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ON VARIETAL FUZZY SUBGROUPS

A. JAVADI [(1)] AND A. GHOLAMI [(2)]

Abstract. Varieties of groups and fuzzy subgroups are two important concepts in mathematics. In this paper, after presenting concepts of verbal and marginal fuzzy subgroups and discussing some of their most important characteristics of these concepts, we define variety of fuzzy subgroups and study the complete structure of this. At the end , we devote isologism of fuzzy subgroups.

1. Introduction

As in [2] let $F(X)$ be free group over X. Let $T = \{i \in [0,1]|\ i \in I\}$ and let $t \in [0,1]$ be such that $t \geq \vee\{s|\ s \in T\}$. The fuzzy subset $f(X;T,t)$ of $F(X)$ is called fuzzy subgroup of $F(X)$, where for all $y \in F(X)$,

$$f(X;T,t)(y) = \vee\{\wedge\{t \wedge t_i|\ i \in I(w)\}|\ w \in y\}.$$

We call $(F(X), f(X;T,t))$ the free fuzzy subgroup. It is easy to see that every subgroup is homomorphic image of a free fuzzy subgroup. (See also [2])

A fuzzy subset of a set X is a mapping $\mu : X \to [0,1]$ and the fuzzy power set of X is denoted by $FP(X)$.

Fuzzy subset μ of a group G is called a fuzzy subgroup if

(a) $\mu(x,y) \geqslant \mu(x) \wedge \mu(y)\ \forall x,y \in G$

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(b) $\mu(x^{-1}) \geq \mu(x)\ \forall x \in G$

The set of all fuzzy subgroups of G denoted by $F(G)$.

Let $\mu \in F(G)$, then we set

$$\mu_* = \{x \in G|\ \mu(X) = \mu(e)\},$$

$$\mu^* = \{x \in G|\ \mu(X) \geq 0\}$$

where μ^* and μ_* are subgroups of G, μ^* is called the support of μ and (G, μ) is called an Abelian fuzzy subgroup of $(G, F(G))$.

Let $\mu \in FP(G)$ and

$$Z(\mu) = \{x \in G|\ \mu(xy) = \mu(yx)\ and\ \mu(xyz) = \mu(yxz)\ \forall y, z \in G\}.$$

μ is called commutative in G if $Z(\mu) = G$. It is easy to see that

$$Z(\mu) = \{x \in G|\ \mu[x, y] = \mu(e)\ \forall y \in G\}.$$

The following are some of the properties used in the paper.

Lemma 1.1. ([2, Lemma 1.2.5]) *Let $\mu \in F(G)$ then for all $x \in G$ we have,*

(i) $\mu(e) \geq \mu(x)$

(ii) $\mu(x) = \mu(x^{-1})$

(iii) If for all $x, y \in G$, $\mu(x) \neq \mu(y)$ *then* $\mu(xy) = \mu(x) \wedge \mu(y)$

Definition 1.1. Let $\mu \in F(G)$and for all $x, y \in G$: $\mu^z(x) = \mu(z^{-1}xz)$. If for all $x, y \in G$, $\mu^z(x) = \mu(x)$ then μ is called normal fuzzy subgroup of $(G, F(G))$, and denote by $\mu \lhd^F F$(G).

The set of all normal fuzzy subgroup of $(G, F(G))$ denoted by $NF(G)$.

Let $\mu \in F\ (G)$ and θ be a function from G to G then for all $x \in G$, $\mu^\theta(x) = \mu(\theta(x)) = \mu(x^\theta)$ is a fuzzy subset of G.

Theorem 1.1. ([2, Theorem 2.2.3]) *Let $\mu \in F(G)$, then the followings holds,*
(i) If θ is a homomorphism of G into itself, then $\mu^\theta \in F(G)$.
(ii) If θ is a automorphism of G, then $\mu^\theta \in NF(G)$.

A fuzzy subgroup of G is called normal (characteristic, fully invariant) if for all $\theta \in Inn(G)$ $(\theta \in Aut(G), \theta \in End(G))$, $\mu^\theta = \mu$ and is denoted by

$$\mu \lhd^F F(G)(\mu \lhd^{F.c} F(G), \mu \lhd^{F.f.in} F(G)).$$

If $\mu \in F(G)$ and $H \leq G$ then $\mu|_H$ is fuzzy subgroup of H.

Let $\mu \in NF$(G) then we define the set $G/\mu = \{x\mu |\ x \in G\}$ where,

$$(x\mu)(t) = (\mu(e)_{\{x\}} \circ \mu)(t) = \vee\{\mu(e)_{\{x\}}) \wedge \mu(b)|\ t = ab\} \qquad (1)$$

On the other hand, we know

$$(\mu(e)_{\{x\}})(a) = \begin{cases} \mu(e) & a \in \{x\} \\ 0 & a \in G \setminus \{x\} \end{cases} \qquad (2)$$

Now by (1) and (2):
if $a \in G \setminus \{x\}$ then $\mu(e)_{\{x\}} \wedge \mu(b) = 0$ and,
if $a \in \{x\}$ then $b = x^{-1}t$ and $\mu(e)_{\{x\}} \wedge \mu(b) = \mu(b)$
therefore

$$(x\mu)(t) = \begin{cases} \mu(x^{-1}t) & \exists b \in G :\ t = xb \\ 0 & \text{otherwise} \end{cases}$$

similarly

$$(\mu x)(t) = \begin{cases} \mu(tx^{-1}) & \exists b \in G :\ t = bx \\ 0 & \text{otherwise} \end{cases}$$

If $N \lhd G$ and $\mu \in F(G)$, then for all $x \in G$,

$$\mu(xN) = \vee\{\mu(z)|\ z \in xN\}$$

2. Verbal and Marginal Fuzzy subgroups

Let (G, μ) be a fuzzy subgroup of $(G, F(\mathrm{G}))$ and let $(F(X), f(X; T, t))$ be homomorphic image of (G, μ) and $X = \{x_1, x_2, \cdots\}$ be a countable set and let V be a non-empty subset of $F(X)$. If $v = x_{i_1}^{l_1} x_{i_2}^{l_2} \cdots x_{i_r}^{l_r} \in V$ and $g_1, g_2, \cdots, g_r$ are elements of the group G, then $v(g_1, g_2, \cdots, g_r) = g_{i_1}^{l_1} g_{i_2}^{l_2} \cdots g_{i_r}^{l_r} \in G$ is called the value of the word ν at $(g_1, g_2, \cdots, g_r)$. The subgroup of G genarated by all values in G of words in V, i.e

$$V(G) = < v(g_1, g_2, \cdots, g_r) |\ g_i \in G, \nu \in G >,$$

is called verbal subgroup of G.

Definition 2.1. Let $\mu \in F(G)$ and let $V(G)$ be verbal subgroup of G. Suppose μ_V is restriction of μ on $V(G)$. We call $(V(G), \mu_V)$, the **verbal fuzzy subgroup** of $(V(G), F(V(G)))$.

Definition 2.2. Let $\mu \in F(G)$ and V be a non-empty set of word in $x_1, x_2, \cdots$. Then the following subset

$$V_\mu^*(G) = \{a \in G |\ \mu(v(g_1, g_2, ..., g_i a, ..., g_r)) = \mu(v(g_1, g_2, ..., g_r))\ |g_i \in G, 1 \leq i \leq r, v \in V\}$$

of G forms a subgroups of G. Let μ_{V^*} be restriction of μ on $V_\mu^*(G)$. We call $(V_\mu^*(G), \mu_{V^*})$ the **marginal fuzzy subgroup** of $(V_\mu^*(G), F(V_\mu^*(G))$. It is easy to see that $V_\mu^*(G) \leq G$.

The following theorem indicates a connection between marginal and verbal fuzzy subgroups.

Theorem 2.1. *Let V be a non-empty set of word on $x_1, x_2, \cdots$ and let $\mu \in F(G)$. Then $V_\mu^*(G) = G$ if and only if $\mu_V = \mu(e)$.*

Proof. Obviously, $\mu_V = \mu(e)$ implies that $V^*_\mu(G) = G$. Conversely if $V^*_\mu(G) = G$, then for all $\nu \in V$ and all $g_i \in G$

$$\mu(v(g_1, g_2, ..., g_r)) = \mu(v(e, g_2, ..., g_r)) = \cdots = \mu(v(e, e, ..., e)) = \mu(e)$$

hence $\mu_V = \mu(e)$. □

In continue, we provide some preliminary properties and notions concerning verbal and marginal fuzzy subgroups.

Theorem 2.2. *If $V(G) = \{[x_1, x_2]\}$ then $Z(\mu) = V^*_\mu(G)$.*

Proof. By definition of $V(G)$ we have

$$V^*_\mu(G) = \{a \in G|\ \mu[ax, y] = \mu[x, ay] = \mu[x, y];\ \forall x, y \in G\}\ and$$

$$Z(\mu) = \{x \in G|\ \mu[x, y] = \mu(e);\ \forall y \in G\}$$

If $a \in V^*_\mu(G)$ then for all $y \in G$: $\mu[a, y] = \mu[e, y] = \mu(e)$. Therefore $a \in Z(\mu)$. Conversely, if $a \in Z(\mu)$ then for all $x, y \in G$:

$$\mu[ax, y] = \mu(axyx^{-1}a^{-1}y^{-1}) = \mu(a^{-1}axyx^{-1}y^{-1}) = \mu[x, y]\ and$$

$$\mu[x, ay] = \mu(xayx^{-1}y^{-1}a^{-1}) = \mu(axyx^{-1}y^{-1}a^{-1}) = \mu(a^{-1}axyx^{-1}y^{-1}) = \mu[x, y]$$

hence $a \in V^*_\mu(G)$. □

Theorem 2.3. *Let V be a non-empty set of word on $x_1, x_2, \cdots$ and let $\mu \in F(G)$. If H is subgroup of G such that*

$$\frac{H}{V^*_\mu(G)} \subseteq Z(\frac{G}{V^*_\mu(G)})$$

then $\mu[H, V(G)] = \mu(e)$

Proof. By assumption, for every $h \in H$ and $\nu \in V$ and $g \in G$:

$$hV_\mu^*(G)gV_\mu^*(G) = gV_\mu^*(G)hV_\mu^*(G) \;\Rightarrow\; [h,g] \in V_\mu^*(G) \quad (1)$$

hence we have

$$\mu[h, v(g_1, g_2, ..., g_r)] = \mu(hv(g_1, g_2, ..., g_r)h^{-1}v(g_1, g_2, ..., g_r)^{-1})$$

$$= \mu(v(g_1^h, g_2^h, ..., g_r^h)v(g_1, g_2, ..., g_r)^{-1})$$

$$= \mu(v(g_1[g_1, h], g_2[g_2, h], ..., g_r[g_r, h])v(g_1, g_2, ..., g_r)^{-1}) \quad (2)$$

since $V(G) \leq G$ and by insertion (1), (2), we have

$$\mu[h, v(g_1, g_2, ..., g_r)] = \mu(v(g_1, g_2, ..., g_r)v(g_1, g_2, ..., g_r)^{-1}) = \mu(e).$$

□

Definition 2.3. Let V be a non-empty set of word on $x_1, x_2, \cdots$ and let $\mu \in F(G)$ and G be any group with normal subgroup N. We define $[NV^*G]$ to be subgroup of G generated by

$$\{v(g_1, \dots, g_{i-1}, g_i n, \dots, g_r)v(g_1, g_2, ..., g_r)^{-1} |\; v \in V, g_i \in G, n \in N\}$$

and let $\mu[NV^*G]$ be restriction of μ on $[NV^*G]$ and define $[NV_\mu^*G]$ to be the subset of real number, with the following form

$$[NV_\mu^*G] = \{\mu(v(g_1, \dots, g_{i-1}, g_i n, \dots, g_r))\mu(v(g_1, g_2, ..., g_r)^{-1}) |\; v \in V, g_i \in G, n \in N\}$$

Proposition 2.1. *Let V be non-empty set of words on $x_1, x_2, \cdots$ and let $\mu \in F(G)$ and $N \trianglelefteq G$, then the following holds,*

(i) $\mu_V(V_\mu^*(G)) = \mu(e)$

(ii) $N \subseteq V_\mu^*(G) \iff [NV_\mu^*G] = 1$

(iii) $\mu_{[NV^*G]} = \mu(e) \Rightarrow N \subseteq V_\mu^*(G)$

(iv) $V_\mu^*(\frac{G}{V(G)}) = V^*(\frac{G}{V(G)})$

(v) $[G, N] \subseteq V_\mu^*(G) \;\Rightarrow\; \mu(V(G)) \subseteq \mu[V(G), N]$

Proof. (i) We know $\mu_V(V^*_\mu(G))$ is restriction of μ on words of V that their letters is selected of $V^*_\mu(G)$. Therfore:

$$V(V^*_\mu(G)) = \{v(g_1, g_2, ..., g_r)|\ v \in V, g_i \in V^*_\mu(G)\}.$$

Let $v(g_1, g_2, ..., g_r) \in V(V^*_\mu(G))$ then,

$$\mu(v(g_1, g_2, ..., g_r)) = \mu(v(e, g_2, ..., g_r)) = \ldots = \mu(v(e, e, ..., e)) = \mu(e)$$

(ii)

$$[NV^*_\mu G] = 1 \iff$$

$$\forall n \in N; \mu(v(g_1, g_2, ..., g_i n, \ldots, g_r)) = \mu(v(g_1, g_2, ..., g_r)) \iff$$

$$n \in V^*_\mu(G)$$

(iii) Let $n \in N$ and $\nu \in V$ be arbitary. Clearly, if

$$\mu(\nu(g_1, g_2, ..., g_i n, \ldots, g_r)) = \mu(v(g_1, g_2, ..., g_r))$$

then $n \in V^*_\mu(G)$. Otherwise

$$\mu(v(g_1, g_2, ..., g_i n, \ldots, g_r)) \neq \mu(v(g_1, g_2, ..., g_r)) \quad (1)$$

by assumption

$$\mu(e) = \mu(v(g_1, \ldots, g_{i-1}, g_i n, \ldots, g_r) v(g_1, g_2, ..., g_r)^{-1})$$

now by (1) and (1.1) (i,ii)

$$\mu(e) = \mu(v(g_1, \ldots, g_{i-1}, g_i n, \ldots, g_r)) \wedge \mu(v(g_1, g_2, ..., g_r)^{-1}) \quad (2)$$

therfore, by (2)

$$\mu(e) \leq \mu(v(g_1, \ldots, g_{i-1}, g_i n, \ldots, g_r)) \textit{ and } \mu(e) = \mu(v(g_1, \ldots, g_r))$$

now by (1.1) (i)

$$\mu(v(g_1,\ldots,g_{i-1},g_i n,\ldots,g_r))=\mu(e)=\mu(v(g_1,\ldots,g_r))$$

but it is in contradiction to (1).

(iv) In [3] was proved that $\frac{G}{V(G)}=V^*(\frac{G}{V(G)})$. So we need to prove $V^*_\mu(\frac{G}{V(G)})=\frac{G}{V(G)}$.

$$V^*_\mu(\frac{G}{V(G)})=$$

$$\{gV(G)|\mu(v(g_1V(G),\ldots,g_iV(G)gV(G),\ldots,g_rV(G)))=$$

$$\mu(v(g_1V(G),\ldots,g_rV(G)));g_i\in G,v\in V\}=$$

$$\{gV(G)|\mu(v(g_1,\ldots,g_ig,\ldots,g_r)V(G))\mu(v(g_1,\ldots,g_r)V(G));g_i\in G,v\in V\}=$$

$$\{gV(G)|\ \mu(V(G))=\mu(V(G))\}=$$

$$\frac{G}{V(G)}$$

(v) Let $v(g_1,g_2,...,g_r)\in V$ and $n\in N$:

$$\mu[v(g_1,g_2,...,g_r),n]=\mu(v(g_1,g_2,...,g_r)^{-1}n^{-1}v(g_1,g_2,...,g_r)n)$$

$$=\mu(v(g_1,g_2,...,g_r)^{-1}v(g_1[g_1,n],g_2[g_r,n],...,g_r[g_r,n]))$$

$$\geq\mu(v(g_1,g_2,...,g_r))\wedge\mu(v(g_1[g_1,n],g_2[g_r,n],...,g_r[g_r,n]))$$

by assumption,

$$\mu[v(g_1,g_2,...,g_r),n]\geq\mu(v(g_1,g_2,...,g_r))$$

wich complete the proof. □

Let (G, λ) and (G, μ) be fuzzy subsets of $(G, F(\mathrm{G}))$. Let (λ, μ) be the fuzzy subset of G defined as follows: $\forall x \in G)$,

$$(\mu, \lambda)(x) = \begin{cases} \vee\{\mu(a) \wedge \mu(b) | \ x = [a, b], a, b \in G\} & x \text{ is commutator} \\ 0 & \text{otherwise} \end{cases}$$

The commutator of λ and μ is the fuzzy subgroup $[\lambda, \mu]$ of G generated by (λ, μ). In [2], was proved $(\mu, \mu)(x) \leq \mu(x)$ therfore $[\mu, \mu] \subseteq \mu$.

Theorem 2.4. *Let* $\mu \in F(G)$*. Then*

$$[\mu, \mu]([x_1, x_2]) \leq \mu([x_1, x_2])$$

Proof. The fuzzy subgroup $[\mu, \mu]$ generated by (μ, μ), so that

$$(\mu, \mu)(t) = \begin{cases} \vee\{\mu(a) \wedge \mu(b) | \ t = [a, b], a, b \in G\} & t \text{ is commutator} \\ 0 & \text{otherwise} \end{cases}$$

therefore

$$(\mu, \mu)([x_1, x_2]) = \vee\{\mu(a) \wedge \mu(b) | \ [x_1, x_2] = [a, b]\} \quad (1)$$

. It is obvious that a function on the same elements, acts the same. Therfore, for all $[a, b]$ in the absence $[x_1, x_2] = [a, b]$,

$$\mu[x_1, x_2] = \mu[a, b] \geq \mu(a) \wedge \mu(b)$$

therefore

$$\mu[x_1, x_2] \geq \vee\{\mu(a) \wedge \mu(b) | \ [x_1, x_2] = [a, b]\} \quad (2)$$

Now by (1) and (2) the proof is complete. □

According to (3.2.14) in [2] we have,

$$\ldots \subseteq [\mu, \mu, \ldots, \mu] \subseteq [\mu, \ldots, \mu] \subseteq \ldots \subseteq [\mu, \mu] \subseteq \mu$$

so we have,

$$[\mu, \mu, \ldots, \mu](x_1, x_2 \ldots, x_s) \leq \mu(x_1, x_2 \ldots, x_s)$$

3. Fuzzy Varieties of Groups

P. Hall introduced the concept of varieties of groups in 1940 and as time went on, it become one of the most important concepts in mathematics. In this chapter, we try to express the meaning of fuzzy varieties.

Definition 3.1. Let μ be a fuzzy subgroup of G. Then the chain

$$\mu = \mu^{(0)} \supseteq \mu^{(1)} \supseteq ... \supseteq \mu^{(n)} \supseteq ...$$

of fuzzy subgroups of G is called the derived chain of μ where, $\forall n \in N$:

$$\mu^{(n+1)} = [\mu^{(n)}, \mu^{(n)}].$$

Let μ be a fuzzy subgroup of G. Then the chain

$$\mu = Z_0(\mu) \supseteq Z_1(\mu) \supseteq ... \supseteq Z_n(\mu) \supseteq ...$$

of fuzzy subgroups of G is called the descending central chain of μ where, $Z_{n+1}(\mu) = [Z_n(\mu), \mu]$.

Definition 3.2. Let G be a group. A **Fuzzy variety** on $(G, F(\mathrm{G}))$ is an equationally defined class of fuzzy subgroups. If W is a set of words in $x_1, x_2, \ldots$ the class of all fuzzy subgroups of $(G, F(\mathrm{G}))$ such that $V(\mu) = e_t$, (G, μ) is the fuzzy subgroup with tip t, is called **Fuzzy variety** $FV(W)$ determined by W. We also say that W is a set of lows for the fuzzy variety $FV(W)$.

Now let $W = \{[x_1, x_2]\}$, then $FV(W)$ is the class of abelian fuzzy sub groups and is called **Abelian fuzzy variety**.
More precisely, if $[\mu, \mu] = e_t$ then fuzzy subgroup (G, μ) settle on abelian fuzzy variety.

If $W = \{[x_1, x_2, \ldots, x_{c+1}]$, where $c \geq 1$, then $FV(W)$ is the class of all nilpotent fuzzy subgroups of at most c and is called **Nilpotent fuzzy variety**

Clearly, if the descending central chain of μ is such that $Z_c(\mu) = e_t$ for $c \in N$, then fuzzy subgroup (G, μ) settle on nilpotent fuzzy variety.

If $W = \{x^{(c+1)}\}$, where $c \geq 1$, then $FV(W)$ is the class of all solvable fuzzy subgroups of at most c and is called **Solvable fuzzy variety**

More precisely, if the derived chain of μ is such that $\mu^{(c+1)} = e_t$ for $c \in N$, then fuzzy subgroup (G, μ) settle on solvable fuzzy variety.

4. Isologisms of Fuzzy subgroups

P. Hall introduced the notion of isologism, an equivalence relation on the class of all groups. This equivalence relation depends on some fixed variety V and has the property that the groups in the variety V form a single equivalence classes. Now we introduce and study the concept of fuzzy isologism on some fixed fuzzy variety.

Let f be a function from X into Y, and let $\mu \in FP(\mathrm{X})$ and $\lambda \in FP(\mathrm{Y})$. Define the fuzzy subsets $f(\mu) \in FP(\mathrm{Y})$ and $f^{-1}(\lambda) \in FP(\mathrm{X})$ by $\forall y \in Y$,

$$f(\mu)(y) = \begin{cases} \vee\{\mu(x) |\ t = [a, b], x \in X, f(x) = y\} & f^{-1}(y) \neq \varnothing \\ 0 & \text{otherwise} \end{cases}$$

and $\forall x \in X$

$$f^{-1}(\lambda)(x) = \lambda(f(x)).$$

Then $f(\mu)$ is called the image of μ under f and $f^{-1}(\lambda)$ is called the preimage (or inverse image) of λ under f.

Let $\mu \in F(G)$ and H be a group. Suppose that f is a homomorphism of G into H. Then $f(\mu) \in F(\mathrm{H})$ and if $\lambda \in F(\mathrm{H})$ then $f^{-1}(\lambda) \in F(\mathrm{G})$.

Definition 4.1. Let $(F(X); T_1, t_1)$ be the homomorphic image of (G, μ) and let $(F(X); T_2, t_2)$ be the homomorphic image of (H, λ). Let V be a fuzzy variety.

A $FV-$ **Isologism** between (G,μ) and (H,λ) is a pair of isomorfism (α,β) if, $\alpha : \dfrac{G}{V^*_\mu(G)} \rightarrow \dfrac{H}{V^*_\lambda(H)}$ and $\beta : \mu_V(G) \rightarrow \lambda_V(H)$, such that for all $s > 0$ and all $v(x_1, x_2 \ldots, x_s) \in V$ and all $g_1, g_2, \ldots, g_s \in G$, it hold that

$$\beta(\mu v(g_1, g_2, ..., g_r)) = \lambda v(h_1, ..., h_r) \ \ whenever \ \ h_i \in \alpha(g_i V^*_\mu(G)), i = 1, ..., r.$$

We write $(G,\mu) \approx (H,\lambda)$

Lemma 4.1. *Let (α,β) be a FV-isologism between (G_1,μ_1) and (G_2,μ_2). If $V^*_{\mu_1}(G_1) \leq H_1 \leq G_1$ and $\alpha(\dfrac{H_1}{V^*_{\mu_1}(G_1)}) = \dfrac{H_2}{V^*_{\mu_2}(G_2)}$, Then $(H_1,\lambda_1) \approx (H_2,\lambda_2)$ where, $\lambda_1 = \mu_1|_{H_1}, \lambda_2 = \mu_2|_{H_2}$.*

Proof. Since (α,β) is a FV-isologism between (G_1,μ_1) and (G_2,μ_2), both α,β are isomorfisms such that,

$$\alpha : \frac{G_1}{V^*_{\mu_1}(G_1)} \longrightarrow \frac{G_2}{V^*_{\mu_2}(G_2)}$$
$$\beta : \mu_{1_V}(G_1) \longrightarrow \mu_{2_V}(G_2)$$

and

$$\beta(\mu_1 v(g_1, g_2, ..., g_r)) = \mu_2 v(g_1^{'}, ..., g_r^{'}) \ \ whenever \ \ g_i^{'} \in \alpha(g_i V^*_{\mu_1}(G_1)), i = 1, ..., r.$$

Since $V^*_{\mu_1}(G_1) \leq H_1 \leq G_1$, we conclude $V^*_{\mu_1}(G_1) \leq V^*_{\lambda_1}(H_1)$.
Similarly, since $\alpha(\dfrac{H_1}{V^*_{\mu_1}(G_1)}) = \dfrac{H_2}{V^*_{\mu_2}(G_2)}$, we conclude $V^*_{\mu_2}(G_2) \leq H_2 \leq G_2$. Therefore, $V^*_{\mu_2}(G_2) \leq V^*_{\lambda_2}(H_2)$. We define two isomorphisms

$$\alpha^{'} : \frac{H_1}{V^*_{\lambda_1}(H_1)} \longrightarrow \frac{H_2}{V^*_{\lambda_2}(H_2)}$$

$$\beta^{'} : \lambda_{1_V}(H_1) \longrightarrow \lambda_{2_V}(H_2),$$

as follows,

$$\alpha^{'}(h_1 V^*_{\lambda_1}(H_1)) = h_2 V^*_{\lambda_2}(H_2), \ if \ h_1 \in H_1 \ and \ h_2 \in \alpha(h_1 V^*_{\lambda_1}(H_1)),$$

$$\beta^{'}(\lambda_1(v(h_1, \ldots, h_r))) = \lambda_2(v(h_1^{'}, \ldots, h_r^{'}))) \ and$$

$$\beta^{'}(\lambda_1(v(h_1,\ldots,h_r))) = \beta(\lambda_1(v(h_1,\ldots,h_r)))$$

It is easy to see that the pair $(\alpha^{'},\beta^{'})$ is an FV-isologism between (H_1,λ_1) and (H_2,λ_2).

□

Lemma 4.2. *Let (α,β) be a FV-isologism between (G,μ) and (H,λ). If $g\in G$ and $h\in\alpha(gV^*_\mu(G))$, then $\beta(\mu(v^g)) = \lambda(v^h)$.*

Proof. Let $v\in V$, say $v = v(x_1,x_2,\ldots,x_s)$. Let $g_1,g_2,\ldots,g_s\in G$ and choose

$$h\in\alpha(gV^*_\mu(G)),\ h_i\in\alpha(g_iV^*_\mu(G)) \text{ for all } 1\le i\le s$$

we have

$$(h_i)^h = h^{-1}h_ih\in\alpha(g^{-1}g_igV^*_\mu(G))$$

thus $(h_i)^h\in\alpha((g_i)^gV^*_\mu(G))$, because α is homomorphism. Now we have

$$\beta(\mu(v(g_1,g_2,\ldots,g_s)^g)) = \beta(\mu(v(g_1^g,g_2^g,\ldots,g_s^g))) = \lambda(v(h_1^h,h_2^h,\ldots,h_s^h)) = \lambda(v^h)$$

□

Proposition 4.1. *Let $\mu\in F(G)$ and $H\le G$ and $\mu^{'}=\mu|_H$, then the following holds,*
(i) $V^*_\mu(HV^*_\mu(G)) = V^*_{\mu^{'}}(H)V^*_\mu(G)$
(ii) $\mu^{'}_V(H) = \mu_V(HV^*_\mu(G))$

Proof. (i) by definition 2.2, we have

$$V^*_\mu(HV^*_\mu(G)) = \{ha|\ \mu(v(h_1a_1,\ldots,h_ia_iha,\ldots,h_ra_r)) =$$

$$\mu(v(h_1a_1,\ldots,h_ia_i,\ldots,h_ra_r));\forall v\in V;\forall h,h_i\in H;\forall a,a_i\in V^*_\mu(G)\} =$$

$$\{ha|\mu(v(h_1,\ldots,h_ihh^{-1}a_ih,\ldots,h_r)) =$$

$$\mu(v(h_1,\ldots,h_i,\ldots,h_r));\forall v\in V;\forall h,h_i\in H;\forall a_i\in V^*_\mu(G)\}$$

$V^*_\mu(G)$ is normal subgroup of G, therefore $h^{-1}a_ih\in V^*_\mu(G)$. So,

$$V^*_\mu(HV^*_\mu(G)) = \{ha|\mu(v(h_1,\ldots,h_iha^{'}_i,\ldots,h_r)) =$$

$$\mu(v(h_1,\ldots,h_i,\ldots,h_r)); \forall v \in V; \forall h, h_i \in H; \forall a_i^{'} \in V_{\mu}^{*}(G)\} =$$

$$\{ha|\mu(v(h_1,\ldots,h_ih,\ldots,h_r)) =$$

$$\mu(v(h_1,\ldots,h_i,\ldots,h_r)); \forall v \in V; \forall h, h_i \in H\} =$$

$$\{ha|\ h \in V_{\mu'}^{*}(H)\} = V_{\mu'}^{*}(H)V_{\mu}^{*}(G))$$

(ii) For all $v(h_1a_1,\ldots,h_ra_r) \in V(HV_{\mu}^{*}(G))$

$$\mu(v(h_1a_1,\ldots,h_ra_r)) = \mu(v(h_1,\ldots,h_r))$$

therefore $\mu_V^{'}(H) = \mu_V(HV_{\mu}^{*}(G))$. □

Theorem 4.1. *Let $H \leq G$ and $\mu \in F(G)$ and $\mu^{'} = \mu|_H$. Then $(H,\mu^{'}) \approx (HV_{\mu}^{*}(G))$. In particular, if $G = HV_{\mu}^{*}(G)$, then $(G,\mu) \approx (H,\mu^{'})$. Conversely, if $\dfrac{G}{V_{\mu}^{*}(G)}$ satisfies the ascending chain condition on subgroups and $(G,\mu) \approx (H,\mu^{'})$, then $G = HV_{\mu}^{*}(G)$*

Proof. We difine a map α by putting $\alpha(hV_{\mu'}^{*}(H)) = hV_{\mu}^{*}(HV_{\mu}^{*}(G))$ $(h \in H)$. By Proposition 4.4 (i), since $V_{\mu}^{*}(HV_{\mu}^{*}(G)) = V_{\mu'}^{*}(H)V_{\mu}^{*}(G)$, α is an isomorphism from $\dfrac{H}{V_{\mu'}^{*}(H)}$ onto $\dfrac{HV_{\mu}^{*}(G)}{V_{\mu}^{*}(HV_{\mu}^{*})}$. By Proposition 4.4 (ii), since $\mu_V^{'}(H) = \mu_V(HV_{\mu}^{*}(G))$ and α induces the identity on $\mu_V^{'}(H)$, the pair $(\alpha, id_{\mu_V^{'}(H)})$ is a FV-isologism between $(H,\mu^{'})$ and $(HV_{\mu}^{*}(G),\mu)$.

Now suppose that $H \leq G$ and $(G,\mu) \approx (H,\mu^{'})$. There is the pair (α_0,β_0) such that,

$$\alpha_0 : \frac{G}{V_{\mu}^{*}(G)} \longrightarrow \frac{H}{V_{\mu'}^{*}(H)},$$

$$\beta_0 : \mu_V(G) \longrightarrow \mu_V^{'}(H)$$

Define $H_1 \leq H$ by $\alpha_0(\dfrac{H}{V_{\mu}^{*}(G)}) = \dfrac{H_1}{V_{\mu'}^{*}(G)}$. So $V_{\mu}^{*}(G) \leq H \leq G$ and by (4.5) we have $(H,\mu^{'}) \approx (H_1,\mu_1^{'})$ where $\mu^{'} = \mu|_H$ and $\mu_1^{'} = \mu^{'}|_{H_1}$, thus $(G,\mu) \approx (H_1,\mu_1^{'})$.

since $(H,\mu^{'}) \approx (H_1,\mu_1^{'})$, there is the pair (α_1,β_1) such that,

$$\alpha_1 : \frac{H}{V_{\mu'}^{*}(H)} \longrightarrow \frac{H_1}{V_{\mu_1'}^{*}(H_1)},$$

$$\beta_1 : \mu'_V(H) \longrightarrow \mu'_1 V(H_1)$$

Define $H_2 \leq H_1$ by $\alpha_1(\frac{H_1}{V^*_{\mu'}(H)}) = \frac{H_2}{V^*_{\mu'_1}(H_1)}$. So $V^*_{\mu'_1}(H_1) \leq H_2 \leq H_1$ and by (4.5) we have $(H_1, \mu'_1) \approx (H_2, \mu'_2)$ where $\mu'_2 = \mu'_1|(H_2)$ so that $(G, \mu) \approx (H_2, \mu'_2)$.

Continuing the above process, we get a sequence of subgroups of H,

$$V^*_\mu(G) \leq V^*_{\mu'}(H) \leq V^*_{\mu'_1}(H_1) \leq \ldots \leq H_2 \leq H_1 \leq H$$

with the property that $(G, \mu) \approx (H_i, \mu'_i)$ for each $i \geq 0$. If however $\frac{G}{V^*_\mu(G)}$, and hence $\frac{H}{V^*_{\mu'}(G)}$, satisfies the descending chain condition on subgroups, then it follows that for some $i \geq 0$ we have $H_i = H_{i+1}$. But this is equivalent to $G = H$, as desired. □

Proposition 4.2. *Let FV be a fuzzy variety and let (α, β) be a FV-isologism between (G, μ) and (H, λ). Let $M \trianglelefteq G$ and put $\alpha(\frac{MV^*_\mu(G)}{V^*_\mu(G)}) = \frac{N}{V^*_\lambda(H)}$. Then*

$$\beta(\mu[MV^*G]) = \lambda[NV^*H].$$

Proof. Since (α, β) is a FV-isologism between (G, μ) and (H, λ), both α, β are isomorfisms such that,

$$\alpha : \frac{G}{V^*_\mu(G)} \longrightarrow \frac{H}{V^*_\lambda(H)}$$
$$\beta : \mu_V(G) \longrightarrow \lambda_V(H)$$

and

$$\beta(\mu(v(g_1, g_2, ..., g_s))) = \lambda(v(h_1, ..., h_s)) \quad whenever \quad h_i \in \alpha(g_i V^*_\mu(G)), i = 1, ..., s.$$

Let $m \in M, g_1, \ldots, g_s \in G$ and $v(x_1, \ldots, x_s) \in V$.choose $h_i \in \alpha(g_i V^*_\mu(G))(i = 1, ..., s)$ and $n \in \alpha(mV^*_\mu)$. By definition of FV-isologism we have that ,

$$\beta(\mu(v(g_1, \ldots, g_i m, ..., g_s))) = \lambda v(h_1, \ldots, h_i n, h_s)$$

therefore

$$\beta(\mu(v(g_1, \ldots, g_i m, ..., g_s) v(g_1, ..., g_s)^{-1})) = \lambda(v(h_1, \ldots, h_i n, h_s) v(h_1, \ldots, h_s)^{-1}).$$

We conclude that $\beta(\mu[MV^*G]) \subseteq \lambda[NV^*H]$ and the reverse inclusion follows by applying the above arguments to β^{-1}. □

There are many concepts in varieties of groups that can be viewed on fuzzy varieties in the next research.

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Department of Mathematics, Faculty of Science, University of Qom, Qom, I. R. Iran

E-mail address: (1) a.gholami@qom.ac.ir

E-mail address: (2) atefehjavadi80@yahoo.com