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NEW RESULTS ON BEHAVIORS OF FUNCTIONAL VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS WITH MULTIPLE TIME-LAGS

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ABSTRACT. The paper deals with a non-linear Volterra integro-differential equation (NVIDE) with multiple time-lags. Conditions are obtained which are sufficient for stability (S), boundedness (B), globally asymptotically stability (GAS) of solutions, and for every solution x of the given (NVIDE) to be belong to the solutions classes, such as $L^1[0,\infty)$ and $L^2[0,\infty)$. We prove some results on stability, boundedness, global asymptotic stability, integrability and square integrability properties of solutions of the considered (NVIDE). The technique of the proofs involves to construct some suitable Lyapunov functionals (LFs). The given conditions involve nonlinear generalizations and extensions of those conditions found in the literature. The obtained results are new and complement that found in the literature.

1. Introduction

In the mathematical literature, a famous mathematical model is known as (VIDE), which appeared after its establishment by Vito Volterra, in 1926. Today,linear (VIDEs) and (NVIDEs) are very important effective mathematical models to describe many real world phenomena related to atomic energy, biology, chemistry, control theory, economy, the engineering technique fields, information theory, medicine,

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population dynamics, physics, etc. (see, Burton [3], Burton and Mahfoud [6], Corduneanu [8], Gripenberg et al. [19], Lakshmikantham and Rama Mohan Rao [27], Peschel and Mende [35], Rahman [39], Staffans [42], Wazwaz [59], the books in [63], [64], [65] and the references therein).

In particular, during the last four decades, motivated by many applications in biology, physics, engineering, medicine, economy, etc., researchers have obtained many interesting results on the qualitative properties of solutions of linear (VIDEs) and (NVIDEs) by fixed point method, perturbation method or the Lyapunov's functional method, etc., (see, Becker [1], Burton ([2], [4)], Burton and Mahfoud [5], Chang and Wang [7], Dung [9], Eloe et al. [10], Engler [11], Funakubo et al. [12], Furumochi and Matsuoka [13], Grace and Akin [14], Graef, and Tunç [15], Graef et al. [16], Grimmer and Seifert [17], Grimmer and Zeman [18], Grossman and Miller [20], Hara et al. ([21],[22]), Hino and Murakami [23], Islam et al. [24], Jin and Luo [25], Lakshmikantham and Rama Mohan Rao [26], Mahfoud ([28],[29],[30]), Martinez [31], Miller [32], Murakami [33], Napoles Valdes [34], Raffoul ([36], [37],[38]), Rama Mohana Rao and Raghavendra [40], Rama Mohana Rao and Srinivas [41], Tunç ([43], [44], [45], [46], [47], [48]), Tunç and Ayhan [49], Tunç and Mohammed ([50], [51],[52], [53]), Tunç and Tunç ([54], [55], [56]), Vanualailai [57], Wang [58], Wang et al. [60], Zhang [61], Da Zhang [62] and many relative papers or books found in the references of these sources).

When we look at the mentioned sources, it can be seen that nearly all of the results therein were proved by means of suitable (LFs). Indeed, this fact shows the effectiveness of the Lyapunov's functional method during investigation of the qualitative properties of the solutions of (VIDEs), which can be raised in many researches and applications. However, to the best of our information from the literature, in a few results the fixed point techiques or perturbation techiques are used for investigation of qualitative properties of certain linear (VIDEs) and (NVIDEs). This case can be seen by looking at the context of the mentioned works and those found in their references. Here, we would not like to give the details of the applications of these methods.

Meanwhile, despite its long history, today the Lyapunov's method still the most effective technique to reduce a complexed system into a relatively simpler system and discuss the qualitative behaviors of solutions of ordinary differential equations, functional differential equations , (VIDEs) and so on. However, the Lyapunov characterizations for retarded (NVIDEs) with non-smooth fuctionals remain as an open problem in the related literature till now. Here, we try to proceed an application of this method for a (NVIDE) with multiple time lags. By this way, it is worth to investigate qualitative properties of solutions of (NVIDEs) with multiple time lags.

First, we would now like to summarize some related papers.

In [1], Becker considered the linear (VIDE) of the form

$$x' = -a(t)x + \int_0^t b(t,s)x(s)ds. \tag{1.1}$$

In [1], a Lyapunov functional is constructed to obtain sufficient conditions for global asymptotic stability of the trivial solution of linear (VIDE) (1.1). It is shown that the Lyapunov functional is uniformly continuous, which in turn implies that solutions of linear (VIDE) (1.1) are uniformly continuous, stable, bounded, etc.. Examples are considered to illustrate the obtained results.

In ([43], [44], [16], [54]), the authors considered (NVIDEs) of the form

$$x' = -a(t)h(x) + \int_0^t b(t,s)g(x(s))ds,$$
$$x' = -a(t)f(x) + \int_{t-\tau}^t B(t,s)g(x(s))ds + p(t),$$
$$x' = -a(t)x + \sum_{i=1}^n \int_{t-\tau_i}^t b_i(t,s)f_i(x(s))ds$$

and

$$\begin{aligned} x' = &- a(t)g(x) + \sum_{i=1}^{n} \int_{t-\tau_i}^{t} K_i(t,s)g_i(s,x(s))ds \\ &+ \sum_{i=1}^{n} r_i(t,x,x(t-\tau_i)), \end{aligned}$$

respectively.

In ([43], [44], [16], [54]), defining appropriate (LFs), the authors established sufficient conditions under which the solutions of these (NVIDEs) have some qualitative properties such as stability, boundedness, convergence, global asymptotic stability and so on.

Motivated by the mentioned papers, books and the works proceeded, in this paper, we consider the following (NVIDE) with multiple time-lags

$$x' = -g(t,x) + \sum_{i=1}^{n} \int_{t-\tau_i}^{t} b_i(t,s,x(s))g_i(x(s))ds, \tag{1.2}$$

where $t \in \Re^+$, $\Re^+ = [0,\infty)$, such that $t-\tau_i \geq 0$, τ_i is the fixed constant delay, $x \in \Re$, $\Re = (-\infty,\infty)$, $g(t,0) = 0$, $g_i(0) = 0$ and $b_i(t,s,0) = 0$. We assume that the function $g : \Re^+ \times \Re \to \Re$ is continuously differentiable and the functions $b_i : \Omega \times \Re \to \Re$ and $g_i : \Re \to \Re$ are continuous, where $\Omega := \{(t,s) : 0 \leq s \leq t < \infty\}$.

Let

$$g_0(t,x) = \begin{cases} \frac{g(t,x)}{x}, & x \neq 0 \\ g_x(t,0), & x = 0. \end{cases}$$

Hence, from (NVIDE) (1.2), we can write

$$x' = -a(t)g_0(t,x)x + \sum_{i=1}^{n} \int_{t-\tau_i}^{t} b_i(t,s,x(s))g_i(x(s))ds.$$

The following notations are needed throughout this paper.

Let $\tau = \max_{1 \leq i \leq n} \tau_i$. For $\phi \in C[t_0-\tau, t_0]$, it is supposed that $|\phi|_{t_0} := \sup\{|\phi(t)| : t_0 - \tau \leq t \leq t_0\}$. In addition, let $L^1[0,\infty)$ and $L^2[0,\infty)$ represent the set of all continuous and real-valued functions g and h, for which, we have $\int_0^\infty |g(s)|ds < \infty$ and $\int_0^\infty |h(s)|^2 ds < \infty$, respectively.

It should be noted that when we need x represents $x(t)$

2. Main results

We assume the following-hypotheses hold.

A. Hypotheses

(A1): *The function* $g : \Re^+ \times \Re \to \Re$ *is continuously differentiable and the functions* $g_i : \Re \to \Re$ *and* $b_i : \Omega \times \Re \to \Re$ *are continuous for each* $i = 1, ..., n$.

(A2): *Let* α *and* β *be positive constants such that* $1 \leq g_0(t, x) \leq \beta$ *and* $|g_i(x)| \leq \alpha|x|$ *for all* $x \in \Re$ *and each* $i = 1, ..., n$.

(A3): $g_0(s, x(s)) - \sum_{i=1}^n \int_{s-\tau_i}^t \alpha|b_i(t, u, x(u))|du \geq 0$ *for all* $t \geq s - \tau \geq 0$.

(A4): $g_0(t, x) - \sum_{i=1}^n \int_0^t \alpha|b_i(t, s, x(s))|ds \geq 0$ *for all* $t \geq 0$.

(A5): $\int_0^\infty |b_i(t, s, x(s))|ds \leq L$ *for some constant* $L, L > 0$, *and* $t \in \Re$

Theorem 2.1. If hypotheses (A1)-(A5) hold, then all solutions of (NVIDE) (1.2) are bounded and the zero solution of (NVIDE) (1.2) is stable.

Proof. Let $t_0 \geq 0$, $\phi \in C[t_0 - \tau, t_0]$ be an initial function and $x(t) = x(t, t_0, \phi)$ represent the solution of (NVIDE) (1.2) on $[t_0 - \tau, \infty)$ such that $x(t) = \phi(t)$ for $t \in [t_0 - \tau, t_0]$.

We define a (LF)

$$W : [0, \infty) \times C[0, \infty) \to [0, \infty)$$

by

$$W(t) = W(t, x(.)) = x^2 + \int_0^t \{g_0(s, x(s)) - \sum_{i=1}^n \int_{s-\tau_i}^t \alpha|b_i(t, u, x(u))|du\} x^2(s)ds. \tag{2.1}$$

In view of assumption $(A4)$, from (2.1) we see that

$$W(t, x(.)) \geq x^2$$

for all $t \geq s - \tau \geq 0$.

From calculations of the time derivative of (LF) W, we can obtain

$$
\begin{aligned}
W'(t) = & - g_0(t,x)x^2 + 2x \sum_{i=1}^{n} \int_{t-\tau_i}^{t} b_i(t,s,x(s))g_i(x(s))ds \\
& - x^2 \sum_{i=1}^{n} \int_{t-\tau_i}^{t} \alpha |b_i(t,u,x(u))|du - \int_0^t \sum_{i=1}^{n} \alpha |b_i(t,s,x(s))|x^2(s)ds \\
\leq & - g_0(t,x)x^2 + 2|x| \sum_{i=1}^{n} \int_{t-\tau_i}^{t} \alpha |b_i(t,s,x(s))||x(s)|ds \\
& - x^2 \sum_{i=1}^{n} \int_{t-\tau_i}^{t} \alpha |b_i(t,u,x(u))|du - \sum_{i=1}^{n} \int_0^t \alpha |b_i(t,s,x(s))|x^2(s)ds \\
\leq & - g_0(t,x)x^2 + \sum_{i=1}^{n} \int_0^t \alpha |b_i(t,s,x(s))||x(s)|(x^2(t) + x^2(s))ds \\
& - x^2 \sum_{i=1}^{n} \int_{t-\tau_i}^{t} \alpha |b_i(t,u,x(u))|du - \sum_{i=1}^{n} \int_0^t \alpha |b_i(t,s,x(s))|x^2(s)ds \\
= & - [g_0(t,x) - \sum_{i=1}^{n} \int_0^t \alpha |b_i(t,s,x(s))|ds]x^2 \\
& - [\sum_{i=1}^{n} \int_{t-\tau_i}^{t} \alpha |b_i(t,u,x(u))|du]x^2.
\end{aligned}
$$

Hence, we find

$$
\begin{aligned}
W'(t) & \leq -[g_0(t,x) - \sum_{i=1}^{n} \int_0^t \alpha |b_i(t,s,x(s))|ds + \sum_{i=1}^{n} \int_{t-\tau_i}^{t} \alpha |b_i(t,u,x(u))|du]x^2 \\
& \leq -[g_0(t,x) - \sum_{i=1}^{n} \int_0^t \alpha |b_i(t,s,x(s))|ds]x^2 \qquad (2.2)
\end{aligned}
$$

for all $t \geq 0$.

In view assumption $(A4)$, from (2.2) we can conclude that

$$W'(t) \leq 0 \ for \ all \ t \geq t_0 \geq 0.$$

Integrating the last inequality from 0 to t and considering the estimate $W(t) \geq x^2$, we obtain

$$x^2 \leq W(t) \leq W(t_0) \tag{2.3}$$

for all $t \geq t_0 \geq 0$.

Hence, we can see from the estimate (2.3) that all solutions of (NVIDE) (1.2) are bounded for all $t \geq t_0 \geq 0$.

Indeed, from (2.3) and the estimate

$$\begin{aligned} W(t_0) &= \phi^2(t_0) + \int_0^{t_0} \{g_0(s, \phi(s)) - \sum_{i=1}^{n} \int_{s-\tau_i}^{t_0} \alpha |b_i(t, u, \phi(u))| du\} \phi^2(s) ds \\ &\leq |\phi|_{t_0}^2 M(t_0), \end{aligned}$$

we can get

$$|x| \leq |\phi|_{t_0} \sqrt{M(t_0)} \tag{2.4}$$

for all $t \geq t_0 \geq s - \tau$, where

$$M(t_0) := 1 + \int_0^{t_0} \{g_0(s, \phi(s)) - \sum_{i=1}^{n} \int_{s-\tau_i}^{t_0} \alpha |b_i(t, u, \phi(u))| du\} ds.$$

The obtained result, that is, the inequality (2.4), implies that the zero solution of (NVIDE) (1.2) is stable. That is, for any given $\varepsilon > 0$, there exists a positive constant $\delta = \frac{\varepsilon}{\sqrt{M(t_0)}}$ such that for $\phi \in C[t_0 - \tau, t_0]$, $|\phi|_{t_0} < \delta$ implies that

$$|x| \leq \delta \sqrt{M(t_0)} = \varepsilon$$

for all $t \geq t_0 \geq s - \tau$.

B. Hypotheses

We assume that following hypotheses are satisfied.

(H1): $g_0(t, x) - \sum_{i=1}^{n} \int_0^t \alpha |b_i(t, s, x(s))| ds \geq k$ *for* $t_1 \geq t_0, k \in \Re, k > 0.$

(H2): $g_0(s, x(s)) - \sum_{i=1}^{n} \int_{s-\tau_i}^{t} \alpha |b_i(t, u, x(u))| du \geq k$

for all $t \geq s - \tau \geq t_0$.

Theorem 2.2. In addition to hypotheses $(A1) - (A5)$, if either hypothesis $(H1)$ or $(H2)$ holds, then every solution of (NVIDE) (1.2) is square integrable and all solutions of (NVIDE) (1.2) are bounded for all $t \in [0, \infty)$

Proof. We have from Theorem 2.1 that any solution $x(t)$ of (NVIDE) (1.2) is bounded and satisfies the estimates (2.3) and (2.4). If hypothesis $(H1)$ holds, then from (2.2) we have

$$W'(t) \leq -kx^2$$

for all $t \geq t_1$.

Integrating the last estimate from t_1 to t, we get

$$W(t) - W(t_1) \leq -k \int_{t_1}^{t} x^2(s) ds.$$

Therefore, we can write

$$k \int_{t_1}^{t} x^2(s) ds \leq W(t_1) - W(t) \leq W(t_1).$$

Let $M = W(t_1) > 0$, $M \in \Re$. Hence, we can conclude that

$$\int_{t_1}^{\infty} x^2(s) ds \leq k^{-1} M \ when \ t \to \infty.$$

That is, $x \in L^2[0, \infty)$.

If hypothesis $(H2)$ holds, then from the definition of the functional W, we can get

$$x^2 + k \int_{t_1}^{\infty} x^2(s) ds \leq W(t_1).$$

Then, we conclude that all solutions of (NVIDE) (1.2) are bounded and $x \in L^2[0, \infty)$. These results complete the proof of Theorem 2.2.

C. Hypothesis

The following hypothesis is needed to show the (GAS) of the zero solution of (NVIDE) (1.2).

(C1): *We assume that there exists a positive constant K such that*

$$g_0(t,x)+\sum_{i=1}^{n}\int_{s-\tau_i}^{t}\alpha|b_i(t,s,\phi(s))|ds\le K$$

holds for all $t\ge t_1$.

Theorem 2.3. In addition to hypotheses $(A1)-(A5)$ and $(C1)$, if either hypothesis (H1) or (H2) holds, then the zero solution of (NVIDE) (1.2) is (GAS).

Proof. From Theorem 2.2, we have that every solution $x(t)$ of (NVIDE) (1.2) is square integrable. In addition, from (NVIDE) (1.2) and (2.4), it follows that

$$\begin{aligned}|x'(t)|&\le|g_0(t,x)||x|+\sum_{i=1}^{n}\int_{t-\tau_i}^{t}|b_i(t,s,x(s))||g_i(x(s))|ds\\&\le|g_0(t,x)||x|+\sum_{i=1}^{n}\int_{t-\tau_i}^{t}\alpha|b_i(t,s,x(s))||x(s)|ds\\&\le|g_0(t,x)||\phi|_{t_0}\sqrt{M(t_0)}+\sum_{i=1}^{n}\int_{t-\tau_i}^{t}\alpha|b_i(t,s,x(s))|ds|\phi|_{t_0}\sqrt{M(t_0)}\\&\le K|\phi|_{t_0}\sqrt{M(t_0)}.\end{aligned}$$

Hence, we can conclude that $x'(t)$ is bounded. Since $x'(t)$ is bounded and $x\in L^2[0,\infty)$, both of these results together imply that $x(t)\to0$ as $t\to\infty$. That is, the zero solution of (NVIDE) (1.2) is (GAS). The proof of Theorem 2.3 is completed.

D. Hypothesis

We assume that the following hypothesis holds

(D1): *There exist constants $k_1>0$ and β_1, $0\le\beta_1<1$, such that*

$$g_0(t,x)\ge k_1,\beta_1g_0(t,x)-\sum_{i=1}^{n}\int_{s-\tau_i}^{t}\alpha|b_i(t,u,x(u))|du\ge0$$

for all $t\ge s-\tau$.

Theorem 2.4. If hypotheses $(A2), (A3)$ and $(A5)$ hold, then all solutions of (NVIDE) (1.2) are bounded and the zero solution of (NVIDE) (1.2) is stable. In addition, if hypothesis $(D1)$ holds, then every solution of (NVIDE) (1.2) is integrable on the interval $[0,\infty)$ Further, if hypothesis $(C1)$ holds, then every solution of (NVIDE) (1.2) is square integrable on the interval $[0,\infty)$ and the zero solution of (NVIDE) (1.2) is (GAS).

Proof. Define the (LF)

$$W_1 : [0,\infty)\times C[0,\infty) \to [0,\infty)$$

by

$$\begin{aligned} W_1(t) = W_1(t,x(.)) :=&|x(t)| \\ &+\int_0^t \{g_0(s,x(s)) - \sum_{i=1}^{n}\int_{s-\tau_i}^{t} \alpha|b_i(t,u,x(u))|du\}|x(s)|ds. \end{aligned} \tag{2.5}$$

It is clear that $W_1(t,x(.)) \geq |x|$ for all $t \geq s-\tau$.

It is known that for a continuously differentiable function $h(t)$, $|h(t)|$ has a right derivative, and this right derivative $D_r|h(t)|$ is defined by

$$D_r|h(t)| = \begin{cases} h'(t)sgnh(t), \ if \ h(t) \neq 0 \\ |h'(t)|, \ if \ h(t) = 0. \end{cases}$$

Hence, the right derivative of W_1 is given by

$$\begin{aligned} D_rW_1(t) =&D_r|x| + D_r\int_0^t \{g_0(s,x(s)) - \sum_{i=1}^{n}\int_{s-\tau_i}^{t} \alpha|b_i(t,u,x(u))|du\}|x(s)|ds \\ =& -g_0(t,x)|x| + \sum_{i=1}^{n}\int_{t-\tau_i}^{t} b_i(t,s,x(s))g_i(x(s))ds \\ &+ g_0(t,x)|x| - \sum_{i=1}^{n}\int_{t-\tau_i}^{t} \alpha|b_i(t,u,x(u))|du|x| \\ &- \sum_{i=1}^{n}\int_0^t \alpha|b_i(t,s,x(s))||x(s)|ds \end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{i=1}^{n}\int_{0}^{t}\alpha|b_i(t,s,x(s))||x(s)|ds-\sum_{i=1}^{n}\int_{0}^{t}\alpha|b_i(t,u,x(u))||du|x| \\
&\quad -\sum_{i=1}^{n}\int_{0}^{t}\alpha|b_i(t,s,x(s))||x(s)|ds \\
&=-\sum_{i=1}^{n}\int_{t-\tau_i}^{t}\alpha|b_i(t,u,x(u))||du|x|\leq 0.
\end{aligned}
$$

Then, we can see that

$$D_rW_1(t)\leq 0.$$

This estimate implies that the zero solution of (NVIDE) (1.2) is (S).

Further, by integrating the last inequality from t_0 to t and considering the (LF) (2.5), we can obtain

$$|x|\leq W_1(t)\leq W_1(t_0)$$

for all $t\geq t_0\geq t_0-\tau\geq 0$. Let $W_1(t_0)=M_1$. Since the (LF) W_1 is positive definite, then it is clear that

$$|x|\leq W_1(t_0)=M_1>0, M_1\in\Re.$$

Thus, we can conclude that all of solutions (NVIDE) (1.2) are bounded.

To complete the rest of the proof of Theorem 2.4, we will modify the functional W_1.

Define

$$
\begin{aligned}
&W_\beta(t)=W_\beta(t,x(.))=|x| && (2.6)\\
&\qquad +\int_{0}^{t}\{\beta_1g_0(s,x(s))-\sum_{i=1}^{n}\int_{s-\tau_i}^{t}\alpha|b_i(t,u,x(u)|du\}|x(s)|ds. && (2.7)
\end{aligned}
$$

Hence, by hypothesis $(D1)$, we have

$$W_\beta(t)\geq |x|.$$

On the other hand, the right derivative of functional W_β can be calculated as

$$
\begin{aligned}
D_r W_\beta(t) =& D_r|x| + D_r \int_0^t \{\beta_1 g_0(s,x(s)) - \sum_{i=1}^n \int_{s-\tau_i}^t \alpha |b_i(t,u,x(u))| du\} |x(s)| ds \\
=& - g_0(t,x)|x| + \sum_{i=1}^n \int_{t-\tau_i}^t b_i(t,s,x(s)) g_i(x(s)) ds \\
&+ \beta_1 g_0(t,x)|x| - \sum_{i=1}^n \int_{t-\tau_i}^t \alpha |b_i(t,u,x(u))| du |x| \\
&- \sum_{i=1}^n \int_0^t \alpha |b_i(t,s,x(s))||x(s)| ds
\end{aligned}
$$

From the hypotheses of Theorem 2.4, it can be easily shown that

$$
\begin{aligned}
D_r W_\beta(t) \leq& - g_0(t,x)|x| + \sum_{i=1}^n \int_{t-\tau_i}^t |b_i(t,s,x(s))||g_i(x(s))| ds \\
&+ \beta_1 g_0(t,x)|x| - |x| \sum_{i=1}^n \int_{t-\tau_i}^t \alpha |b_i(t,u,x(u))| du \\
&- \sum_{i=1}^n \int_0^t \alpha |b_i(t,s,x(s))||x(s)| ds \\
\leq& - g_0(t,x)|x| + \sum_{i=1}^n \int_0^t \alpha |b_i(t,s,x(s))||x(s)| ds \\
&+ \beta_1 g_0(t,x)|x| - |x| \sum_{i=1}^n \int_{t-\tau_i}^t \alpha |b_i(t,u,x(u))| du
\end{aligned}
$$

$$
\begin{aligned}
&- \sum_{i=1}^n \int_0^t \alpha |b_i(t,s,x(s))||x(s)| ds \\
=& - (1-\beta_1) g_0(t,x)|x| - \sum_{i=1}^n \int_{t-\tau_i}^t \alpha |b_i(t,u,x(u))| du |x| \\
\leq& - (1-\beta_1) k_1 |x| - \sum_{i=1}^n \int_{t-\tau_i}^t \alpha |b_i(t,u,x(s))| du |x| \\
\leq& - (1-\beta_1) k_1 |x|. \qquad (2.8)
\end{aligned}
$$

By the hypothesis $0 \leq \beta_1 < 1$, it is now clear from (2.8) that

$$D_r W_\beta(t) \leq 0.$$

In addition, both of an integration of the estimate (2.8) and consideration of $W_\beta(t) \geq |x|$ yield that

$$(1-\beta_1)k_1 \int_{t_0}^{t} |x(s)|ds \leq W_\beta(t_0 - W_\beta(t).$$

Therefore, the improrer integral

$$\int_0^\infty |x(s)|ds$$

converges. That is, $x \in L^1[t_0, \infty)$. The proof of Theorem 2.4 is completed.

3. Conclusion

We consider a functional (NVIDE) of first order with multiple time-lags. The (S), (B), (GAS) properties of solutions and the solutions classes, such as $L^1[0,\infty)$ and $L^2[0,\infty)$ are investigated by defining suitable (LFs). The obtained results have a contribution to the literature, and they improve or generalize the results of Becker [1], Graef et al. [16] and Tunç ([43],[44]) , Tunç and Tunç [54] and some the other results that can be found in the relevant literature.

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