Jordan Journal of Mathematics and Statistics (JJMS) 12(3), 2019, pp 351 - 374

ASYMPTOTIC PROPERTIES OF THE CONDITIONAL HAZARD FUNCTION AND ITS MAXIMUM ESTIMATION UNDER RIGHT-CENSORING AND LEFT-TRUNCATION

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Abstract. Gneyou[6, 7] considered the estimation of the maximum hazard rate under random censorship with covariate random and established strong representation and strong uniform consistency with rate of the estimate. Then he studied the asymptotic normality of his estimator. Agbokou et al.[2] generalize this work to the case of right censored and left truncated data with covariate and established strong representation and strong uniform consistency with rate of the estimate of the said estimator and of a non-parametric estimator of its maximum value. The aim of this paper is to study the asymptotic normality result of the two non-parametric estimators.

1. Introduction

Survival analysis is a widely used method in a variety of disciplines to assess the properties of durations between specific events. Important examples of durations are unemployment spells, life times, and durations between subsequent transactions in a financial security. A useful tool in survival analysis is the so-called hazard rate, which reflects the instantaneous probability that a duration will end in the next time instant. An increasing hazard rate indicates that the probability that a spell

1991 *Mathematics Subject Classification.* 62N01, 62N02, 62P10, 62P12.

Key words and phrases. Conditional hazard rate, maximum conditional hazard rate, non-parametric estimation, kernel, right censoring, left truncation, asymptotic normality.

Received: April 24, 2018 Accepted: Aug. 1, 2018 .

will be completed is increasing with the duration of the event; this is called positive duration dependence. Similarly, a decreasing hazard rate reflects negative duration dependence. Parametric, semi-parametric, and non-parametric methods have been proposed to estimate hazard rates. Parametric methods impose an explicit parametric structure on the hazard rate, such as an exponential, Weibull, or lognormal distribution and have different degrees of flexibility with respect to duration dependence. For instance, the exponential distribution has a constant hazard rate, the Weibull hazard is either monotonically increasing or decreasing, and the lognormal hazard rate is non-monotonic. All parametric and semi-parametric estimation techniques impose certain restrictions on the functional form of the hazard rate, which are often too restrictive. Non-parametric methods are more flexible and allow for hazard rate estimation without strong parametric assumptions. Surveys of non-parametric kernel rate estimation are provided by Singpurwalla and Wong[11], as well as Hassani and al.[8].

In practice, the hazard rate will often depend on certain covariates. For instance, the survival time of a patient will be affected by characteristics such as age and gender. A frequently used semi-parametric method to estimate a conditional hazard rate is Coxs proportional hazards model. This model assumes that the conditional hazard rate is a multiplicative function of time (the so-called baseline hazard) and a vector of covariates. An attractive feature of this method is that can be estimated by means of Coxs partial likelihood method without specification of the baseline hazard. However, this semi-parametric method imposes proportionality on the hazard rate. Unfortunately, in many cases the proportional hazards model is too restrictive. Often other semi-parametric models such as the accelerated lifetime model are not flexible enough either. When parametric and semi-parametric models fail, non-parametric hazard rate models are more appropriate.

In this paper we focus on the investigation of the maximum hazard rate with covariate. More precisely, we consider a sample $(T_i)_{i=1,\cdots,n}$ of non-negative variables, and a sample $(C_i)_{i=1,\cdots,n}$ of non-negative censoring times. Then we observe a sample $(Y_i, \delta_i)_{i=1,\cdots,n}$ with: $Y_i = \min(T_i, C_i)$, $\quad \delta_i = 1_{\{T_i \leqslant C_i\}}$, where 1_A denotes the indicator function of the event A. So $\delta_i = 1$ indicates that the ith subject's observed time is not censored.

The hazard conditional rate $\lambda(t|x)$ of T given $X = x$ is defined by

$$\begin{aligned}\lambda(t|x) &= \lim_{\delta t \to 0} \frac{\mathbb{P}[T \leqslant t + \delta t | T \geqslant t, X = x]}{\delta t} \\ &= \frac{f(t|x)}{1 - F(t|x)}, \quad F(t|x) < 1, \ \forall\ (t, x),\end{aligned}$$

where $f(t|x)$ and $F(t|x) = \mathbb{P}[T \leqslant t | X = x]$ denote the density and the unknown continuous distribution function of T. Denote by σ the time in an interval $[a_x, b_x] \in \mathbb{R}^+$ corresponding to the maximum of the conditional hazard rate function, that is,

$$\sigma(x) = \operatorname{Arg\,max}_{t \in [a_x, b_x]} \lambda(t|x). \tag{1.1}$$

We consider lifetime data with covariates which are subject to both left truncation and right censorship. In this context, it is interesting to study the conditional hazard function of the lifetime and its corresponding maximum value. Many biomedical studies are interested in predicting the survival time of a patient for a given vector of covariates of this individual (age, sex, cholesterol, etc.). Frequently, in works where the survival time is the variable of interest, two different problem appear: the first one, when a subject is not included in the study because its lifetime origin precedes the starting time of the study dying before this moment (for instance a short period of illness), these subjects are referred to as left truncated (LT); on the other hand, when a patient is into the study but its lifetime may not be completely observed due to different causes (death for a reason unrelated to the study or change of address), these subjects are called right censored (RC). More specifically, let (Y, T, C) be a random vector, where Y is the lifetime, T is the random left truncation time and C denotes the random right censoring time. In addition Y is assumed to be independent of (T, C). In a random LTRC (left truncation and right censorship) model one observes (Z, T, δ) if $Z \geqslant T$, where $Z = \min(Y, C)$ and $\delta = 1_{\{Y \leqslant C\}}$. When $Z < T$ nothing is observed. Take $\alpha = \mathbb{P}[T \leqslant Z]$, then necessarily, we assume $\alpha > 0$. Finally, we work with non-negative variables as is usual in survival analysis.

Non-parametric estimation of the hazard rate function was first introduce in the statistical literature by Watson and Leadbetter[14] and Watson[15]. The topic was developed by other authors among Singpurwalla and Wong[11], Tanner and Wong[12]. The conditional case was considered later by Van Keilegom and Veraverbeke[13] and by Ferraty et al.[5].

Concerning the maximum hazard rate estimation, Quintela-del-Rìo[10] considered a non-parametric estimator under dependence conditions in uncensored case. Gneyou[7] and Agbokou and al.[1] considered a kernel-type estimator in the model of right censored data with covariate and establish strong uniform consistency results and Gneyou[6] established a basic almost sure asymptotic representation for the maximum value of the hazard rate function estimator which leaded to some main results such as weak convergence and asymptotic normality.

The aim of this paper is to address the asymptotic normality results of the non-parametric estimator of Agbokou et al.[2] as in Gneyou[6] in the case of right-censoring and left truncation data. The paper is organized as follows. In the next section we recall the definitions of the non-parametric estimator of the conditional hazard rate function $\hat{\lambda}_n$ and the corresponding estimator of its maximum value $\hat{\sigma}_n$ and we state the assumptions under which the results will be derived. Section 3 describes the main results and detailed proofs are given in Section 4.

2. Notations, definitions and assumptions

Let $(X;Y;T;C)$ be a random vector, where Y is the lifetime, T is the random left truncation time, C denotes the random right censoring time and X is a covariate related with Y, $Z = \min\{Y;C\}$ and $\delta = 1_{\{Y \leqslant C\}}$. It is assumed that Y and (T,C) are conditionally independent given $X = x$ and $\alpha(x) = P(T \leqslant Z|X = x) > 0$. In this model, one observes $(X;Z;T;\delta)$ if $Z \geqslant T$. When $Z < T$ nothing is observed.

Let $(X_i;Z_i;T_i;\delta_i)$, $i = 1;\cdots;n$, be an i.i.d. random sample from $(X;Z;T;\delta)$ which one observes (then $T_i \leqslant Z_i$, for all i). $F(t|x) = \mathbb{P}(Y \leqslant t|X = x)$ denotes the conditional distribution function of Y given $X = x$.

Let us introduce some notations.

2.1. Notations.

(a) $M(x) = P(X \leqslant x)$, represents the distribution function of the covariate X.

(b) $L(t|x) = P(T \leqslant t|X = x)$, is the conditional distribution function of T given $X = x$.

(c) $H(t|x) = P(Z \leqslant t|X = x)$, is the conditional distribution function of Z given $X = x$.

(d) $H_1(t|x) = P(Z \leqslant t, \delta = 1|X = x)$, is the conditional sub-distribution function of the uncensored observation (when $Z = Y$ and $\delta = 1$) of Z given $X = x$.

(e) $C(t|x) = P(T \leqslant t \leqslant Z|X = x)$.

(f) The conditional cumulative hazard function of Y given $X = x$, is defined by:

$$\Lambda(t|x) = \int_{-\infty}^{t} \frac{dF(s|x)}{1 - F(s|x)}. \tag{2.1}$$

and notice that $\Lambda(t|x)$ uniquely determines the unknown conditional distribution $F(t|x)$.

(g) Recall that $F(t|x) = P(Y \leqslant t|X = x)$, is the conditional distribution function of Y given $X = x$, and

(h) $\alpha(x) = P(T \leqslant Z|X = x)$, is the conditional probability of absence of truncation given $X = x$.

Moreover, for any positive random variable η with distribution function $W(t) = P(\eta \leqslant t)$, we denote the left and right support endpoints by $a_W = \inf\{t : W(t) > 0\}$ and $b_W = \inf\{t : W(t) = 1\}$, respectively. Specifically, we will use the notation: $a_{L(.|x)}$, $a_{H(.|x)}$, $b_{L(.|x)}$ and $b_{H(.|x)}$ for the support endpoints of functions $L(t|x)$ and $H(t|x)$, considering L and H as functions of the variable t for a fixed x value. Finally, we define $W^{\#}(t) = P(\eta \leqslant t|T \leqslant Z)$. So, we set:

(i) $M^{\#}(x) = P(X \leqslant x|T \leqslant Z)$

(j) $H_1^{\#}(t|x) = P(Z \leqslant t, \delta = 1|X = x, T \leqslant Z)$.

2.2. **Definitions.** The conditional cumulative hazard function of Y given $X = x$ is denoted by

$$\Lambda(t|x) = \int_0^t \lambda(s|x)ds. \tag{2.2}$$

Define

$$H_1^{\#}(t|x) \;=\; \mathbb{P}(Z \leqslant t,\, \delta = 1\,|X = x,\, T \leqslant Z),$$

the conditional sub-distribution function of the uncensored observation $(Z, \delta = 1)$. and

$$C(t|x) \;=\; \mathbb{P}(T \leqslant t \leqslant Z|X = x,\, T \leqslant Z).$$

The that T; Y and C are independent conditionally on X, $\Lambda(t|x)$ can be written in the following form

$$\Lambda(t|x) \;=\; \int_{-\infty}^{t} \frac{dH_1^{\#}(s|x)}{C(s|x)} \tag{2.3}$$

Let $(Z_i, T_i, X_i, \delta_i)_{1\leqslant i\leqslant n}$ be a sample of n i.i.d. random variables, K and N be the kernels on $\mathbb{R}$, (h_n) and (a_n), $(n \in \mathbb{N})$ be two sequences of positive non increasing real numbers which will be connected with the smoothing parameters of the estimators. Set for all $x \in \mathbb{R}$, $h > 0$ and $a > 0$, $K_h(x) = \frac{1}{h}K\left(\frac{x}{h}\right)$, and $N_a(x) = \frac{1}{a}N\left(\frac{x}{a}\right)$. By the relation (2.3), we can write $\Lambda(t|x)$ as a function of empirically estimable expressions, we have:

$$\Lambda_n(t|x) = \int_{-\infty}^{t} \frac{dH_{1n}^{\#}(s|x)}{C_n(s|x)}, \text{ for } t \leqslant b_{H(.|x)}$$

where

$$H_{1n}^{\#}(t|x) = \sum_{i=1}^{n} 1_{\{Z_i \leqslant t,\, \delta_i = 1\}} B_i(x, h_n)$$

and

$$C_n(t|x) = \sum_{i=1}^{n} 1_{\{T_i \leqslant t \leqslant Z_i\}} B_i(x, h_n)$$

where for $i = 1, \cdots, n$

$$B_i(x, h_n) = \frac{K_{h_n}(x - X_i)}{\sum_{i=1}^{n} K_{h_n}(x - X_i)}.$$

For simplicity set $B_i(x, h_n) = B_{ni}(x)$. The $B_{ni}(x)$ are the so-called Nadaraya-Watson weights and $H_{1n}(t|x)$ and $C_n(t|x)$ are respectively the kernel estimators of Iglesias-Prez and González-Manteiga[9] of $H_1(t|x)$ and $C(t|x)$, deduced from estimators of Watson[15] and Watson-Leadbetter[14], obtained by regression.

and the following non-parametric estimator of the conditional hazard rate function $\lambda(t|x)$ and its maximum value estimator for right censoring and left truncated data, deduced from (2.2), are defined by

$$\hat{\lambda}_n(t|x) = \sum_{i=1}^{n} \frac{B_{ni}(x)\delta_i N_{a_n}(t - Z_i)}{\sum_{j=1}^{n} 1_{\{T_j \leqslant Z_i \leqslant Z_j\}} B_{nj}(x)}. \tag{2.4}$$

and

$$\hat{\sigma}_n(x) = \operatorname{Arg\,max}_{t \in [a_x, b_x]} \hat{\lambda}_n(t|x). \tag{2.5}$$

The hypotheses which will be needed to prove the results are the same as those Agbokou et al.[2] and Gneyou[6] used to derive theirs results.

2.3. **Assumptions.** Let m denotes the density of X, and $M^{\#}$ the conditional distribution function of X when $T \leqslant Z$, with density $m^{\#}$, then

$$m^{\#}(x) = m(x)i(x)$$

where $i(x) = \dfrac{\alpha(x)}{\alpha}$ is an index of truncation in x with $\alpha = \mathbb{P}(T \leqslant Z)$. We need to consider x values with $i(x) \neq 0$.

A_1 : X, Y, T and C are absolutely continuous random variables and random variables Y, T, C are conditionally independent at $X = x$.

A_2 :

A_2(a). The random variable X takes values in an interval $I = [x_1, x_2]$ contained in the support of $m^{\#}$, such that

$$0 < \gamma = \inf[m^{\#}(x) : x \in I_\varepsilon] < \sup[m^{\#}(x) : x \in I_\varepsilon] = \Gamma < \infty,$$

where $I_\varepsilon = [x_1 - \varepsilon, x_2 + \varepsilon]$ with $\varepsilon > 0$ and $0 < \varepsilon\Gamma < 1$.

A_2(b). Moreover, as regards the random variables Y; T and C, we consider:

(i) $a_{L(.|x)} \leqslant a_{H(.|x)}$, for all $x \in I_\varepsilon$.

(ii) The random variable Y moves in an interval $[a; b]$ such that

$$\inf[\alpha^{-1}(x)(1 - H(b|x))L(a|x) : \, x \in I_\varepsilon] \geqslant \theta > 0.$$

Note that, if $a_{L(.|x)} < y < a_{H(.|x)}$ then $C(t|x) = \alpha^{-1}(x)(1-H(t|x))L(t|x) > 0$, therefore condition (ii) says $C(t|x) \geqslant \theta > 0$ in $[a, b] \times I_\varepsilon$.

A_3 : $a < a_{H(.|x)}$, for all $x \in I_\varepsilon$.

A_4 : The corresponding (improper) densities of the distribution (sub-distributions) functions $L(t)$, $H(t)$ and $H_1(t)$ are bounded away from 0 in $[a, b]$.

F_1 : The first derivatives of functions $m(x)$ and $\alpha(x)$ exist and are continuous in $x \in I_\varepsilon$ and the first derivatives with respect to x of functions $L(t|x)$, $H(t|x)$ et $H_1(t|x)$ exist and are continuous and bounded in $(t, x) \in [0, \infty[\times I_\varepsilon$.

F_2 : The second derivatives of functions $m(x)$ and $\alpha(x)$ exist and are continuous in $x \in I_\varepsilon$ and the second derivatives with respect to x of functions $L(t|x)$, $H(t|x)$ et $H_1(t|x)$ exist and are continuous and bounded in $(t,x) \in [0,\infty[\times I_\varepsilon$.

F_3 : The first derivatives with respect to t of functions $L(t|x)$, $H(t|x)$ and $H_1(t|x)$ exist and are continuous in $(t,x) \in [a,b] \times I_\varepsilon$.

F_4 : The second derivatives with respect to t of functions $L(t|x)$, $H(t|x)$ and $H_1(t|x)$ exist and are continuous in $(t,x) \in [a,b] \times I_\varepsilon$.

F_5 : The second derivatives with respect to x and with respect to t of functions $L(t|x)$, $H(t|x)$ and $H_1(t|x)$ exist and are continuous in $(t,x) \in [a,b] \times I_\varepsilon$.

K_1 : The kernel function K is a symmetrical density vanishing outside $(-1,1)$ and the total variation of K is less than some $\mu < +\infty$. Moreover

(i) $\int_{\mathbb{R}} K(x)dx = 1$,

(ii) $\int_{\mathbb{R}} xK(x)dx = 0$,

(iii) $\int_{\mathbb{R}} x^2K(x)dx = \alpha(K) > 0$,

K_2 : N is a symmetric Kernel of bounded variation on $\mathbb{R}$ vanishing outside the interval $[-M,+M]$ for some $M > 0$ and satisfying

(i) $\int_{\mathbb{R}} N(u)du = 1$,

(ii) $\int_{\mathbb{R}} uN(u)du = 0$,

(iii) $\int_{\mathbb{R}} u^2N(u)du = \alpha(N) > 0$,

(iv) N is twice differentiable, the derivative N' is of bounded variation and satisfies

$\int_{\mathbb{R}} N'^2(u)\mathrm{du} < \infty$.

H_1 : The bandwidth parameter $(h_n)_{n\in\mathbb{N}}$ is a non increasing sequence of positive real numbers such that:

(i) $h_n \longrightarrow 0$,

(ii) $nh_n \longrightarrow \infty$

(iii) $\dfrac{\log n}{nh_n} \longrightarrow 0,$

(iv) $\dfrac{nh^5}{\log n} = O(1)$

H_2 : The bandwidth parameter $(a_n)_{n\in\mathbb{N}}$ is a non increasing sequence of positive real numbers such that:

(i) $a_n \longrightarrow 0$ (therefore $a_n^2 \longrightarrow 0$),

(ii) $\dfrac{\log n}{na_n^\beta h_n} \longrightarrow 0$, for all $\beta \in [1, 8|3]$.

H_3 (i) $na_nh_n \longrightarrow +\infty$, $na_n^3h_n \longrightarrow +\infty$ and $na_nh_n^5 \longrightarrow 0$.

(ii) $\dfrac{\log^2 n}{na_nh_n} \longrightarrow 0$, $\dfrac{\log^3 n}{na_n^2h_n} \longrightarrow 0$ and $na_n^5h_n \longrightarrow 0$.

Remark 1. Based on the above assumptions, it is followed that the first and the second derivatives with respect to t of $\lambda(t|x)$ exist and are continuous in $(t, x) \in [a, b] \times I_\varepsilon$. We denote these derivatives as $\lambda'(t|x)$ and $\lambda''(t|x)$, respectively.

This leads us to the following hypotheses:

F_6 : There exists an interval $[a_x, b_x] \subset [a, b]$, with unique $\sigma = \sigma(x)$ satisfying $\lambda(\sigma|x) = \sup\limits_{a_x \leqslant t \leqslant b_x} \lambda(t|x)$.

F_7 : The function $t \longmapsto \lambda(t|x)$ is of class $\mathcal{C}^2$ with respect (w.r.) to t such that

(i) $\lambda'(\sigma|x) = 0$;

(ii) $d_x = \inf\limits_{a_x \leqslant t \leqslant b_x} |\lambda''(t|x)| > 0$.

The assumptions $A_1 - A_4$, $F_1 - F_5$, K_1 and H_1 are quite standard. $A_1 - A_2(a)$, $F_4 - F_5$, K_1 and H_1 insure the strong uniform convergence of the estimators $H_{1n}(t|x)$ and $C_n(t|x)$ to $H_1(t|x)$ and $C(t|x)$ respectively as in Iglesias-Prez and González-Manteiga[9] while K_2 and H_2 ensure the almost sure representation and the strong uniform consistency of $\hat{\lambda}_n(t|x)$ to $\lambda(t|x)$, when $F_6 - F_7$ make sure of the strong uniform convergence of $\hat{\sigma}_n(x)$ to $\sigma(x)$. The hypotheses H_3 and K ensure the asymptotic normality of the both estimators $\hat{\lambda}_n(t|x)$ and $\hat{\sigma}_n(x)$.

3. Main results

Agbokou et al.[2] proved the strong uniform convergence of the conditional hazard rate function and its maximum location estimators $\hat{\lambda}_n(t|x)$ and $\hat{\sigma}_n(x)$. This lead us to the investigation on the asymptotic normality results. We need to consider the process

$$\xi(Z,T,\delta,t,x)=\frac{1_{\{Z\leqslant t,\,\delta=1\}}}{C(Z|x)}-\int_0^t\frac{1_{\{T\leqslant u\leqslant Z\}}}{C^2(u|x)}dH_1^{\#}(u|x). \tag{3.1}$$

$\xi(Z,T,\delta,t,x)$ is a centred random process which play a major role in our investigation. The following theorem offers the asymptotic normality of the estimator $\hat{\lambda}_n(t|x)$. After we enforce it to pull the asymptotic normality of $\hat{\sigma}_n(x)$.

Theorem 3.1. *Assume that the assumptions* A_1-A_4, F_1-F_5, K_1-K_2 *and* H_1-H_3 *hold. Then for all* $x\in I$ *and* $t\in[a_x,b_x]$, *we have:*

$$\sqrt{na_nh_n}[\hat{\lambda}_n(t|x)-\lambda(t|x)]\xrightarrow{D}\mathcal{N}(0,s^2(t|x)) \tag{3.2}$$

with

$$s^2(t|x)=\frac{\lambda(t|x)\left(\int_{\mathbb{R}}K^2(z)dz\right)\left(\int_{\mathbb{R}}N^2(v)dv\right)}{C(t|x)m^{\#}(x)}. \tag{3.3}$$

The proofs of the Theorem 3.1 and its corollaries below are given in the next section. As a consequence of Theorem 3.1, we get the following asymptotic normality result for the estimator $\hat{\sigma}_n$.

Corollary 3.1. *Under the assumptions of Theorem* 3.1, *we assume that the hypotheses* F_6-F_7 *are hold, for all* $x\in I$, *we have*

$$\sqrt{na_n^3h_n}[\hat{\sigma}_n(x)-\sigma(x)]\xrightarrow{D}\mathcal{N}(0,s^2(\sigma|x)) \tag{3.4}$$

with

$$s^2(\sigma|x) = \frac{\lambda(\sigma|x)\left(\int_{\mathbb{R}} K^2(z)dz\right)\left(\int_{\mathbb{R}} N'^2(v)dv\right)}{\lambda''^2(\sigma|x)C(\sigma|x)m^{\#}(x)} \tag{3.5}$$

4. Proofs

The asymptotic normality of the conditional hazard rate estimator given in Theorem 3.1 is based on the following lemmas:

4.1. **Proofs of the lemmas.**

Lemma 4.1. *Define for all $x \in I$ and $t \in [a_x, b_x]$,*

$$\chi_i(t|x) = \frac{1}{a_n}\int_{\mathbb{R}} \xi_i(t - a_n u|x)dN(u),$$

then

$$\mathbb{E}[\chi_i(t|x)|X = x] = 0 \tag{4.1}$$

and if the assumptions K_2 and $H_1 - i$ are satisfied then

$$\mathbb{V}ar[\chi_i(t|x)|X = x] = \frac{1}{a_n}\left[\lambda^{\#}(t|x)\int_{\mathbb{R}} N^2(u)du\right] + o(1) \tag{4.2}$$

where

$$\lambda^{\#}(t|x) = \frac{\lambda(t|x)}{C(t|x)}$$

Proof of Lemma 4.1 By Fubini Theorem, it is easily seen that

$$\mathbb{E}[\chi_i(t|x)|X = y] = \frac{1}{a_n}\int_{\mathbb{R}} \mathbb{E}[\xi_i(t - a_n u|x)|T \leqslant Z, X = y]dN(u),$$

with

$$\mathbb{E}[\xi_i(t - a_n u|x)|T \leqslant Z, X = y] = \int_0^{t-a_n u} \frac{dH_1^{\#}(s|y)}{C(s|x)} - \int_0^{t-a_n u} \frac{C(s|y)}{C^2(s|x)}dH_1^{\#}(s|x),$$

so we have:

$$\begin{aligned}\mathbb{E}[\xi_i(t-a_nu|x)|T \leqslant Z, X=x] &= \int_0^{t-a_nu} \frac{dH_1^{\#}(s|x)}{C(s|x)} - \int_0^{t-a_nu} \frac{C(s|x)}{C^2(s|x)} dH_1^{\#}(s|x) \\ &= 0\end{aligned}$$

hence

$$\mathbb{E}[\chi_i(t|x)|X=x] = 0$$

Thus the first part (4.1) of the lemma is proved. For the second part (4.2), we have as well by Fubini Theorem for all $t, t' \in [a_x, b_x]$ and $x \in I$

$$\mathbb{C}ov[\chi_i(t|x), \chi_i(t'|x)] = \frac{1}{a_n^2} \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbb{E}[\xi_i(t-a_nu|x)\xi_i(t'-a_nv|x)|T \leqslant Z, X=x] dN(u)dN(v),$$

so

$$\begin{aligned}\mathbb{V}ar[\chi_i(t|x)|X=x] &= \mathbb{C}ov[\chi_i(t|x), \chi_i(t'|x)]|_{t'=t}, \\ &= \frac{1}{a_n^2} \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbb{E}[\xi_i(t-a_nu|x)\xi_i(t-a_nv|x)|T \leqslant Z, X=x] dN(u)dN(v),\end{aligned}$$

with

$$\begin{aligned}\xi_i(t-a_nu|x)\xi_i(t-a_nv|x) &= \left[\frac{1_{\{Z \leqslant t-a_nu\}}}{C(Z|x)} - \int_0^{t-a_nu} \frac{1_{\{T \leqslant s \leqslant Z\}}}{C^2(s|x)} dH_1^{\#}(s|x)\right] \\ &\quad \times \left[\frac{1_{\{Z \leqslant t-a_nv\}}}{C(Z|x)} - \int_0^{t-a_nv} \frac{1_{\{T \leqslant s \leqslant Z\}}}{C^2(s|x)} dH_1^{\#}(s|x)\right], \\ &= \underbrace{\frac{1_{\{Z \leqslant (t-a_nu)\wedge(t-a_nv)\}}}{C^2(Z|x)}}_{A(x)} \\ &\quad + \underbrace{\left(\int_0^{t-a_nu} \frac{1_{\{T \leqslant s \leqslant Z\}}}{C^2(s|x)} dH_1^{\#}(s|x)\right)\left(\int_0^{t-a_nv} \frac{1_{\{T \leqslant s \leqslant Z\}}}{C^2(s|x)} dH_1^{\#}(s|x)\right)}_{B(x)}\end{aligned}$$

$$- \underbrace{\left(\frac{1_{\{Z\leqslant t-a_nu\}}}{C(Z|x)}\right)\left(\int_0^{t-a_nv}\frac{1_{\{T\leqslant s\leqslant Z\}}}{C^2(s|x)}dH_1^{\#}(s|x)\right)}_{C(x)}$$
$$- \underbrace{\left(\frac{1_{\{Z\leqslant t-a_nv\}}}{C(Z|x)}\right)\left(\int_0^{t-a_nu}\frac{1_{\{T\leqslant s\leqslant Z\}}}{C^2(s|x)}dH_1^{\#}(s|x)\right)}_{D(x)}.$$

Moreover, by Fubini theorem and by straightforward calculations we check that

$$\mathbb{E}[B(x) - C(x) - D(x)|T \leqslant Z, X = x] = 0.$$

Thus

$$\begin{aligned}
\mathbb{E}[\xi_i(t-a_nu|x)\xi_i(t-a_nv|x)|T \leqslant Z, X = x] &= \mathbb{E}[A(x)|T \leqslant Z, X = x] \\
&= \mathbb{E}\left[\frac{1_{\{Z\leqslant(t-a_nu)\wedge(t-a_nv)\}}}{C^2(Z|x)}|T \leqslant Z, X = x\right], \\
&= \int_0^{(t-a_nu)\wedge(t-a_nv)}\frac{dH_1^{\#}(s|x)}{C^2(s|x)}, \\
&= \int_0^{(t-a_nu)\wedge(t-a_nv)}\frac{\lambda(s|x)}{C(s|x)}ds.
\end{aligned}$$

Let us,

$$\begin{aligned}
\Lambda^{\#}(t|x) &= \int_0^t \frac{\lambda(s|x)}{C(s|x)}ds, \\
&= \int_0^t \lambda^{\#}(s|x)ds,
\end{aligned}$$

so finally we have

$$\mathbb{E}[\xi_i(t-a_nu|x)\xi_i(t-a_nv|x)|T \leqslant Z, X = x] = \Lambda^{\#}((t-a_nu)\wedge(t-a_nv)|x).$$

Thus by integrating by parts under the assumptions K_2, we can write

$$\begin{aligned}
\mathbb{V}ar[\chi_i(t|x)|X = x] &= \frac{1}{a_n^2}\int_{\mathbb{R}}\int_{\mathbb{R}}\Lambda^{\#}((t-a_nu)\wedge(t-a_nv)|x)dN(u)dN(v), \\
&= \frac{1}{a_n^2}\int_{\mathbb{R}}\int_{u>v}\Lambda^{\#}(t-a_nu|x)dN(u)dN(v)
\end{aligned}$$

$$
\begin{aligned}
& +\frac{1}{a_n^2}\int_{\mathbb{R}}\int_{u<v}\Lambda^{\#}(t-a_nv|x)dN(u)dN(v), \\
= & \frac{1}{a_n}\int_{\mathbb{R}}\int_{v}^{M}\lambda^{\#}(t-a_nu|x)N(u)dudN(v), \\
= & \frac{1}{a_n}\int_{\mathbb{R}}\lambda^{\#}(t-a_nv|x)N^2(v)dv,
\end{aligned}
$$

where M is the upper boundary of the support of the kenel N. A Taylor's expansion of the function $\lambda^{\#}(t-a_nv|x)$ in order one at a neighbourhood of t yields

$$
\mathbb{V}ar[\chi_i(t|x)|X=x] = \frac{1}{a_n}\lambda^{\#}(t|x)\int_{\mathbb{R}}N^2(v)dv + o(1)
$$

which ends the proof of the Lemma 4.1.

Lemma 4.2. *Under the assumptions of the Theorem* 3.1*, then for all* $x \in I$ *and* $t \in [a_x, b_x]$*, we have*

$$
\hat{\lambda}'_n(t|x) - \lambda'(t|x) = -\sum_{i=1}^{n} B_{ni}(x)\zeta_i(t|x) + \Upsilon_n(t|x) + \mathcal{O}\left(a_n^2\right) \ a.s.,
$$

with

$$
\zeta_i(t|x) = \zeta(T_i, Z_i, \delta_i, t, x) = \frac{1}{a_n^2}\int_{\mathbb{R}}\xi_i(t-a_nu|x)N'(u)du,
$$

and

$$
\sup_{t\in[a_x,b_x]}|\Upsilon_n(t|x)| \longrightarrow 0 \ a.s. \ quand\, n \longrightarrow +\infty.
$$

Proof

The Theorem 4.1 of Agbokou and al.[2] allow us to write

$$
\hat{\lambda}_n(t|x) - \lambda(t|x) = \frac{1}{a_n}\int_{\mathbb{R}}N\left(\frac{t-s}{a_n}\right)d(\Lambda_n(s|x)-\Lambda(s|x)) + \mathcal{O}\left(a_n^2\right).
$$

Using the theorem of differentiability (under the integral sign) and an integration by parts, we have

$$\begin{aligned}
\hat{\lambda}'_n(t|x) - \lambda'(t|x) &= +\frac{1}{a_n^2}\int_{\mathbb{R}} N'\left(\frac{t-s}{a_n}\right) d(\Lambda_n(s|x) - \Lambda(s|x)) + \mathcal{O}\left(a_n^2\right) \\
&= -\frac{1}{a_n^2}\int_{\mathbb{R}} (\Lambda_n(t-a_nu|x) - \Lambda(t-a_nu|x))dN'(u) + \mathcal{O}\left(a_n^2\right), \\
&= -\frac{1}{a_n^2}\int_{\mathbb{R}} A_n(t-a_nu|x)dN'(u) \\
&\quad -\frac{1}{a_n^2}\int_{\mathbb{R}} R_n(t-a_nu|x)dN'(u) + \mathcal{O}\left(a_n^2\right),
\end{aligned}$$

where

$$A_n(t-a_nu|x) = \sum_{i=1}^{n} B_{ni}(x)\xi_i(t-a_nu|x).$$

By integrating each member, we obtain

$$\begin{aligned}
-\frac{1}{a_n^2}\int_{\mathbb{R}} A_n(t-a_nu|x)dN'(u) &= -\frac{1}{a_n^2}\int_{\mathbb{R}} \sum_{i=1}^{n} B_{ni}(x)\xi_i(t-a_nu|x)dN'(u) \\
&= -\sum_{i=1}^{n} B_{ni}(x)\zeta_i(t|x).
\end{aligned}$$

We have

$$\begin{aligned}
\Upsilon_n(t|x) &= -\frac{1}{a_n^2}\int_{\mathbb{R}} R_n(t-a_nu|x)dN'(u), \\
&= -\frac{1}{a_n^2}\int_{\mathbb{R}} R_{n1}(t-a_nu|x)dN'(u) - \frac{1}{a_n^2}\int_{\mathbb{R}} R_{n2}(t-a_nu|x)dN'(u), \\
&= \Omega_{n1}(t|x) + \Omega_{n2}(t|x),
\end{aligned}$$

with

$$\Omega_{n1}(t|x) = -\frac{1}{a_n^2}\int_{\mathbb{R}}\int_0^{t-a_nu} \frac{(C(s|x) - C_n(s|x))^2}{C_n(s|x)C^2(s|x)} dH_1^{\#}(s|x)dN'(u),$$

and

$$\Omega_{n2}(t|x) = -\frac{1}{a_n^2}\int_{\mathbb{R}}\int_0^{t-a_nu} \left(\frac{1}{C_n(s|x)} - \frac{1}{C(s|x)}\right) d(H_1^{\#}(s|x) - H_1^{\#}(s|x))dN'(u).$$

We prove, on the basis of the proof of the Theorem 4.1 of Agbokou et al.[2], that

$$\sup_{t\in[a_x,b_x]} |\Omega_{n1}(t|x)| \longrightarrow 0 \text{ p.s. quand } n \longrightarrow +\infty,$$

and

$$\sup_{t\in[a_x,b_x]} |\Omega_{n2}(t|x)| \longrightarrow 0 \text{ p.s. quand } n \longrightarrow +\infty.$$

Hence,

$$\sup_{t\in[a_x,b_x]} |\Upsilon_n(t|x)| \leqslant \sup_{t\in[a_x,b_x]} |\Omega_{n1}(t|x)| + \sup_{t\in[a_x,b_x]} |\Omega_{n2}(t|x)| \longrightarrow 0 \text{ p.s. quand } n \longrightarrow +\infty.$$

This completes the proof of this lemma.

4.2. **Proofs of the main results.**

Proof of the Theorem 3.1

According to the Theorem 4.1 of Agbokou et al.[2], we have

$$\hat{\lambda}_n(t|x) - \lambda(t|x) = \sum_{i=1}^{n} B_{ni}(x)\chi_i(t|x) + r_n(t|x),$$

this allow us to write

$$\sqrt{na_nh_n}\sum_{i=1}^{n} B_{ni}(x)\chi_i(t|x) = \frac{\sqrt{na_nh_n}\sum_{i=1}^{n}\frac{1}{nh_n}K\left(\frac{x-X_i}{h}\right)\chi_i(t|x)}{m_n^{\#}(x)},$$

where $m_n^{\#}(x)$ is the Parzen-Rosenblatt estimator of the conditional density of X when $T \leqslant Z$. Using that $m_n^{\#}(x)$ converges in probability to $m^{\#}(x)$ together with Theorem 5.1 in Billingsley[3], we have only to study the limiting distribution of

$$\begin{aligned}\sqrt{na_nh_n}\sum_{i=1}^{n}\frac{1}{nh_n}K\left(\frac{x-X}{h}\right)\chi_i(t|x) &= \sqrt{na_nh_n}\sum_{i=1}^{n}\frac{1}{nh_n}\\ &\times\left\{K\left(\frac{x-X_i}{h}\right)\chi_i(t|x) - \mathbb{E}\left[K\left(\frac{x-X_i}{h}\right)\chi_i(t|x)\right]\right\}\\ &+\sqrt{na_nh_n}\sum_{i=1}^{n}\frac{1}{nh_n}\mathbb{E}\left[K\left(\frac{x-X_i}{h}\right)\chi_i(t|x)\right],\\ &= J_{1n}(t|x) + J_{2n}(t|x),\end{aligned}$$

where

$$\begin{aligned} J_{2n}(t|x) &= \sqrt{na_nh_n}\sum_{i=1}^{n}\frac{1}{nh_n}\mathbb{E}\left[K\left(\frac{x-X_i}{h}\right)\chi_i(t|x)\right], \\ &= \sqrt{na_nh_n}\mathbb{E}\left[\frac{1}{h_n}K\left(\frac{x-X_i}{h}\right)\chi_i(t|x)\right], \end{aligned}$$

$$= \sqrt{na_nh_n}\mathbb{E}\left\{\frac{1}{h_n}K\left(\frac{x-X_i}{h}\right)\mathbb{E}[\chi_i(t|x)|T\leqslant Z, X=x]\right\},$$

with

$$\begin{aligned} &\mathbb{E}\left\{\frac{1}{h_n}K\left(\frac{x-X}{h}\right)\mathbb{E}[\chi_i(t|x)|T\leqslant Z, X=x]\right\} \\ &\qquad = \int_{\mathbb{R}}\frac{1}{h_n}K\left(\frac{x-u}{h}\right)\mathbb{E}[\chi_i(t|x)|T\leqslant Z, X=u]m^{\#}(u)du, \\ &\qquad = \int_{\mathbb{R}}\frac{1}{h_n}K\left(\frac{x-u}{h}\right)\beta(u)m^{\#}(u)du, \\ &\qquad = \int_{\mathbb{R}}K(v)\beta(x-vh_n)m^{\#}(u)du, \end{aligned}$$

where

$$\beta(u) = \mathbb{E}[\chi_i(t|x)|T\leqslant Z, X=u].$$

Under the assumption K_1, developing the function $\beta(x-vh_n)$ by Taylor's theorem in order two at a neighbourhood of x yields,

$$\begin{aligned} &\mathbb{E}\left\{\frac{1}{h_n}K\left(\frac{x-X_i}{h}\right)\mathbb{E}[\chi_i(t|x)|T\leqslant Z, X=x]\right\} \\ &\qquad = \beta(x)m^{\#}(x) + \frac{h_n^2}{2}\left(\int_{\mathbb{R}}v^2K(v)\right)\left[\beta(x)m^{\#"}(x)+2\beta'(x)m^{\#'}(x)+\beta"(x)m^{\#}(x)\right] \\ &\qquad\quad + o(h_n^2). \end{aligned}$$

The hypotheses $F_1 - F_5$ and K_2v prove that the function β has bounded first and second derivatives, because it can be write as a function of $H_1(t|x)$ and $C(t|x)$ which have bounded first and second derivatives functions. Moreover, under the first part of the Lemma 4.1, we get $\beta(x) = 0$, so have under the assumption $H_2 iii$

$$\begin{aligned} J_{2n}(t|x) &= \sqrt{na_n h_n}\frac{h_n^2}{2}\alpha(K)\left[2\beta'(x)m^{\#'}(x) + \beta''(x)m^{\#}(x)\right] + o(h_n^2), \\ &= \mathcal{O}\left((na_n h_n)^{1|2}h^2\right) = \mathcal{O}\left((na_n h_n^5)^{1|2}\right), \\ &= o(1). \end{aligned}$$

Thus the asymptotic normality distribution of $J_{1n}(t|x)$ and $J_{2n}(t|x)$ is the same as that $J_{1n}(t|x)$. We have now

$$\begin{aligned} J_{1n}(t|x) &= \sqrt{na_n h_n}\sum_{i=1}^{n}\frac{1}{nh_n}\left\{K\left(\frac{x-X_i}{h}\right)\chi_i(t|x) - \mathbb{E}\left[K\left(\frac{x-X_i}{h}\right)\chi_i(t|x)\right]\right\}, \\ &= \sum_{i=1}^{n}\sqrt{\frac{a_n}{nh_n}}\left\{K\left(\frac{x-X_i}{h}\right)\chi_i(t|x) - \mathbb{E}\left[K\left(\frac{x-X_i}{h}\right)\chi_i(t|x)\right]\right\}, \\ &= \sum_{i=1}^{n}\eta_{i,n}(t|x), \end{aligned}$$

with

$$\eta_{i,n}(t|x) = \sqrt{\frac{a_n}{nh_n}}\left\{K\left(\frac{x-X_i}{h}\right)\chi_i(t|x) - \mathbb{E}\left[K\left(\frac{x-X_i}{h}\right)\chi_i(t|x)\right]\right\},$$

where for each $i = 1, \cdots, n$, $\eta_{i,n}(t|x)$ are n independent random variable with mean equal to 0. Now we will show that the random variable $\{\chi_i(t|x)\}_{i=1}^{n}$ satisfies Lindenberg's Central Limit Theorem, which states that:

suppose that $X_1, \cdots, X_n$ *are independent random variables such that* $\mathbb{E}(X_i) = \mu_i$ *and* $Var(X_i) = \sigma_i^2 < \infty$ *for all* $i = 1, \cdots, n$. *Define*

$$Y_i = X_i - \mu_i, \qquad T_n = \sum_{i=1}^{n} Y_i, \qquad s_n^2 = Var(T_n) = \sum_{i=1}^{n}\sigma_i^2.$$

The sufficient condition that ensure $\frac{T_n}{s_n} \xrightarrow{d} \mathcal{N}(0,1)$ *according to Lindenberg is:*

$$\text{for every}\, \epsilon > 0,\ \frac{1}{s_n^2}\sum_{i=1}^{n}\mathbb{E}\left[Y_i^2 1_{\{|Y_i| \geqslant \epsilon s_n\}}\right] \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

Now we will see that the variance is finite. Proceeding as in (4.2), we obtain that

$$\mathbb{V}ar[\eta_{i,n}(t|x)] = \frac{a_n}{n}\mathbb{V}ar\left[\frac{1}{\sqrt{h_n}}K\left(\frac{x-X_i}{h}\right)\chi_i(t|x)\right]$$

$$= \frac{a_n}{n}\left(\int_{\mathbb{R}} K^2(z)dz\right)\mathbb{V}ar[\chi_i(t|x)|T \leqslant Z, X = x]m^{\#}(x) + \mathcal{O}\left(h_n^2\right). \tag{4.3}$$

With the part two of the Lemma 4.1, the result (4.3) lead to

$$\begin{aligned}
\sigma_n^2(t|x) &= \sum_{i=1}^{n}\mathbb{V}ar[\eta_{i,n}(t|x)],\\
&= \left(\int_{\mathbb{R}} K^2(z)dz\right)\left[\lambda^{\#}(t|x)\int_{\mathbb{R}} N^2(u)du\right]m^{\#}(x) + o(1),\\
&= \sigma^2(t|x) + o(1) \quad < \infty,
\end{aligned}$$

this prove the finiteness of the variance $\sigma_n^2(t|x)$. So we go now to satisfy the Lindeberg's theorem. Let us for all $\epsilon > 0$,

$$\begin{aligned}
\delta_{i,n} &= 1_{\{|\eta_{i,n}(t|x)|>\epsilon\sigma_n\}} = 1_{\{\eta_{i,n}^2(t|x)>\epsilon^2\sigma_n^2\}}\\
&= 1_{\left\{\frac{a_n}{nh_n}\left[K\left(\frac{x-X_i}{h}\right)\chi_i(t|x) - \mathbb{E}\left[K\left(\frac{x-X_i}{h}\right)\chi_i(t|x)\right]\right]^2 > \epsilon^2\sigma_n^2\right\}},\\
&= 1_{\left\{\frac{1}{na_nh_n}\left[K\left(\frac{x-X_i}{h}\right)\int_{\mathbb{R}}\xi_i(t-a_nu|x)dN(u) - W\right]^2 > \epsilon^2\sigma_n^2\right\}},
\end{aligned}$$

where

$$W = \mathbb{E}\left(K\left(\frac{x-X_i}{h}\right)\int_{\mathbb{R}}\xi_i(t-a_nu|x)dN(u)\right).$$

Under the assumption H_3, $(na_nh_n)^{-1} \longrightarrow 0$, moreover the functions K, N and ξ are bounded under assumptions A_2, F_3 and $K_1 - K_2$, then we obtain $\lim_{n\longrightarrow+\infty}\delta_{i,n} = 0$.

The fact that $\sigma_n^2(x) < \infty$, this last equality implies

$$\lim_{n\longrightarrow\infty} \frac{1}{\sigma_n^2(x)} \mathbb{E}\left[\sum_{i=1}^{n} \eta_{i,n}^2(t|x)\delta_{i,n}\right] = \lim_{n\longrightarrow\infty} \frac{1}{\sigma_n^2(x)} \sum_{i=1}^{n} \mathbb{E}\left[\eta_{i,n}^2(t|x)\delta_{i,n}\right] = 0.$$

Finally, we have the Lindeberg's condition which is satisfied. This ends the proof of the Theorem 3.1.

Proof of the Corollary 3.1

Note that by definition of σ et de $\hat{\sigma}_n$, we have $\lambda'(\sigma|x) = \hat{\lambda}'_n(\hat{\sigma}_n|x) = 0$, $\lambda''(\sigma|x) < 0$ and $\hat{\lambda}''_n(\hat{\sigma}_n|x) < 0$. A Taylor's expansion of $\hat{\lambda}'_n(.|x)$ in the neighbourhood of σ, gives

$$\hat{\lambda}'_n(\hat{\sigma}_n|x) - \hat{\lambda}'_n(\sigma|x) = (\hat{\sigma}_n - \sigma)\hat{\lambda}''_n(\sigma^{\#}|x),$$

where $\sigma_n^{\#}$ is between σ and $\hat{\sigma}_n$. This means that

$$\begin{aligned} \hat{\sigma}_n - \sigma &= -\frac{\hat{\lambda}'_n(\sigma|x)}{\hat{\lambda}''_n(\sigma_n^{\#}|x)}, \\ &= -\frac{1}{\hat{\lambda}''_n(\sigma_n^{\#}|x)}(\hat{\lambda}'_n(\sigma|x) - 0), \\ &= -\frac{1}{\hat{\lambda}''_n(\sigma_n^{\#}|x)}(\hat{\lambda}'_n(\sigma|x) - \lambda'(\sigma|x)), \end{aligned}$$

with the Lemma 4.2, we have finally

$$\hat{\sigma}_n - \sigma = \frac{1}{\hat{\lambda}''_n(\sigma_n^{\#}|x)} \sum_{i=1}^{n} B_{ni}(x)\zeta_i(\sigma|x) + \tilde{\Upsilon}_n(\sigma|x) + \mathcal{O}\left(a_n^2\right), \tag{4.4}$$

where

$$\tilde{\Upsilon}_n(t|x) = -\frac{1}{\hat{\lambda}''_n(\sigma^{\#}|x)}\Upsilon_n(t|x),$$

with

$$\zeta_i(t|x) = \frac{1}{a_n^2}\int_{\mathbb{R}} \xi(t - a_n u|x)(t|x)dN'(u).$$

We have

$$\frac{\sqrt{na_n^3h_n}}{\hat{\lambda}_n(\sigma_n^{\#}|x)}\sum_{i=1}^{n}B_{ni}(x)\zeta_i(\sigma|x) = \frac{\sqrt{na_n^3h_n}\displaystyle\sum_{i=1}^{n}\frac{1}{nh}K\left(\frac{x-X_i}{h_n}\right)\zeta_i(t|x)}{\hat{\lambda''}_n(\sigma_n^{\#}|x)m_n^{\#}(x)}. \tag{4.5}$$

Arguing as in the proof of the Lemma 4.1, we get

$$\mathbb{E}[\zeta_i(t|x)|T \leqslant Z, X = x] = 0, \tag{4.6}$$

and

$$\mathbb{V}ar[\zeta_i(t|x)|T \leqslant Z, X = x] = \frac{1}{a_n^3}\left[\left(\lambda^{\#}(t|x)\int_{\mathbb{R}}N'^2(u)du\right) + o(1)\right]. \tag{4.7}$$

By imitating the proof of Theorem 3.1, we arrive at the following result:

$$\begin{aligned} s_n^2(\sigma|x) &= \left(\int_{\mathbb{R}}K^2(z)dz\right)\left[\lambda^{\#}(\sigma|x)\int_{\mathbb{R}}N'^2(u)du\right]m^{\#}(x) + o(1), \\ &= s_0^2(\sigma|x) + o(1) \quad < \infty. \end{aligned}$$

Taking into account that $\hat{\lambda''}_n(\sigma_n^{\#}|x) \longrightarrow \lambda''(\sigma|x)$ when $n \to +\infty$, we deduce that the denominator of (4.5) tends to $\lambda''(\sigma|x)m^{\#}(x)$ when $n \to +\infty$. Hence we obtain the asymptotic normality of $\hat{\sigma}_n$ from the Lindeberg's theorem given by

$$\sqrt{na_n^3h_n}[\hat{\sigma}_n(x) - \sigma(x)] \xrightarrow{D} \mathcal{N}(0, s^2(\sigma|x)),$$

with

$$s^2(\sigma|x) = \frac{\lambda(\sigma|x)\left(\int_{\mathbb{R}}K^2(z)dz\right)\left(\int_{\mathbb{R}}N'^2(v)dv\right)}{\lambda''^2(\sigma|x)C(\sigma|x)m^{\#}(x)}.$$

Acknowledgement

We would like to thank the editor and the referees

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