace{\left(\phi_4(t)\frac{1}{(1+t^2)}\right)} + \frac{1}{4}e^{-t^2}.$$

We can think of $F(x)$ s a mixture of two absolutely continuous distributions, where:

(1) $\underbrace{\left(\phi_4(t)\,\frac{1}{(1+t^2)}\right)}$ is the characteristic function that corresponds to the convolution between $F(x)$and the distribution function of Laplace random variable, since $\frac{1}{(1+t^2)}$ is the characteristic function for the Laplace random variable with support $(-\infty,\,\infty)$.

(2) e^{-t^2} is the characteristic function that corresponds to the distribution function of Normal $(\mu=0,\,\sigma^2=2)$ with support $(-\infty,\,\infty)$.

Based on the above discussion, it is suggested that the support is $(-\infty,\,\infty)$.

4. Concluding Remarks and Discussion

To summarize, in this article, we have presented a method to determine the support of a random variable if only its corresponding characteristic function is given. The tail behavior of the characteristic function reveals most of the information about the support. It turns out that it is not an easy problem to fully determine the support of the distribution function if we are given its corresponding characteristic function, even though the characteristic function has all the information we need.

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