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HOMOTOPY REGULARIZATION METHOD TO SOLVE THE SINGULAR VOLTERRA INTEGRAL EQUATIONS OF THE FIRST KIND

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Abstract. In this paper, by combining the regularization method (RM) and the homotopy analysis method (HAM) a new approach is presented to solve the generalized Abel's integral equations (AIEs) of the first kind which is called the HAM-regularization method (HRM). The RM is applied to transform the first kind integral equations (FKIEs) to the second kind (SK) which depends on the regularization parameter and the HAM is used to find the solution of AIEs of the SK. The validation of results are illustrated by presenting a convergence analysis theorem. Also, the efficiency and accuracy of the HRM are shown by solving three examples.

1. Introduction

Many applicable problems were presented in several fields of sciences and engineering such as mechanics, physics, plasma diagnostics, physical electronics, nuclear physics, optics, astrophysics and mathematical physics [2, 8, 9, 11, 19] which they can be transformed to the linear and non-linear forms of the AIEs [21, 22].

In the recent years, several computational and semi-analytical schemes to solve the singular problems of the Abel's kind were presented [4, 15, 25, 26]. Furthermore, the

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Chebyshev polynomials [16], the spectral method [18] and the Legendre wavelets [24] were applied to estimate the solution of the first and second kinds AIEs.

The HAM is one of the semi-analytical and flexible methods which was presented by Liao [12, 13] and it was applied to solve different problems [6, 7, 15, 17]. The RM was constructed by Tikhonov [20] and it was extended by others [5, 14, 23].

In this paper, in order to solve the first kind generalized AIEs of the linear or non-linear form, the HAM and the RM are combined. The RM is used to convert the first kind AIEs to the SK. When we apply the HAM to solve the AIE, the approximate solution will be obtained. But, by using the HRM, the exact solution can be produced. According to [23], this method is depend on the parameter τ and the exact solution of IE can be obtained when $\tau \to 0$.

The organization of this paper is in the following form: the HRM is introduced to solve the AIE of the first kind in section 2. In this section, a simple technique is applied to transform the non-linear IEs to the linear form. The convergence theorem is presented in section 3. In section 4, some applicable examples of singular IEs are solved based on the proposed algorithm. Finally, section 5 is conclusion.

2. The HAM-Regularization method

The non-linear form of generalized AIE of the first kind is introduced as [21, 22]:

$$g(z) = \int_a^z \frac{P(\psi(s))}{(p(z)-p(s))^\beta} ds, \ 0 < \beta < 1, \tag{2.1}$$

where a is a given real value, p and g are known, $P(\psi(s))$ is a nonlinear function of $\psi(s)$ that ψ is an unknown function. Furthermore p is a differentiable, monotone and strict increasing function in $[a, b]$, and $p' \neq 0$ for every s in the interval.

By using the transformation

$$\omega(z) = P(\psi(z)) \tag{2.2}$$

the non-linear IE (2.1) can be changed to the linear form as follows:

$$g(z) = \int_a^z \frac{\omega(s)}{(p(z)-p(s))^\beta} ds, \ 0 < \beta < 1. \tag{2.3}$$

Now, the RM is applied to convert Eq. (2.3) to the SK

$$\tau\omega_\tau(z) = g(z) - \int_a^z \frac{\omega_\kappa(s)}{(p(z)-p(s))^\beta} ds, \tag{2.4}$$

which is depend on the regularization parameter τ. Now, we can write

$$\omega_\tau(z) = \frac{1}{\tau} g(z) - \frac{1}{\tau} \int_a^z \frac{\omega_\tau(s)}{(p(z)-p(s))^\beta} ds, \tag{2.5}$$

and the HAM is employed to solve Eq. (2.5). Let

$$\widehat{\mathcal{N}}\big[W_\tau(z;\ell)\big] = W_\tau(z;\ell) - \frac{1}{\tau} g(z) + \frac{1}{\tau} \int_a^z \frac{W_\tau(s;\ell)}{(p(z)-p(s))^\beta} ds, \tag{2.6}$$

is an operator of non-linear form where the homotopy parameter is shown by $\ell \in [0,1]$ and $W_\tau(z;\ell)$ is an unknown function.

The following zero order deformation equation can be produced by the prevalent homotopy method [12, 13] as

$$(1-\ell)\widehat{\mathcal{L}}\left[W_\tau(z;\ell) - \omega_{\tau,0}(z)\right] = \ell\hbar\widehat{H}(z)\widehat{\mathcal{N}}\left[W_\tau(z;\ell)\right], \tag{2.7}$$

where the convergence-control parameter is demonstrated by $\hbar \neq 0$, the auxiliary function is displayed by $\widehat{H}(z) \neq 0$, the auxiliary operator of linear form is shown by $\widehat{\mathcal{L}}$, the primary conjecture of $\omega_\tau(z)$ is shown by $\omega_{\tau,0}(z)$ and $W_\tau(z;\ell)$ is an unknown function. By putting $\ell = 0, 1$ into Eq. (2.7), since $\hbar \neq 0$ and $\widehat{H}(z) \neq 0$ we have $W_\tau(z;0) = \omega_{\tau,0}(z)$, and $W_\tau(z;1) = \omega_\tau(z)$. Now, by using Taylor's theorem, we get

$$W_\tau(z;\ell) = \omega_{\tau,0}(z) + \sum_{d=1}^{\infty} \omega_{\tau,d}(z)\ell^d, \tag{2.8}$$

where

$$\omega_{\tau,d}(z) = \frac{1}{d!} \frac{\partial^d W_\tau(z;\ell)}{\partial r^d}\bigg|_{\ell=0}. \tag{2.9}$$

In order to show the convergence of series (2.8) at $\ell = 1$, we assume that $\hbar$, $\widehat{H}(z), \omega_{\tau,0}(z)$ and $\widehat{\mathcal{L}}$ are so correctly selected. Therefore, the following series solution can be obtained:

$$\omega_{\tau,d}(z) = W_\tau(z;1) = \omega_{\tau,0}(z) + \sum_{d=1}^{\infty} \omega_{\tau,d}(z). \tag{2.10}$$

According to [5, 12, 13, 15], the d-th order deformation equation is defined as follows:

$$\widehat{\mathcal{L}}\big[\omega_{\tau,d}(z) - \upsilon_d \omega_{\tau,d-1}(z)\big] = \hbar \widehat{H}(z)\big[\Re_d(\omega_{\tau,d-1})\big], \tag{2.11}$$

where

$$\Re_d(\omega_{\tau,d-1}) = \omega_{\tau,d-1}(z) + \frac{1}{\tau}\int_a^z \frac{\omega_{\tau,d-1}(s)}{(p(z)-p(s))^\beta} ds - (1-\upsilon_d)\frac{1}{\tau} g(z), \tag{2.12}$$

and

$$\upsilon_d = \begin{cases} 0, \ d \leqslant 1, \\ \\ 1, \ d > 1. \end{cases} \tag{2.13}$$

By choosing $\widehat{\mathcal{L}}\omega_\tau = \omega_\tau$, $\widehat{H}(z) = 1$ and $\hbar = -1$, the following recursive relation is produced

$$\begin{cases} \omega_{\tau,0}(z) = \dfrac{1}{\tau} g(z), \\ \\ \omega_{\tau,d}(z) = -\dfrac{1}{\tau}\displaystyle\int_a^z \frac{\omega_{\tau,d-1}(s)}{(p(z)-p(s))^\beta} ds, \ d \geq 1. \end{cases} \tag{2.14}$$

The $\jmath$-th order of regularized approximate solution can be estimated by

$$\omega_{\tau,\jmath}(z) = \sum_{d=0}^{\jmath} \omega_{\tau,d}(z), \tag{2.15}$$

and the solution of IE (2.3) can be obtained by $\tau \to 0$. Finally, the transformation

$$\psi = P^{-1}(\omega), \tag{2.16}$$

is applied to find the exact solution of Eq. (2.1).

3. Theoretical discussion and algorithm

By presenting the following theorem, the convergence of the HRM to solve the AIE of the first kind (2.3) is discussed. The existence and uniqueness theorems were demonstrated in [1, 3, 10].

Theorem 1. Let the series solution

$$V_\tau(z) = \omega_{\tau,0}(z) + \sum_{d=1}^{\infty} \omega_{\tau,d}(z), \tag{3.1}$$

is convergent where $\omega_{\tau,d}(z)$ is produced by the d-th order deformation equation (2.11) then the regularized SK IE (2.5) has an exact solution which is the solution of the series (3.1).

Proof Since the series (3.1) is convergent, then

$$\lim_{d\to\infty} \omega_{\tau,d}(z) = 0. \tag{3.2}$$

The linear operator $\widehat{\mathcal{L}}$ is applied to left hand side of d-th order deformation equation (2.11) as follows

$$\sum_{d=1}^{\infty} \widehat{\mathcal{L}}\Big[\omega_{\tau,d}(z) - \upsilon_d \omega_{\tau,d-1}(z)\Big] = \widehat{\mathcal{L}}\Big[\sum_{d=1}^{\infty} \big(\omega_{\tau,d}(z) - \upsilon_d \omega_{\tau,d-1}(z)\big)\Big], \tag{3.3}$$

where we have

$$\sum_{d=1}^{q} \Big[\omega_{\tau,d}(z) - \upsilon_d \omega_{\tau,d-1}(z)\Big] = \omega_{\tau,1}(z) + (\omega_{\tau,2}(z) - \omega_{\tau,1}(z)) + \ldots + (\omega_{\tau,q}(z) - \omega_{\tau,q-1}(z)) = \omega_{\tau,q}(z), \tag{3.4}$$

and

$$\sum_{d=1}^{\infty} \Big[\omega_{\tau,d}(z) - \upsilon_d \omega_{\tau,d-1}(z)\Big] = \lim_{q\to\infty} \omega_{\tau,q}(z) = 0. \tag{3.5}$$

Therefore Eq. (3.3) is equal with zero and by using Eq. (2.11) the following relation can be obtained

$$\sum_{d=1}^{\infty}\widehat{\mathcal{L}}\Big[\omega_{\tau,d}(z)-\upsilon_d\omega_{\tau,d-1}(z)\Big]=\hbar\widehat{H}(z)\sum_{d=1}^{\infty}\Re_{d-1}(\overrightarrow{\omega_{\tau,d-1}}(z))=0. \tag{3.6}$$

Since $\hbar,\widehat{H}(z)$ are not equal with zero thus $\sum_{d=1}^{\infty}\Re_{d-1}(\overrightarrow{\omega_{\tau,d-1}}(z))=0$ and we get

$$\begin{aligned}
&\sum_{d=1}^{\infty}\Re_{d-1}(\omega_{\tau,d-1}(z))\\
&=\sum_{d=1}^{\infty}\left[\omega_{\tau,d-1}(z)+\frac{1}{\tau}\int_a^z\frac{\omega_{\tau,d-1}(s)}{(p(z)-p(s))^{\beta}}ds-(1-\upsilon_d)\frac{1}{\tau}g(z)\right]\\
&=-\frac{1}{\tau}g(z)+\sum_{d=0}^{\infty}\omega_{\tau,d}(z)+\frac{1}{\tau}\int_a^z\frac{\sum_{d=0}^{\infty}\omega_{\tau,d}(s)}{(p(z)-p(s))^{\beta}}ds\\
&=-\frac{1}{\tau}g(z)+V_{\tau}(z)+\frac{1}{\tau}\int_a^z\frac{V_{\tau}(s)}{(p(z)-p(s))^{\beta}}ds.
\end{aligned} \tag{3.7}$$

Now, by using Eqs. (3.6) and (3.7) we can write

$$V_{\tau}(z)=\frac{1}{\tau}g(z)-\frac{1}{\tau}\int_a^z\frac{V_{\tau}(s)}{(p(z)-p(s))^{\beta}}ds, \tag{3.8}$$

therefore $V_{\tau}(z)$ is the exact solution of Eq. (2.5) . According to [20] when $\tau\to 0$ the solution of regularized AIE of the SK (2.5) is convergent to the solution of the first kind singular IE (2.3). ■

In order to implement the examples, the following algorithm is presented based on the proposed approach.

Algorithm:

Step 1: Transform the AIE (2.1) to the linear form by using Eq. (2.2).

Step 2: Apply the RM to construct the regularized AIE of the SK (2.5).

Step 3: Use Eqs. (2.14) to calculate the regularized series solution (2.15).

Step 4: Find the exact solution of linear IE (2.3) by $\tau\to 0$.

Step 5: Apply the transformation (2.16) to find the exact solution of non-linear IE (2.1).

4. Sample examples

In this section, some examples of the first kind generalized AIEs are solved. By using the proposed technique, at first, the non-linear given AIE is converted to the linear form. Then, the HRM is applied to calculate the exact solution based on the proposed algorithm. The programs have been provided by Mathematica 10.

Example 1: Let the following non-linear AIE of the first kind [21]

$$\pi + z = \int_0^z \frac{\sin^{-1}(\psi(s))}{(z^2 - s^2)^{\frac{1}{2}}} ds, \; 0 < z < 2. \tag{4.1}$$

The transformation

$$\omega(z) = \sin^{-1}(\psi(s)), \psi(z) = \sin(\omega(z)), \tag{4.2}$$

and the RM, converts Eq. (4.1) to the linear SK IE

$$\omega_\tau(z) = \frac{1}{\tau}(\pi + z) - \frac{1}{\tau}\int_0^z \frac{\omega_{\tau(s)}}{(z^2 - s^2)^{\frac{1}{2}}} ds. \tag{4.3}$$

By using recursive formula (2.14) which is obtained from HRM, we get

$$\begin{cases} \omega_{\tau,0}(z) = \dfrac{1}{\tau} g(z) = \dfrac{\pi + z}{\tau}, \\ \\ \omega_{\tau,1}(z) = -\dfrac{1}{\tau}\displaystyle\int_0^z \frac{\omega_{\tau,0}(s)}{(z^2 - s^2)^{\frac{1}{2}}} ds = -\dfrac{\pi^2 + 2z}{2\tau^2}, \\ \\ \omega_{\tau,2}(z) = -\dfrac{1}{\tau}\displaystyle\int_0^z \frac{\omega_{\tau,1}(s)}{(z^2 - s^2)^{\frac{1}{2}}} ds = \dfrac{\pi^3 + 4z}{4\tau^3}, \\ \vdots \\ \omega_{\tau,m}(z) = -\dfrac{1}{\tau}\displaystyle\int_0^z \frac{\omega_{\tau,m-1}(s)}{(z^2 - s^2)^{\frac{1}{2}}} ds. \end{cases} \tag{4.4}$$

Therefore, the solution of linear IE (4.3) is obtained as follows:

$$\begin{aligned}\omega_\tau(z) &= \frac{z}{\tau}+\frac{\pi}{\tau}-\frac{z}{\tau^2}-\frac{\pi^2}{2\tau^2}+\frac{z}{\tau^3}+\frac{\pi^3}{4\tau^3}+\cdots \\ &= \frac{z}{\tau}\left[1-\frac{1}{\tau}+\frac{1}{\tau^2}-\frac{1}{\tau^3}+\cdots\right]+\frac{\pi}{\tau}\left[1-\frac{\pi}{2\tau}+\left(\frac{\pi}{2\tau}\right)^2-\left(\frac{\pi}{2\tau}\right)^3+\cdots\right] \\ &= z\left(\frac{1}{\tau+1}\right)+\pi\left(\frac{2}{2\tau+\pi}\right).\end{aligned} \tag{4.5}$$

We know the obtained solution of the HRM is convergent to the exact solution when $\tau \to 0$. So, we obtain

$$\omega(z) = \lim_{\tau\to 0}\omega_\tau(z) = z+2, \tag{4.6}$$

and by substituting $\omega(z)$ in (4.2) the exact solution of nonlinear IE (4.1) can be obtained as

$$\psi(z) = \sin(z+2). \tag{4.7}$$

Example 2: Consider the non-linear AIE [21]

$$\frac{4}{3}z^{\frac{3}{2}} = \int_0^z \frac{\ln(\psi(s))}{\sqrt{z-s}}ds. \tag{4.8}$$

By transformation

$$\omega(z) = \ln(\psi(z)),\ \psi(z) = e^{\omega(z)},$$

and applying the RM, the FKIE (4.8) can be converted to the linear SK AIE as

$$\omega_\tau(z) = \frac{4}{3\tau}z^{\frac{3}{2}} - \frac{1}{\tau}\int_0^z \frac{\omega_\tau(s)}{\sqrt{z-s}}ds. \tag{4.9}$$

By applying the HAM and the following recursive formula

$$\begin{cases} \omega_{\tau,0}(z) = \frac{1}{\tau} g(z) = \frac{4}{3\tau} z^{\frac{3}{2}}, \\ \omega_{\tau,1}(z) = -\frac{1}{\tau} \int_0^z \frac{\omega_{\tau,0}(s)}{\sqrt{z-s}} ds = -\frac{\pi z^2}{2\tau^2}, \\ \omega_{\tau,2}(z) = -\frac{1}{\tau} \int_0^z \frac{\omega_{\tau,1}(s)}{z^{\frac{3}{2}}} ds = \frac{8\pi z^{\frac{5}{2}}}{15\tau^3}, \\ \vdots \\ \omega_{\tau,m}(z) = -\frac{1}{\tau} \int_0^z \frac{\omega_{\tau,m-1}(s)}{z^{\frac{3}{2}}} ds, \end{cases} \tag{4.10}$$

the regularized series solution of IE (4.9) can be produced as

$$\omega_\tau(z) = \sum_{m=0}^{\infty} \omega_{\tau,m}(z) = \frac{4}{3\tau} z^{\frac{3}{2}} - \frac{\pi z^2}{2\tau^2} + \frac{8\pi z^{\frac{5}{2}}}{15\tau^3} + \cdots = 1.33 z^{\frac{3}{2}} \left(\frac{1}{\tau + 1.33 z^{\frac{1}{2}}} \right). \tag{4.11}$$

Since $\omega(z) = \lim_{\tau \to 0} \omega_\tau(z) = z$ therefore, the solution of Eq. (4.8) is obtained in the following form

$$\psi(z) = e^{\omega(z)} = e^z. \tag{4.12}$$

Example 3: In this example, the following generalized AIE [21]

$$\frac{3}{10} z^{\frac{10}{3}} = \int_0^z \frac{\psi^3(s)}{(z^4 - s^4)^{\frac{1}{6}}} ds, \tag{4.13}$$

is considered. The transformation $\omega(z) = \psi^3(z)$ and $\psi(z) = \omega^{\frac{1}{3}}(z)$ are changed the non-linear IE (4.13) to the linear form as

$$\omega_\tau(z) = \frac{3}{10\tau} z^{\frac{10}{3}} - \frac{1}{\tau} \int_0^z \frac{\omega_\tau(s)}{(z^4 - s^4)^{\frac{1}{6}}} ds. \tag{4.14}$$

By applying the formula (2.14), the series solution of HRM can be constructed as follows:

(4.15)

$$\omega_\tau(z) = \sum_{m=0}^{\infty} \omega_{\tau,m}(z) = \frac{3 z^{\frac{10}{3}}}{10\tau} - \frac{3 z^{\frac{11}{3}} \Gamma\left[\frac{5}{6}\right] \Gamma\left[\frac{13}{12}\right]}{40\tau^2 \Gamma\left[\frac{23}{12}\right]} + \frac{\pi z^4 \Gamma\left[\frac{5}{6}\right] \Gamma\left[\frac{13}{12}\right]}{160\tau^3 \Gamma\left[\frac{23}{12}\right]} + \cdots = \frac{\frac{3}{10} z^{\frac{10}{3}}}{\tau + \frac{3}{10} z^{\frac{1}{3}}}.$$

Now, in order to calculate the exact solution of SK IE (4.14) we find

$$\omega(z) = \lim_{\tau \to 0} \omega_\tau(z) = \lim_{\tau \to 0} \frac{\frac{3}{10} z^{\frac{10}{3}}}{\tau + \frac{3}{10} z^{\frac{1}{3}}} = z^3. \tag{4.16}$$

Thus $\psi(z) = \omega(z)^{\frac{1}{3}} = z$ is the solution of Eq. (4.13).

5. Conclusions

Generalized AIEs are special form of Volterra IEs which is modelled in sciences and engineering problems. In this work, an applicable and efficient approach based on the HAM and the RM was applied to solve the first kind AIEs. In this approach, the RM is combined as an auxiliary scheme with the HAM to find the exact solution. Also, the convergence theorem was proved to illustrate the solution obtained from the proposed algorithm is satisfied to the exact solution of the AIE. Finally, some examples of the non-linear generalized AIEs were solved based on the proposed algorithm. As furture works, solving the Volterra integro-differential equations, system of IEs and high dimensional IEs with singular kernel by using the HRM are suggested.

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