ODD VERTEX EQUITABLE EVEN LABELING OF LADDER GRAPHS

P. JEYANTHI (1), A. MAHESWARI (2) AND M. VIJAYALAKSHMI (3)

ABSTRACT. Let $G$ be a graph with $p$ vertices and $q$ edges and $A=\{1,3,\ldots,q\}$ if $q$ is odd or $A=\{1,3,\ldots,q+1\}$ if $q$ is even. A graph $G$ is said to admit an odd vertex equitable even labeling if there exists a vertex labeling $f : V(G) \rightarrow A$ that induces an edge labeling $f^*$ defined by $f^*(uv) = f(u) + f(v)$ for all edges $uv$ such that for all $a$ and $b$ in $A$, $|v_f(a) - v_f(b)| \leq 1$ and the induced edge labels are $2, 4, \ldots, 2q$ where $v_f(a)$ be the number of vertices $v$ with $f(v) = a$ for $a \in A$. A graph that admits an odd vertex equitable even labeling is called an odd vertex equitable even graph [2]. In this paper we investigate the odd vertex equitable even labeling behavior of some ladder graphs.

1. Introduction

We consider only simple, finite, connected and undirected graphs and follow the basic notations and terminology of graph theory as in [1]. Let $G(V, E)$ be a graph with $p$ vertices and $q$ edges. The vertex set and the edge set of a graph are denoted by $V(G)$ and $E(G)$ respectively. The notion of vertex equitable labeling was due to Lourdusamy and Seenivasan et al.[4]. Let $G$ be a graph with $p$ vertices and $q$ edges and $A = \{0,1,2,\ldots,\left\lfloor \frac{q}{2} \right\rfloor \}$. A graph $G$ is said to be vertex equitable if there

1991 Mathematics Subject Classification. 05C78.
Key words and phrases. vertex equitable labeling; vertex equitable graph; odd vertex equitable even labeling, odd vertex equitable even graph.
Copyright © Deanship of Research and Graduate Studies, Yarmouk University, Irbid, Jordan. Received: Jan. 3, 2018 Accepted: Nov. 11, 2018.
exists a vertex labeling $f : V(G) \to A$ that induces an edge labeling $f^*$ defined by $f^*(uv) = f(u) + f(v)$ for all edges $uv$ such that for all $a$ and $b$ in $A$, $|v_f(a) - v_f(b)| \leq 1$ and the induced edge labels are $1, 2, 3, \ldots, q$ where $v_f(a)$ is the number of vertices $v$ with $f(v) = a$ for $a \in A$. The vertex labeling $f$ is known as vertex equitable labeling. Motivated by the concept of vertex equitable labeling [4], Jeyanthi, Maheswari and Vijayalakshmi extended this concept and introduced a new concept namely odd vertex equitable even labeling in [2] and proved that the graphs, path, $P_n \odot P_m(n, m \geq 1)$, $K_{1,n} \cup K_{1,n-2}(n \geq 3)$, $K_{2,n}(n \geq 1)$, $T_p$-tree, cycle $C_n(n \equiv 0$ or $1(\text{mod } 4))$, quadrilateral snake $Q_n(n \geq 1)$, ladder $L_n(n \geq 1)$, $L_n \odot K_1(n \geq 1)$, arbitrary super subdivision of any path $P_n$ are odd vertex equitable even graphs. They also proved that the graph $K_{1,n}$ is an odd vertex equitable even graph if and only if $n \leq 2$. Let $G$ be a graph with $p$ vertices and $q$ edges and $p \leq \left[\frac{q}{2}\right] + 1$, then $G$ is not an odd vertex equitable even graph. In addition, they proved that if every edge of a graph $G$ is an edge of a triangle, then $G$ is not an odd vertex equitable even graph. In [3], Jeyanthi and Maheswari proved that some cyclic snake related graphs admit odd vertex equitable even labeling.

We use the following definitions and known results in the subsequent section.

**Definition 1.1.** The graph $\langle L_n \hat{\times} K_{1,m} \rangle$ is the graph obtained from ladder $L_n$ and $2n$ copies of $K_{1,m}$ by identifying a non central vertex of $i^{th}$ copy of $K_{1,m}$ with $i^{th}$ vertex of $L_n$.

**Definition 1.2.** The graph $P_n \times P_2$ is called a ladder graph.

**Definition 1.3.** Let $G$ be a graph. The subdivision graph $S(G)$ is obtained from $G$ by subdividing each edge of $G$ with a vertex.

**Definition 1.4.** The corona $G_1 \odot G_2$ of the graphs $G_1$ and $G_2$ is defined as a graph obtained by taking one copy of $G_1$ (with $p$ vertices) and $p$ copies of $G_2$ and then joining the $i^{th}$ vertex of $G_1$ to every vertex of the $i^{th}$ copy of $G_2$. 


Definition 1.5. Let $G_1$ be a graph with $p$ vertices and $G_2$ be any graph. A graph $G_1 \tilde{\odot} G_2$ is obtained from $G_1$ and $p$ copies of $G_2$ by identifying one vertex of $i^{th}$ copy of $G_2$ with $i^{th}$ vertex of $G_1$.

Theorem 1.6. [2] Cycle $C_n$ is an odd vertex equitable even graph if $n \equiv 0 \text{ or } 1 \pmod{4}$.

Theorem 1.7. [2] $K_{1,n} \cup K_{1,n-2}$ is an odd vertex equitable even graph for any $n \geq 3$.

2. Main Results

In this section, we prove that $S(L_n)$, $L_m \tilde{\odot} P_n$, $L_n \tilde{\odot} \overline{K_m}$, $\langle L_n \tilde{\odot} K_{1,m} \rangle$ are odd vertex equitable even graphs.

Theorem 2.1. The subdivision graph $S(L_n)$ is an odd vertex equitable even graph.

Proof. Let $V(L_n) = \{u_i, v_i/1 \leq i \leq n\}$, $E(L_n) = \{u_i u_{i+1}, v_i v_{i+1}/1 \leq i \leq n-1\} \cup \{u_i v_i/1 \leq i \leq n\}$. Let $v'_i$ be the newly added vertex between $v_i$ and $v_{i+1}$, $u'_i$ be the newly added vertex between $u_i$ and $u_{i+1}$. Let $w_i$ be the newly added vertex between $v_i$ and $u_i$. Clearly $S(L_n)$ has $5n - 2$ vertices and $6n - 4$ edges.

Let $A = \begin{cases} 1, 3, 5, ..., 5n - 2 & \text{if } n \text{ is odd} \\ 1, 3, 5, ..., 5n - 1 & \text{if } n \text{ is even} \end{cases}$.

Define the vertex labeling $f : V(S(L_n)) \to A$ as follows:

- $f(u_1) = 1$, $f(v_1) = f(v'_1) = 3$.
- For $1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$, $f(u_{2i}) = 12i - 5$, $f(w_{2i}) = 12i - 3$, $f(v_{2i}) = 12i - 7$.
- $f(w_{2i}) = 12i - 5$.
- For $1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$, $f(w_{2i-1}) = 12i - 11$.
- For $1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil$, $f(u_{2i+1}) = f(u'_{2i}) = 12i - 1$, $f(w_{2i}) = 12i + 3$, $f(v_{2i+1}) = 12i + 1$.

For the vertex labeling $f$, the induced edge labeling $f^*$ is as follows:

- $f^*(u_1 w_1) = 2$, $f^*(w_1 v_1) = 4$, $f^*(u_1 u'_1) = 10$, $f^*(v'_1 v_2) = 8$.
- For $1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor - 1$ $f^*(u_{2i} u_{2i+1}) = 24i - 6$, $f^*(w_{2i} v_{2i}) = 24i - 12$.
\[ f^*(v_{2i}v'_{2i}) = 24i - 4, \quad f^*(v'_{2i}v_{2i+1}) = 24i + 4, \quad f^*(u_{2i}u_{2i+1}) = 24i - 2, \]
\[ f^*(u_{2i+1}w_{2i+1}) = 24i, \quad f^*(w_{2i+1}v_{2i+1}) = 24i + 2. \]
For \(1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor - 1\),
\[ f^*(u_{2i+1}u'_{2i+1}) = 24i + 8, \quad f^*(v'_{2i+1}v_{2i+2}) = 24i + 10. \]
For \(1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor\),
\[ f^*(v_{2i-1}v'_{2i-1}) = 24i - 18, \quad f^*(u'_{2i-1}u_{2i}) = 24i - 8, f^*(u_{2i}w_{2i}) = 2(12i - 5). \]

It can be verified that the induced edge labels of \(S(L_n)\) are 2, 4, \ldots, 12n - 8 and \(|v_f(i) - v_f(j)| \leq 1\) for all \(i, j \in A\). Clearly \(f\) is an odd vertex equitable even labeling of \(S(L_n)\). Thus, \(S(L_n)\) is an odd vertex equitable even graph. An odd vertex equitable even labeling of \(S(L_6)\) is shown in Figure 1.

\[ \begin{array}{cccccccc}
3 & 3 & 5 & 15 & 13 & 17 & 17 & 27 \\
1 & 9 & 7 & 11 & 11 & 21 & 19 & 23 \\
1 & 7 & 13 & 19 & 25 & 31 \\
3 & 5 & 17 & 15 & 13 & 17 & 29 & 29 \\
1 & 9 & 7 & 11 & 11 & 21 & 19 & 23 \\
3 & 5 & 17 & 15 & 13 & 17 & 29 & 29 \\
1 & 9 & 7 & 11 & 11 & 21 & 19 & 23 \\
3 & 5 & 17 & 15 & 13 & 17 & 29 & 29 \\
\end{array} \]

Figure 1. An odd vertex equitable even labeling of \(S(L_6)\)

\[ \square \]

**Theorem 2.2.** The graph \(L_m \hat{\hat{P}}_n\) is an odd vertex equitable even graph.

**Proof.** Let \(u_1, u_2, \ldots, u_m\) and \(v_1, v_2, \ldots, v_m\) be the vertices of the ladder \(L_m\).

Let \(v_{ij}, u_{ij} (1 \leq i \leq m, 1 \leq j \leq n)\) be the vertices of \(m\) copies of \(P_n\).

Let vertex set \(V(L_m \hat{\hat{P}}_n) = \{u_i, v_i, u_{ij}, v_{ij} : 1 \leq i \leq m, 1 \leq j \leq n\}\) and edge set
\[ E(L_m \hat{\hat{P}}_n) = \{u_i v_i : 1 \leq i \leq m\} \cup \{u_{i+1} v_{i+1} : 1 \leq i \leq m-1\} \cup \{u_{ij} v_{ij} v_{ij+1} : 1 \leq i \leq m, 1 \leq j \leq n-1\} \quad \text{and} \quad u_m = u_i, v_m = v_i : 1 \leq i \leq m. \]

Clearly \(L_m \hat{\hat{P}}_n\) has \(2mn\) vertices and \(2mn + m - 2\) edges.

Let \(A = \begin{cases} 
1, 3, 5, \ldots, 2mn + m - 2 & \text{if } m \text{ is odd} \\
1, 3, 5, \ldots, 2mn + m - 1 & \text{if } m \text{ is even} 
\end{cases} \)

Define the vertex labeling \(f : V(L_m \hat{\hat{P}}_n) \rightarrow A\) as follows:
Case 1. n is odd.

For 1 \leq j \leq \left\lfloor \frac{n}{2} \right\rfloor and i is odd, set
\[ f(u_i(2j-1)) = n - (2j - 2) + (4n + 2) \left\lfloor \frac{i}{2} \right\rfloor, \]
\[ f(v_i(2j-1)) = n + (2j - 2) + (4n + 2) \left\lfloor \frac{i}{2} \right\rfloor, \]

For 1 \leq j \leq \left\lfloor \frac{n}{2} \right\rfloor and i is odd, set
\[ f(u_i(2j)) = n - 2j + (4n + 2) \left\lfloor \frac{i}{2} \right\rfloor, \]
\[ f(v_i(2j)) = n + 2j + (4n + 2) \left\lfloor \frac{i}{2} \right\rfloor. \]

Case 2. n is even.

For 1 \leq j \leq \frac{n}{2} and i is odd, set
\[ f(u_i(2j-1)) = f(u_i(2j)) = n - 2j + 1 + (4n + 2) \left\lfloor \frac{i}{2} \right\rfloor, \]
\[ f(v_i(2j-1)) = f(v_i(2j)) = n + 1 + (2j - 2) + (4n + 2) \left\lfloor \frac{i}{2} \right\rfloor, \]

For 1 \leq j \leq \frac{n}{2} and i is even, set
\[ f(u_i(2j-1)) = 3n + 1 + (2j - 2) + (4n + 2) \left\lfloor \frac{i}{2} \right\rfloor, \]
\[ f(u_i(2j)) = 3n + 1 + 2j + (4n + 2) \left\lfloor \frac{i}{2} \right\rfloor, \]
\[ f(v_i(2j-1)) = 3n + 1 - (2j - 2) + (4n + 2) \left\lfloor \frac{i}{2} \right\rfloor, \]
\[ f(v_i(2j)) = (3n - 1) - (2j - 2) + (4n + 2) \left\lfloor \frac{i}{2} \right\rfloor. \]

For the vertex labeling \( f \), the induced edge labeling \( f^* \) is as follows:
\[ f^*(u_iv_i) = (4n + 2)(i - 1) + 2n \] if \( 1 \leq i \leq m \)
\[ f^*(u_iu_{i+1}) = (4n + 2)i \] if \( 1 \leq i \leq m - 1 \)
\[ f^*(v_iv_{i+1}) = (4n + 2)i - 2 \] if \( 1 \leq i \leq m - 1. \)

For \( 1 \leq i \leq m, 1 \leq j \leq n - 1 \)
\[ f^*(v_{ij}v_{i(j+1)}) = \begin{cases} 
2(n + j + (4n + 2) \left\lfloor \frac{i}{2} \right\rfloor) & \text{if } i \text{ is odd} \\
2(3n + (4n + 2) \left\lfloor \frac{i}{2} \right\rfloor) - 2(j - 1) & \text{if } i \text{ is even.} 
\end{cases} \]
If $n$ is even $f^*(u_{ij}u_{i(j+1)}) = \begin{cases} 2(n-j+(4n+2) \left\lfloor \frac{i}{2} \right\rfloor) & \text{if } i \text{ is odd} \\ 2(3n+2+(4n+2) \left\lfloor \frac{i-1}{2} \right\rfloor) + 2(j-1) & \text{if } i \text{ is even}. \end{cases}$

If $n$ is odd $f^*(u_{ij}u_{i(j+1)}) = \begin{cases} 2(n-j+(4n+2) \left\lfloor \frac{i}{2} \right\rfloor) & \text{if } i \text{ is odd} \\ 2(3n+1+(4n+2) \left\lfloor \frac{i-1}{2} \right\rfloor) + 2j & \text{if } i \text{ is even}. \end{cases}$

It can be verified that the induced edge labels of $L_m\hat{O}P_n$ are $2, 4, \ldots, 4mn + 2m - 4$ and $|v_f(i) - v_f(j)| \leq 1$ for all $i, j \in A$. Clearly $f$ is an odd vertex equitable even labeling of $L_m\hat{O}P_n$. Thus, $L_m\hat{O}P_n$ is an odd vertex equitable even graph. An odd vertex equitable even labeling of $L_5\hat{O}P_6$ is shown in Figure 2.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{An odd vertex equitable even labeling of $L_5\hat{O}P_6$}
\end{figure}

Theorem 2.3. The graph $L_n \odot \overline{K_m}$ is an odd vertex equitable even graph if $m > 1$.

Proof. Let $u_1, u_2, \ldots, u_n$ and $v_1, v_2, \ldots, v_n$ be the vertices of the ladder $L_n$.

Let $u_{ij}, v_{ij}$ ($1 \leq i \leq n, 1 \leq j \leq m$) be the vertices of $n$ copies of $\overline{K_m}$. 

\end{proof}
Clearly $L_n \odot \overline{K_m}$ has $2n + 2mn$ vertices and $2mn + 3n - 2$ edges.

Let $A = \begin{cases} 
1, 3, 5, ..., 2mn + 3n - 2 & \text{if } n \text{ is odd} \\
1, 3, 5, ..., 2mn + 3n - 1 & \text{if } n \text{ is even} 
\end{cases}$.

Define the vertex labeling $f : V(L_n \odot \overline{K_m}) \to A$ as follows:

- $f(u_{2i-1}) = 4(m + 1)(i - 1) + 2i - 1$ if $1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$, 
- $f(u_{2i}) = 4(m + 1)i + 2i - 1$ if $1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$, 
- $f(v_{2i-1}) = 4(m + 1)(i - 1) + 2m + 2i - 1$ if $1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$, 
- $f(v_{2i}) = 4(m + 1)(i - 1) + 2m + 2i + 1$ if $1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$, 
- $f(u_{2i-1,j}) = 4(m + 1)(i - 1) + 2j + 2i - 3$ if $1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$ and $1 \leq j \leq m$, 
- $f(v_{2i-1,j}) = 4(m + 1)(i - 1) + 2j + 2i - 1$ if $1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$ and $1 \leq j \leq m$, 
- $f(u_{2i,j}) = f(v_{2i,j}) = 4(m + 1)i + 2j + 2i - 2m - 3$ if $1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$ and $1 \leq j \leq m$.

For the vertex labeling $f$, the induced edge labeling $f^*$ is as follows:

- For $1 \leq i \leq n$, $f^*(u_i v_i) = 2((2m + 3)(i - 1) + (m + 1))$, 
- For $1 \leq i \leq n - 1$, $f^*(u_i u_{i+1}) = 2(2m + 3)i$, $f^*(v_i v_{i+1}) = 2(2m + 3)i - 2$, 
- For $1 \leq i \leq n$, $1 \leq i \leq m$
  
\[
f^*(u_i u_{i,j}) = \begin{cases} 
2(2m + 3)(i - 1) + 2j & \text{if } i \text{ is odd} \\
2((2m + 3)(i - 1) + m + 1) + 2j & \text{if } i \text{ is even.} 
\end{cases}
\]

\[
f^*(v_i v_{i,j}) = \begin{cases} 
2((2m + 3)(i - 1) + m + 1) + 2j & \text{if } i \text{ is odd} \\
2((2m + 3)(i - 1)) + 2j & \text{if } i \text{ is even.} 
\end{cases}
\]

It can be verified that the induced edge labels of $L_n \odot \overline{K_m}$ are 2, 4, ..., 4mn + 6n - 4 and $|v_f(i) - v_f(j)| \leq 1$ for all $i, j \in A$. Clearly $f$ is an odd vertex equitable even labeling of $L_n \odot \overline{K_m}$. Thus, $L_n \odot \overline{K_m}$ is an odd vertex equitable even graph. An odd vertex equitable even labeling of $L_6 \odot \overline{K_4}$ is shown in Figure 3.
**Theorem 2.4.** The graph \( L_n \hat{\Omega} K_{1,m} \) is an odd vertex equitable even graph.

**Proof.** Let \( u_1, u_2, \ldots, u_n \) and \( v_1, v_2, \ldots, v_n \) be the vertices of the ladder \( L_n \).

Let vertex set \( V(\langle L_n \hat{\Omega} K_{1,m} \rangle) = \{u_i, v_i, u_{i0}, v_{i0}, v_{ij}, u_{ij} : 1 \leq i \leq n, 1 \leq j \leq m\} \) and edge set \( E(\langle L_n \hat{\Omega} K_{1,m} \rangle) = \{u_iv_i : 1 \leq i \leq n\} \cup \{u_iu_{i+1}, v_{i+1}v_{i} : 1 \leq i \leq n - 1\} \cup \{u_iu_{i}, v_{i}v_{i0} : 1 \leq i \leq n\} \cup \{u_{i0}v_{ij}, v_{i0}v_{i0} : 1 \leq i \leq n, 1 \leq j \leq m \) and \( u_{im} = u_i, v_{im} = v_i : 1 \leq i \leq n\}. \) Clearly \( \langle L_n \hat{\Omega} K_{1,m} \rangle \) has \( 2n + 2mn \) vertices and \( 2mn + 3n - 2 \) edges.

Let \( A = \begin{cases} 1, 3, 5, \ldots, 2mn + 3n - 2 & \text{if } n \text{ is odd} \\ 1, 3, 5, \ldots, 2mn + 3n - 1 & \text{if } n \text{ is even} \end{cases} \).

Define the vertex labeling \( f : V(\langle L_n \hat{\Omega} K_{1,m} \rangle) \to A \) as follows:

For \( 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \):
- \( f(u_{2i-1}) = f(u_{2i-1}) = 4(m + 1)(i - 1) + 2i - 1, \)
- \( f(v_{2i-1}) = f(v_{2i-1}) = 4(m + 1)(i - 1) + 2m + 2i - 1, \)
- \( f(u_{2i}) = f(u_{2i}) = 4(m + 1)(i - 1) + 2j + 2i - 1 \) if \( 1 \leq j \leq m, \)

For \( 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \):
- \( f(u_{2i-1}) = f(u_{2i-1}) = 4(m + 1)i + 2i - 3, \)
- \( f(v_{2i-1}) = f(v_{2i-1}) = 4(m + 1)(i - 1) + 2m + 2i + 3, \)
- \( f(u_{2i}) = f(u_{2i}) = 4(m + 1)i + 2i - 1, \)
- \( f(v_{2i}) = f(v_{2i}) = 4(m + 1)(i - 1) + 2m + 2i + 1, \)
f(v_{2i,j}) = 4(m + 1)i - 2j + 2i - 3 \text{ if } 1 \leq j \leq m,
\]
f(u_{2i,j}) = 4(m + 1)i - 2j + 2i - 1 \text{ if } 1 \leq j \leq m.

For the vertex labeling $f$, the induced edge labeling $f^*$ is as follows:

For $1 \leq i \leq n,$

\[ f^*(u_i v_i) = (4(m + 1) + 2)(i - 1) + 2(m + 1) \]
\[ f^*(u_i u_{i+1}) = (4(m + 1) + 2)(i - 1) + 2(i + 1) \]

For $1 \leq i \leq n - 1,$

\[ f^*(v_i v_i) = (4(m + 1) + 2)(i - 1) + 2(2m + 1) \]
\[ f^*(v_i v_{i+1}) = (4(m + 1) + 2)(i - 1) + 4m + 8 \]

It can be verified that the induced edge labels of $L_n \hat{O} K_{1,m}$ are $2, 4, \ldots, 4mn + 6n - 4$ and $|v_f(i) - v_f(j)| \leq 1$ for all $i, j \in A$.

Clearly $f$ is an odd vertex equitable even labeling of $L_n \hat{O} K_{1,m}$.

Thus, $L_n \hat{O} K_{1,m}$ is an odd vertex equitable even graph.

An odd vertex equitable even labeling of $L_4 \hat{O} K_{1,5}$ is shown in Figure 4.

\[ \square \]

**Theorem 2.5.** The cycle $C_n$ is an odd vertex equitable even graph if and only if $n \equiv 0 \text{ or } 1 \pmod{4}$.

**Proof.** The necessary condition is already proved in [2]. Conversely assume that $n \equiv 2$ or $3 \pmod{4}$. Let $f$ be an odd vertex equitable even labeling of the cycle $C_n$. Then
2 \sum_{u \in V} f(u) = n(n+1). .......(1)

Case (i): \( n \equiv 2(\text{mod } 4) \).

Take \( n = 4k+2 \). Then \( A = \{1, 3, \ldots, n+1\} \) and \(|A| = \frac{n}{2} + 1\). Since \( f \) is an odd vertex equitable even labeling and \( p = q = n \), we must have \( v_f(a) = 2 \) for \( \frac{n}{2} - 1 \) elements \( a \) of \( A \) and \( v_f(b) = 1 \) for remaining two elements \( b \) of \( A \). Let \( a_1, a_2, \ldots, a_{\frac{n}{2} - 1} \) be the elements of \( A \) such that \( v_f(a_i) = 2 \) for \( 1 \leq i \leq \frac{n}{2} - 1 \) and \( v_f(a_{\frac{n}{2}}) = v_f(a_{\frac{n}{2}+1}) = 1 \).

Let \( v_i^1 \) and \( v_i^2 \) be the vertices of \( C_n \) so that \( f(v_i^1) = f(v_i^2) = a_i \) for \( 1 \leq i \leq \frac{n}{2} - 1 \), \( f(v_{\frac{n}{2}}^1) = a_{\frac{n}{2}} \) and \( f(v_{\frac{n}{2}+1}^1) = a_{\frac{n}{2}+1} \). Then the vertex set \( V \) of \( C_n \) can be written as

\( V = V_1 \cup V_2 \) where \( V_1 = \{v_i^1 : 1 \leq i \leq \frac{n}{2} + 1\} \) and \( V_2 = \{v_i^2 : 1 \leq i \leq \frac{n}{2} - 1\} \). Hence (1) can be written as \( 2 \sum_{u \in V_1} f(u) + 2 \sum_{u \in V_2} f(u) = n(n+1) \).

Since \( 2 \sum_{u \in V_1} f(u) = 2(1 + 3 + \ldots + (n+1)) = 2\left(\frac{(n+1)(n+2)}{2}\right) - 2(1 + 2 + \ldots + \frac{n}{2}) = \frac{(n+2)^2}{2} \).

Then \( 2 \sum_{u \in V_2} f(u) = n(n+1) - \frac{(n+2)^2}{2} \) which implies that \( 2 \sum_{u \in V_2} f(u) = \frac{n^2 - 2n - 4}{2} \).

Thus we have \( \sum_{u \in V_1} f(u) = 2(2k^2 + k) - 1 \).
Now, \( |V_2| = \frac{n}{2} - 1 = 2k \). Hence \( \sum_{u \in V_2} f(u) \) is the sum of 2k odd numbers and hence it is an even number. But \( 2(2k^2 + k) - 1 \) is always an odd number which is a contradiction.

**Case (ii):** \( n \equiv 3 \pmod{4} \).

Take \( n = 4k + 3 \). Then \( A = \{1, 3, \ldots, n\} \) and \( |A| = \frac{n+1}{2} \). Since \( f \) is an odd vertex equitable even labeling and \( p = q = n \), we must have \( v_f(a) = 2 \) for \( \frac{n-1}{2} \) elements \( a \) of \( A \) and \( v_f(b) = 1 \) for remaining two elements \( b \) of \( A \). Let \( a_1, a_2, \ldots, a_{\frac{n-1}{2}} \) be the elements of \( A \) such that \( v_f(a_i) = 2 \) for \( 1 \leq i \leq \frac{n-1}{2} \) and \( v_f(a_{\frac{n-1}{2}+1}) = 1 \). Let \( v_1^i \) and \( v_2^i \) be the vertices of \( C_n \) so that \( f(v_1^i) = f(v_2^i) = a_i \) for \( 1 \leq i \leq \frac{n-1}{2} \), and \( f(v_{\frac{n-1}{2}+1}) = a_{\frac{n+1}{2}} \). Then the vertex set \( V \) of \( C_n \) can be written as \( V = V_1 \cup V_2 \) where \( V_1 = \{v_1^i : 1 \leq i \leq \frac{n-1}{2}\} \) and \( V_2 = \{v_2^i : 1 \leq i \leq \frac{n-1}{2}\} \). Hence (1) can be written as

\[
2 \sum_{u \in V_1} f(u) + 2 \sum_{u \in V_2} f(u) = n(n+1).
\]

Since \( 2 \sum_{u \in V_1} f(u) = 2(1 + 3 + \ldots + n) = 2\left(\frac{n(n+1)}{2}\right) - 2(1 + 2 + \ldots + \frac{n-1}{2}) = \frac{(n+1)^2}{2} \).

Then \( 2 \sum_{u \in V_2} f(u) = n(n+1) - \frac{(n+1)^2}{2} \) which implies that \( 2 \sum_{u \in V_2} f(u) = \frac{n^2-1}{2} \). Thus we have \( \sum_{u \in V_2} f(u) = 2(k+1)(2k+1) \).

Now, \( |V_2| = \frac{n-1}{2} = 2k + 1 \). Hence \( \sum_{u \in V_2} f(u) \) is the sum of \( 2k + 1 \) odd numbers and hence it is an odd number. But \( 2(k+1)(2k+1) \) is always an even number which is a contradiction.

In both cases we get a contradiction. Hence \( f \) can not be an odd vertex equitable even labeling of \( C_n \) if \( n \equiv 2 \) or \( 3 \pmod{4} \).

**Theorem 2.6.** The graph \( G = K_{1,n+k} \cup K_{1,n} \) is an odd vertex equitable even graph if and only if \( k = 1, 2 \).

**Proof.** Let \( V(G) = \{u, v, u_j, v_i : 1 \leq j \leq n+k \text{ and } 1 \leq i \leq n\} \) and \( E(G) = \{uu_j, vv_i : 1 \leq j \leq n+k \text{ and } 1 \leq i \leq n\} \). Then \( G \) has \( 2n+k+2 \) vertices and \( 2n+k \) edges. Let \( A = \{1, 3, 5, \ldots 2n+k \text{ or } 2n+k+1\} \) according as \( k \) is odd or even. Let \( f \) be a an odd vertex equitable even labeling of the graph \( G = K_{1,n+k} \cup K_{1,n} \).

To get an edge label 2, there must be two adjacent vertices with vertex labels 1 and
1. So we can take \( f(u) = 1 \) and \( f(u_1) = 1 \). To get an edge label 4, there must be two adjacent vertices with vertex labels 1 and 3 and so we have \( f(u_2) = 3 \). Since all the edge labels are distinct, the pendent vertices \( u_1, u_2, u_3, u_4, \ldots, u_{n+k} \) should receive the distinct labels from the set \( A \). So we have \( f(u_3) = 5, f(u_4) = 7, \ldots, f(u_{n+k}) = 2n + 2k - 1 \).

If \( k \) is odd then the maximum of \( A \) is \( 2n + k \). Hence \( 2n + 2k - 1 \leq 2n + k \) which implies \( k \leq 1 \). If \( k \) is even then the maximum of \( A \) is \( 2n + k + 1 \). Hence \( 2n + 2k - 1 \leq 2n + k + 1 \) which implies \( k \leq 2 \).

If \( k = 0 \) then \( q = 2n \) and \( A = 1, 3, 5, \ldots, 2n + 1 \). Since \( |A| = n + 1, p = 2n + 2 \) and \( f \) is odd vertex equitable even labeling, \( v_f(a) = 2 \) for all \( a \in A \). Hence, the pendent vertices of the first component receive the labels 1, 3, 5, \ldots, 2n–1, centre vertex receives the label 1 and the pendent vertices of the second component receive the labels 3, 5, \ldots, 2n–1, 2n + 1.

To get an edge label 4n, there must be two adjacent vertices with vertex labels 2n + 1 and 2n–1. So we can take \( f(v) = 2n + 1 \) or \( 2n - 1 \).

If \( f(v) = 2n - 1 \) then \( v_f(2n - 1) = 3 \) and \( v_f(2n + 1) = 1 \). If \( f(v) = 2n + 1 \) then the edge \( vv_n \) receives the label 4n + 2. In both cases we get a contradiction. Thus, if \( k = 0 \) then \( G \) is not an odd vertex equitable even graph.

If \( k = 1 \) then \( A = \{1, 3, 5, \ldots, 2n+1\} \). The vertex labeling \( f : V(G) \rightarrow A \) is defined as follows: \( f(u) = 1; f(u_j) = 2j - 1 \) for \( 1 \leq j \leq n + 1 \), \( f(v) = 2n + 1 \) and for \( 1 \leq i \leq n, f(v_i) = 2i + 1 \). Hence, \( f(V(G)) = \{1, 1, 3, 5, \ldots, 2n+1\} \cup \{2n+1, 3, 5, \ldots, 2n+1\} \) and also \( f*(E(G)) = \{2, 4, 6, \ldots, 2n + 2\} \cup \{2n + 4, 2n + 6, \ldots, 4n + 2\} \). Hence \( v_f(i) = 2 \) for \( i = 1, 3, 5, \ldots, 2n–1 \) and \( v_f(2n + 1) = 3 \). Thus we have \( |v_f(i) - v_f(j)| \leq 1 \) for all \( i, j \in A \). Hence, \( G = K_{1,n+k} \cup K_{1,n} \) is an odd vertex equitable even graph.

If \( k = 2 \) then \( A = \{1, 3, 5, \ldots, 2n + 3\} \). The vertex labeling \( f : V(G) \rightarrow A \) is defined as follows: \( f(u) = 1; f(u_j) = 2j - 1 \) for \( 1 \leq j \leq n + 2 \), \( f(v) = 2n + 3 \) and for \( 1 \leq i \leq n, f(v_i) = 2i + 1 \).
\[f(V(G)) = \{1, 1, 3, 5, \ldots, 2n+1, 2n+3\} \cup \{2n+3, 3, 5, \ldots, 2n+1\}\] and also \[f(E(G)) = \{2, 4, 6, \ldots, 2n+4\} \cup \{2n+6, 2n+8, \ldots, 4n+4\}\]. Hence \(v_f(i) = 2\) for all \(i \in A\).

Thus we have \(|v_f(i) - v_f(j)| \leq 1\) for all \(i, j \in A\). Hence, \(G = K_{1,n+k} \cup K_{1,n}\) is an odd vertex equitable even graph.

If \(k = 2\) the proof follows from Theorem 1.7 by replacing \(n\) by \(m - 2\). Hence \(G = K_{1,n+k} \cup K_{1,n}\) is an odd vertex equitable even graph. \(\square\)

References


(1) Research Center, Department of Mathematics, Govindammal Aditanar College for Women, Tiruchendur 628215, Tamilnadu, India.

E-mail address: jeyajeyanthi@rediffmail.com

(2) Department of Mathematics, Kamaraj College of Engineering and Technology, Virudhunagar, Tamilnadu, India.

E-mail address: bala_nithin@yahoo.co.in

(3) Department of Mathematics, Dr.G.U. Pope College of Engineering, Sawayerpuram, Tamilnadu, India.

E-mail address: viji_mac@rediffmail.com