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ON TAYLOR DIFFERENTIAL TRANSFORM METHOD FOR THE FIRST PAINLEVÉ EQUATION

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Abstract. We apply the Taylor Differential Transform Method (TDTM) to the initial value problem of the first Painlevé equation. We use the deviation to calculate the accuracy of the solutions and the results are compared with the known results. Four cases of initial values, two of them were not considered before, are considered to illustrate the effectiveness of the method.

1. Introduction

Taylor series method was used by Newton for solving ordinary differential equations [1]. It is one of the first methods used to solve differential equations and it can be used to obtain solution of linear and nonlinear equations [2]. However it was not the method of choice for long time since the calculation of the Taylor coefficients requires tedious calculation of the higher order derivatives. The difficulty of implementing the Taylor series method was overcome by using automatic differentiation or recursive computation of the Taylor series coefficients [3, 4, 5, 6].

The differential transform method (DTM) was first used by Zhou [7] to solve linear and nonlinear problems in electrical circuit. We believe that the differential transform method is a new representation of the Taylor series method depending on recursive computation of the Taylor coefficients and it has been formulated in a similar way to

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the well known transformation methods like the Laplace or the Fourier transformations [2]. As a result we think it is better to call it the Taylor differential transform method (TDTM). The TDTM can be used to find approximating solutions to ordinary and partial differential equations by a finite Taylor series. It can be applied easily and successfully to a large class of ordinary and partial differential equations [2].

Painlevé, Gambier and Fuchs addressed a question raised by E. Picard concerning the second-order first-degree ordinary differential equations of the form

$$v'' = F(z, v, v'), \tag{1.1}$$

where F is rational in v', algebraic in v and locally analytic in z, having the Painlevé property. Within the Möbius transformation, they found fifty such equations. Among all these equations, six of them are irreducible and define the classical Painlevé transcendents, PI, PII,..., PVI [8], and the remaining 44 equations are either solvable in terms of known functions or can be transformed into one of the six equations. The Painlevé equations PI, PII,...,PVI may be regarded as the nonlinear counter parts of some classical special equations. The analytic properties of Painlevé equations has been the subject of many studies. In the contrast, the numerical analysis of these equations needs more attention.

The initial value problem for PI was studied in [9] and it was shown that this problem admits a global meromorphic solution. In this article we use Taylor differential transform method to study the numerical solutions of the initial value problem for PI. The Sage Mathematics Software is used to obtain the numerical results and the results are compared with the known results. We use deviation to test the validity of our results. It turns out that the method is an effective tool for investigating the numerical solutions of PI and hence it can help in solving the challenging problem of the numerical solutions of the Painlevé equations.

2. Taylor Differential Transform and PI equation

In this section we give the definition and basic properties of the Taylor differential transform (TDT) [10, 11, 12, 13]. In addition, we apply the TDT to the first Painlevé equation.

Definition 1. Let $u(t)$ be an analytic function on an interval I containing the point $t = t_0$. The TDT of $u(t)$ is defined by

$$U(0) = u(t_0),\ U(k) = \frac{1}{k!}\left[\frac{d^k}{dt^k}u(t)\right]_{t=t_0},\ k \in \mathbb{N}. \tag{2.1}$$

The inverse TDT of $U(k)$ is defined by

$$u(t) = \sum_{k=0}^{\infty} U(k)(t-t_0)^k. \tag{2.2}$$

The following theorem gives some basic properties of TDT.

Theorem 1. [13] Let $U(k)$, $U_1(k)$, and $U_2(k)$ be the TDT of $u(t)$, $u_1(t)$, and $u_2(t)$ respectively.

(1): If $u(t) = c_1u_1(t) \pm c_2u_2(t)$, then $U(k) = c_1U_1(k) \pm c_2U_2(k)$, for any constants $c_1,\ c_2$.

(2): If $u(t) = \frac{d^n}{dt^n}u_1(t)$, then $U(k) = \frac{(k+n)!}{k!}U_1(k)$.

(3): If $u(t) = u_1(t)u_2(t)$, then $U(k) = \sum_{j=0}^{k} U_1(j)U_2(k-j)$.

(5): If $u(t) = (t-t_0)^n$, then $U(k) = \delta(n-k) = \begin{cases} 1, & k = n, \\ 0, & k \neq n. \end{cases}$

(6): If $u(t) = t^n$, then $U(k) = \sum_{j=0}^{n} \frac{n!\ t_0^{n-j}}{j!(n-j)!}\delta(j-k) = \begin{cases} \frac{n!\ t_0^{n-k}}{k!(n-k)!}, & k \leq n, \\ 0, & k > n. \end{cases}$

(7): If $u(t) = \cos(\omega t + \alpha)$, then $U(k) = \frac{\omega^k}{k!} \cos\left(\frac{\pi k}{2} + \omega t_0 + \alpha\right)$, where ω and α are constants.

(8): If $u(t) = \sin(\omega t + \alpha)$, then $U(k) = \frac{\omega^k}{k!} \sin\left(\frac{\pi k}{2} + \omega t_0 + \alpha\right)$, where ω and α are constants.

2.1. **The First Painlevé Equation.** Consider the first Painlevé (PI) equation

$$u'' = 6u^2 + t. \tag{2.3}$$

It is will known that the solution of (2.3) defines a new transcendental function, namely the first Painlevé function. Given any initial conditions

$$u(t_0) = u_0, \ u'(t_0) = u_1, \tag{2.4}$$

the existence and uniqueness theorem guarantees that the initial value problem (2.3-2.4) has a unique local solution. Moreover, solution $u(t)$ is analytic in some neighborhood of t_0 and it has a convergent Taylor series expansion

$$u(t) = \sum_{k=0}^{\infty} U(k)(t - t_0)^k, \ |t - t_0| < R, \tag{2.5}$$

for some positive number R.

Now we recall the deviation of an approximate solution of a system of differential equation [14]. Let the vector-valued function $\widetilde{\mathbf{u}}(t)$ be an approximate solution of the differential equation

$$\frac{d}{dt}\mathbf{u}(t) = \mathbf{F}(t, \mathbf{u}), \ t_0 \le t \le t_0 + T. \tag{2.6}$$

The deviation of $\widetilde{\mathbf{u}}(t)$ is at most ϵ when $\widetilde{\mathbf{u}}(t)$ is continuous, and satisfies the differential inequality

$$\left| \frac{d}{dt}\widetilde{\mathbf{u}}(t) - \mathbf{F}(t, \widetilde{\mathbf{u}}) \right| \le \epsilon$$

for all except a finite number of points t of the interval $[t_0, t_0 + T]$.

The following theorem gives a relation between the absolute error and the deviation.

Theorem 2. [14] Let $\mathbf{u}(t)$ be an exact solution and $\widetilde{\mathbf{u}}(t)$ be an approximate solution with deviation ϵ, of the equation (2.6). Let $\widetilde{\mathbf{u}}$ satisfy the Lipschitz condition

$$|\mathbf{F}(t,\mathbf{u}) - \mathbf{F}(t,\widetilde{\mathbf{u}})| \leq L|\mathbf{u} - \widetilde{\mathbf{u}}|.$$

Then, for $t \geq t_0$, we have

$$|\mathbf{u}(t) - \widetilde{\mathbf{u}}(t)| \leq |\mathbf{u}(t_0) - \widetilde{\mathbf{u}}(t_0)|e^{L(t-t_0)} + \left(\frac{\epsilon}{L}\right)\left(e^{L(t-t_0)} - 1\right).$$

In the case of the initial value problem (2.3-2.4), we have $\mathbf{F}(t,\mathbf{u}) = \mathbf{F}(t,u,v) = (v, 6u^2 + t)$. Assume $(\widetilde{u}(t_0), \widetilde{v}(t_0)) = (u(t_0), v(t_0))$. It follows, by Theorem 1, that

$$|\mathbf{u}(t) - \widetilde{\mathbf{u}}(t)| \leq \left(\frac{\epsilon}{L}\right)(e^{L(t-t_0)} - 1).$$

This shows that when the deviation is small, the absolute error $|\mathbf{u}(t) - \widetilde{\mathbf{u}}(t)|$ will be small.

2.2. **Application of TDT to PI Equation.** Applying the Taylor differential transform to the initial value problem (2.3-2.4), we obtain

$$\begin{aligned} &U(0) = u_0,\ U(1) = u_1,\ U(2) = 3u_0^2 + \tfrac{1}{2}t_0,\ U(3) = 2u_0u_1 + \tfrac{1}{6}, \\ &U(k) = \frac{6}{k(k-1)}\sum_{j=0}^{k-2} U(j)U(k-j-2),\ k \geq 4. \end{aligned} \tag{2.7}$$

The radius of convergence is given by [15]

$$R = \frac{1}{\limsup_{k\to\infty} \sqrt[k]{U(k)}}. \tag{2.8}$$

Thus it is possible estimate the radius of convergence by $R \approx \dfrac{1}{\sqrt[k]{U(k)}}$.

We use the condition $\left|\displaystyle\sum_{k=N-5}^{N} U(k)\right| \leq 10^{-30}$ to determine a suitable order N of the Taylor polynomial and we estimate the radius of convergence by $R \approx \dfrac{1}{\sqrt[N]{U(N)}}$.

3. Numerical Results

The numerical solutions of the initial value problem (2.3-2.4) was the subject of many studies [16]–[26]. In [16, 17] and [20]–[26], the initial point was $t_0 = 0$. The initial values were $u_0 = 1,\ u_1 = 0$ in [16, 26]. Davis [16] solved the initial value problem by the continuous analytic continuation to calculate the solution for $t_j \in [-1, 1]$, with $t_j = 0.01j$. The solution on the interval $[0, 1]$ was studied in [26] by optimal homotopy asymptotic method.

In [17, 20, 21], [22]–[25], the initial conditions were $u_0 = 0,\ u_1 = 1$. Hesameddini and Peyrovi [17] used variational iterative and homotopy perturbation methods and gave the solution for $t_j \in [0, 1.7]$. Behzadi [20], applied the Adomian decomposition, modified Adomian decomposition, variational iteration, modified variational iteration, homotopy perturbation, modified homotopy perturbation, and homotopy analysis methods to find solution on the interval $[0, 1]$. In [21] optimal homotopy asymptotic method was used while in [22]–[25] feed-forward artificial neural networks were used to study the initial value problem on the interval $[0, 1]$. In [18] and [19], the Padé method is used to study the initial value problem (2.3-2.4).

In this article we study the solutions of the initial value problem (2.3-2.4) by TDTM. We take $t_0 = 0$ and consider the four cases $u_0 = 1,\ u_1 = 0$, $u_0 = 0,\ u_1 = 1$, $u_0 = u_1 = 1$, and $u_0 = u_1 = 0$. We use the deviation $|E(t, u, u', u'')| = |u'' - 6u^2 - t|$ to calculate the accuracy of the solutions and the results are compared with the known results. While the two cases $u_0 = 1,\ u_1 = 0$ and $u_0 = 0,\ u_1 = 1$ were studied before, the cases $u_0 = u_1 = 1$, and $u_0 = u_1 = 0$ were not considered before. It turns out that the application of method is very easy and the obtained results are at least as good as the results obtained by more complicated methods.

The case $u_0 = 1,\ u_1 = 0$**.** We applied the TDTM to the initial value problem (2.3-2.4) with $t_0 = 0,\ u_0 = 1,\ u_1 = 0$. Using the condition $\left|\sum_{k=N-5}^{N} U(k)\right| \leq 10^{-30}$ to determine the order N of the Taylor polynomial, we find $N = 410$ and $U(410) = 9.81087484348291 \times 10^{-32}$. Using $R \approx \frac{1}{\sqrt[410]{U(410)}}$ to estimate the radius of convergence, we find $R \approx 1.19022750216386$. We use a Taylor polynomial of order 410,

$$u(t) \approx \sum_{k=0}^{410} U(k)t^k, \tag{3.1}$$

to find the values of the solutions $u(t)$ at $t \in [-1.2, 1.2]$. The values of the solutions $u(t)$ at and the deviations $|E(t, u, u', u'')|$ at $t \in [-1.2, 1.2]$ are given in Table 1 and the graph of $u(t)$ is given in Figure 1. The deviation becomes large at $t = \pm 1.2$

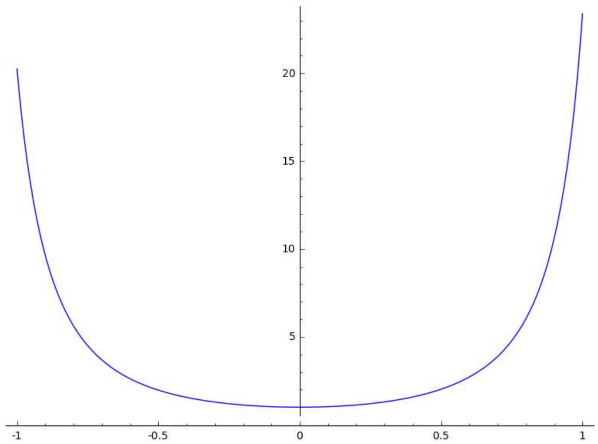

FIGURE 1. The graph of $u(t)$ obtained by TDTM when $u_0 = 1,\ u_1 = 0$

and this agree with the estimation of the radius of convergence. Continuous analytic continuation can be used to investigate whether the solution has singularities near $t = \pm 1.2$.

The same initial value problem was studied in [16] and [26]. The results obtained here are in a good agreement with that obtained in [16]. In addition we see that our results agree with the results obtained by a fourth order Runge-Kutta method in [26].

TABLE 1. Values of $u(t)$ by TDTM when $u_0 = 1,\ u_1 = 0$

t	$u(t)$	$\|E(t,u,u,u'')\|$
-1.2	1994.14199545858	$2.51036972115873 \times 10^{6}$
-1.1	66.8984451130668	$7.89441401138902 \times 10^{-10}$
-1.0	20.2471506239516	$4.09272615797818 \times 10^{-12}$
-0.90	9.63814080222272	$5.68434188608080 \times 10^{-13}$
-0.80	5.62377689776510	$1.13686837721616 \times 10^{-13}$
-0.70	3.69038763592721	$1.42108547152020 \times 10^{-14}$
-0.60	2.61911100566281	$1.42108547152020 \times 10^{-14}$
-0.50	1.97133248764948	0.000000000000000
-0.40	1.55901121493134	$1.24344978758018 \times 10^{-14}$
-0.30	1.29188556928185	$5.32907051820075 \times 10^{-15}$
-0.20	1.12363120350326	$7.99360577730113 \times 10^{-15}$
-0.10	1.03013534198727	$1.77635683940025 \times 10^{-15}$
0.00	1.00000000000000	0.000000000000000
0.10	1.03047070993338	0.000000000000000
0.20	1.12636643137490	$8.88178419700125 \times 10^{-16}$
0.30	1.30145354657786	$7.10542735760100 \times 10^{-15}$
0.40	1.58305494907820	$5.32907051820075 \times 10^{-15}$
0.50	2.02276285430253	$7.10542735760100 \times 10^{-15}$
0.60	2.72124554649151	$1.42108547152020 \times 10^{-14}$
0.70	3.89089292018485	0.000000000000000
0.80	6.03835199271657	$3.97903932025656 \times 10^{-13}$
0.90	10.6226501190303	$2.27373675443232 \times 10^{-13}$
1.0	23.3937131859640	$9.09494701772928 \times 10^{-13}$
1.1	87.7740601626277	$1.53886503539979 \times 10^{-8}$
1.2	14734.2934001342	$7.22544736801918 \times 10^{8}$

The case $u_0 = 0,\ u_1 = 1$**.** We applied the TDTM to the (2.3-2.4) with $t_0 = 0,\ u_0 = 0,\ u_1 = 1$. The results in Table 2 are obtained using a Taylor polynomial of order

144,

$$u(t) \approx \sum_{k=0}^{144} U(k)t^k, \tag{3.2}$$

with $U(144) = 2.10454783221597 \times 10^{-32}$ and radius of convergence $R \approx \frac{1}{\sqrt[144]{U(144)}} = 1.65950308419147$. Table 2 and Table 3 give the values of the solutions $u(t)$ and the deviations $|E(t, u, u', u'')|$ at $t \in [-1.7, 1.7]$ and Figure 2 gives the graph of $u(t)$.

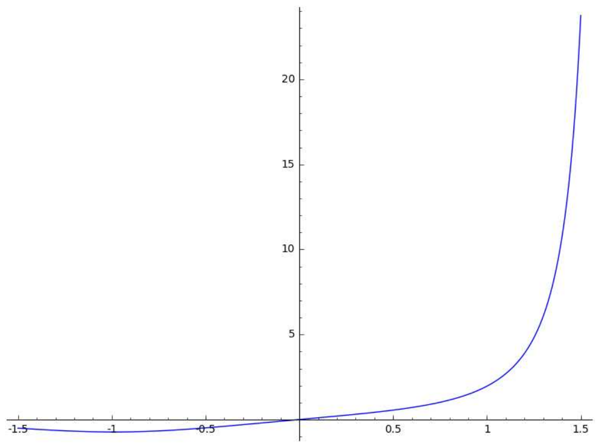

FIGURE 2. The graph of $u(t)$ obtained by TDTM when $u_0 = 0,\ u_1 = 1$

The deviation becomes large for $|t| > 1.4$ even the estimation of the radius of convergence is $R \approx 1.65950308419147$. Continuous analytic continuation can be used to obtain better results for $|t| > 1.4$ and to investigate the singularities of the solution. Another alternative to improve the results is to use a higher order Taylor polynomial. The same initial value was considered in [17], [20] , [21], [22], [23]. The root mean square value of error associated with the differential equation is defined by [23]

$$E_{RMS} = \sqrt{\frac{1}{N}\sum_{j=1}^{N} \left(u''(t_j) - 6u^2(t_j) - t_j\right)^2}, \tag{3.3}$$

where N is the number of the grid points t_j. If we use $t_j = 0.1j,\ j = 1, 2, \ldots 10$, then we find that E_{RMS} has the value $E_{RMS} = 4.02 \times 10^{-15}$. In [23], the root mean square value of error for the same interval and the same grid points was $E_{RMS} > 10^{-5}$.

TABLE 2. Values of $u(t)$ by TDTM when $u_0 = 0,\ u_1 = 1$

t	$u(t)$	$\|E(t,u,u,u'')\|$
-1.7	-16.4349046229765	115224.653138469
-1.6	-0.444346685250494	21.3810949202938
-1.5	-0.508482554937392	0.00230896328357599
-1.4	-0.574736819028458	$1.32597505420051 \times 10^{-7}$
-1.3	-0.635164613511506	$3.68594044175552 \times 10^{-12}$
-1.2	-0.684426658579674	$7.10542735760100 \times 10^{-15}$
-1.1	-0.717674652255229	$3.99680288865056 \times 10^{-15}$
-1.0	-0.731155685894222	$2.66453525910038 \times 10^{-15}$
-0.9	-0.722721782318823	$2.66453525910038 \times 10^{-15}$
-0.8	-0.692104936644286	$5.77315972805081 \times 10^{-15}$
-0.7	-0.640873361309405	$2.66453525910038 \times 10^{-15}$
-0.6	-0.572074776966747	$2.22044604925031 \times 10^{-16}$
-0.5	-0.489659997460620	$6.66133814775094 \times 10^{-16}$
-0.4	-0.397828509821033	$8.88178419700125 \times 10^{-16}$
-0.3	-0.300432728840492	0.000000000000000
-0.2	-0.200530896769527	$1.38777878078145 \times 10^{-16}$
-0.1	-0.100116614277390	$6.93889390390723 \times 10^{-18}$

The initial value problem (2.3-2.4) with $t_0 = 0,\ u_0 = 0,\ u_1 = 1$ has been solved in [17] by homotopy perturbation method (HPM). The solution was given by

$$u(t) \approx \sum_{j=0}^{22} U(k) t^k. \tag{3.4}$$

Using (3.4), the values of the solutions $u(t)$ and the deviations $|E(t, u, u', u'')|$ at $t \in [0, 1.7]$ are given in Table 4.

Comparing the results we see that the results obtained in [17] becomes poor for $t > 0.6$. The agreement of results obtained in [20] and [21] and the results obtained by TDTM becomes week as t becomes larger and larger than 0.

TABLE 3. Values of $u(t)$ by TDTM when $u_0 = 0,\ u_1 = 1$

t	$u(t)$	$\lvert E(t,u,u,u'')\rvert$
0.0	0.000000000000000	0.000000000000000
0.1	0.100216747677405	$6.93889390390723 \times 10^{-18}$
0.2	0.202139452716643	$1.11022302462516 \times 10^{-16}$
0.3	0.308630749167501	0.000000000000000
0.4	0.423986289489073	$2.22044604925031 \times 10^{-16}$
0.5	0.554340118998275	$1.33226762955019 \times 10^{-15}$
0.6	0.708462088047166	$8.88178419700125 \times 10^{-16}$
0.7	0.899249938002205	$8.88178419700125 \times 10^{-16}$
0.8	1.14653172641296	$1.68753899743024 \times 10^{-14}$
0.9	1.48252443049193	$1.06581410364015 \times 10^{-14}$
1.0	1.96312822372003	0.000000000000000
1.1	2.69331866440103	$7.10542735760100 \times 10^{-15}$
1.2	3.89089828116602	$5.68434188608080 \times 10^{-14}$
1.3	6.07245845429223	$2.93312041321769 \times 10^{-11}$
1.4	10.7264596861913	$1.48613423789357 \times 10^{-6}$
1.5	23.7518486737031	0.0414624073428058
1.6	90.3283145054996	845.842403276351
1.7	2703.94394390582	$3.49383693574811 \times 10^{7}$

The case $u_0 = u_1 = 1$. We applied the TDTM to the (2.3-2.4) with $t_0 = 0,\ u_0 = 1,\ u_1 = 1$. The results are obtained using a Taylor polynomial of order 999,

$$u(t) \approx \sum_{k=0}^{999} U(k)t^k, \tag{3.5}$$

with $U(999) = 1.26360516297992 \times 10^{-31}$ and the radius of convergence is $R \approx 1.07381463318788$. The values of the solutions $u(t)$ and the deviations $|E(t,u,u',u'')|$ at $t \in [-1,1]$ are given in Table 5 and the graph of $u(t)$ is given in Figure 3.

Table 4. Comparison between HPM and TDTM

	HPM		TDTM	
t	$u(t)$	$\|E(t,u,u,u'')\|$	$u(t)$	$\|E(t,u,u,u'')\|$
0.0	0.000000000000000	0.000000000000000	0.000000000000000	0.000000000000000
0.1	0.100216747677405	$2.77555756156289 \times 10^{-17}$	0.100216747677405	$6.93889390390723 \times 10^{-18}$
0.2	0.202139452716643	$2.22044604925031 \times 10^{-16}$	0.202139452716643	$1.11022302462516 \times 10^{-16}$
0.3	0.308630749167500	$2.40918396343659 \times 10^{-13}$	0.308630749167501	0.000000000000000
0.4	0.423986289489041	$1.02110542243850 \times 10^{-10}$	0.423986289489073	$2.22044604925031 \times 10^{-16}$
0.5	0.554340118992585	$1.18635652412991 \times 10^{-8}$	0.554340118998275	$1.33226762955019 \times 10^{-15}$
0.6	0.708462087637245	$6.04342890841991 \times 10^{-7}$	0.708462088047166	$8.88178419700125 \times 10^{-16}$
0.7	0.899249922192710	0.0000174160344972307	0.899249938002205	$8.88178419700125 \times 10^{-16}$
0.8	1.14653133989659	0.000331418843030562	1.14653172641296	$1.68753899743024 \times 10^{-14}$
0.9	1.48251773575443	0.00461189856630995	1.48252443049193	$1.06581410364015 \times 10^{-14}$
1.0	1.96303936772973	0.0504396873655963	1.96312822372003	0.000000000000000
1.1	2.69235969075616	0.457793111954992	2.69331866440103	$7.10542735760100 \times 10^{-15}$
1.2	3.88205937954382	3.60288576716354	3.89089828116602	$5.68434188608080 \times 10^{-14}$
1.3	5.99957449576944	25.5459959691514	6.07245845429223	$2.93312041321769 \times 10^{-11}$
1.4	10.1583796091245	169.077891646826	10.7264596861913	$1.48613423789357 \times 10^{-6}$
1.5	19.1605144734806	1079.92069322159	23.7518486737031	0.0414624073428058
1.6	40.2733619752124	6847.31569692990	90.3283145054996	845.842403276351
1.7	92.4976098093683	43869.1803071775	2703.94394390582	$3.49383693574811 \times 10^{7}$

The case $u_0 = u_1 = 0$. We applied the TDTM to the (2.3-2.4) with $t_0 = 0$, $u_0 = u_1 = 0$. The results are obtained using a Taylor polynomial of order 78,

$$u(t) \approx \sum_{k=0}^{78} U(k)t^k, \tag{3.6}$$

with $U(78) = 1.55336397013678 \times 10^{-31}$ and the radius of convergence is $R \approx 2.48303977578992$. The values of the solutions $u(t)$ and the deviations $|E(t,u,u',u'')|$

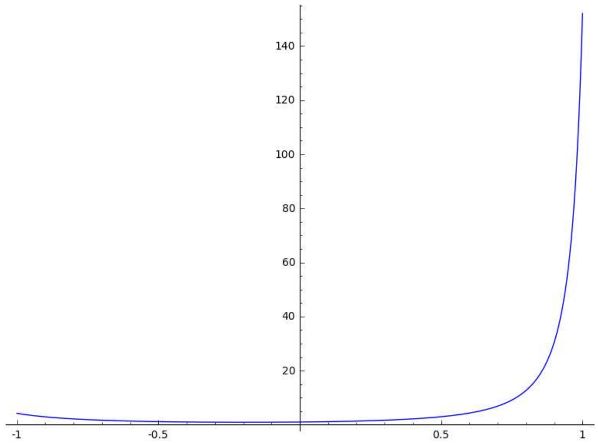

FIGURE 3. The graph of $u(t)$ obtained by TDTM when $u_0 = u_1 = 1$

at $t \in [-2, 2]$ are given in Table 6 and Table 7, and the graph of $u(t)$ is given in Figure 4.

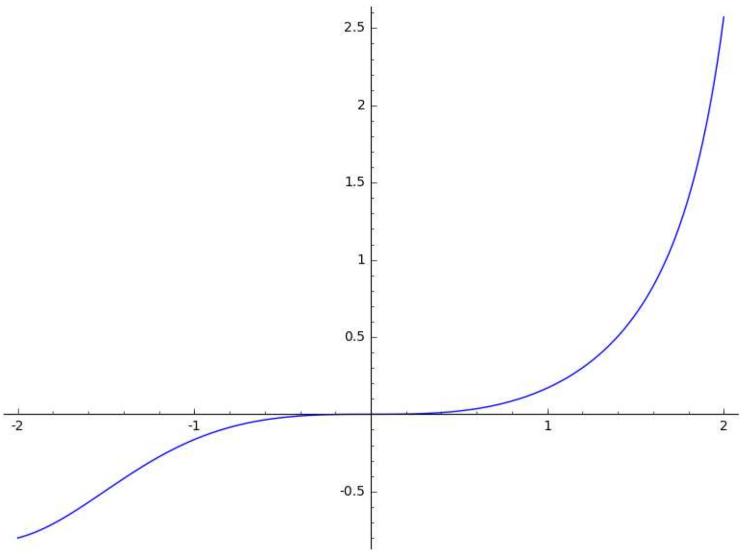

FIGURE 4. The graph of $u(t)$ obtained by TDTM when $u_0 = u_1 = 0$

TABLE 5. Values of $u(t)$ by TDTM when $u_0 = u_1 = 1$

t	$u(t)$	$\lvert E(t,u,u,u'')\rvert$
-1.0	4.21068526111097	$3.99325017497176 \times 10^{-12}$
-0.90	2.91988883115519	$2.84217094304040 \times 10^{-14}$
-0.80	2.15669033044952	$4.97379915032070 \times 10^{-14}$
-0.70	1.67425832972517	$1.42108547152020 \times 10^{-14}$
-0.60	1.35730359206224	$1.42108547152020 \times 10^{-14}$
-0.50	1.14699977958525	$4.44089209850063 \times 10^{-15}$
-0.40	1.01178198355008	$1.77635683940025 \times 10^{-15}$
-0.30	0.934683677115804	$2.66453525910038 \times 10^{-15}$
-0.20	0.907496275592817	$1.77635683940025 \times 10^{-15}$
-0.10	0.928156064887845	$8.88178419700125 \times 10^{-16}$
0.00	1.00000000000000	0.000000000000000
0.10	1.13255215511016	$6.21724893790088 \times 10^{-15}$
0.20	1.34424715658060	$7.10542735760100 \times 10^{-15}$
0.30	1.66856550740472	$7.10542735760100 \times 10^{-15}$
0.40	2.16745379377078	$3.55271367880050 \times 10^{-15}$
0.50	2.96255321026678	$6.39488462184090 \times 10^{-14}$
0.60	4.31635683790429	$1.27897692436818 \times 10^{-13}$
0.70	6.87983988404423	$1.13686837721616 \times 10^{-13}$
0.80	12.6511997392176	$2.27373675443232 \times 10^{-13}$
0.90	30.4890148375368	$1.00044417195022 \times 10^{-11}$
1.0	152.052068741433	$9.31322574615479 \times 10^{-10}$

4. CONCLUSION

In this work we reviewed the well known Taylor series method and one of its new representations known as differential transform method. We proposed the name Taylor differential transform method (TDTM) to unified the two representations.

TABLE 6. Values of $u(t)$ by TDTM when $u_0 = u_1 = 0$

t	$u(t)$	$\lvert E(t, u, u, u'')\rvert$
−2.0	−0.800877132300124	0.0000170175155744801
−1.9	−0.763142529256954	$2.81978849159259 \times 10^{-7}$
−1.8	−0.709564698511054	$3.70169761509942 \times 10^{-9}$
−1.7	−0.643828202522750	$3.75397490870455 \times 10^{-11}$
−1.6	−0.570222274280742	$2.84661183513890 \times 10^{-13}$
−1.5	−0.493069770560089	$1.33226762955019 \times 10^{-15}$
−1.4	−0.416268478365105	$2.22044604925031 \times 10^{-16}$
−1.3	−0.342998807520263	$1.11022302462516 \times 10^{-16}$
−1.2	−0.275600190843592	$2.77555756156289 \times 10^{-16}$
−1.1	−0.215583016840408	0.000000000000000
−1.0	−0.163728213733202	$5.55111512312578 \times 10^{-17}$
−0.90	−0.120228483290190	$2.77555756156289 \times 10^{-17}$
−0.80	−0.0848361014970647	$6.93889390390723 \times 10^{-17}$
−0.70	−0.0569954642130963	$4.16333634234434 \times 10^{-17}$
−0.60	−0.0359500612203996	$1.73472347597681 \times 10^{-18}$
−0.50	−0.0208217122454224	$4.33680868994202 \times 10^{-17}$
−0.40	−0.0106647164465096	$1.04083408558608 \times 10^{-17}$
−0.30	−0.00449980473822604	$2.19550939928315 \times 10^{-18}$
−0.20	−0.00133332571431697	$2.02949094162130 \times 10^{-18}$
−0.10	−0.000166666636904766	$2.49541200217037 \times 10^{-18}$
0.00	0.000000000000000	0.000000000000000
0.10	0.000166666696428576	$6.45071175871313 \times 10^{-18}$
0.20	0.00133334095241221	$6.26296161199830 \times 10^{-18}$
0.30	0.00450019527394066	$1.25089825650515 \times 10^{-17}$
0.40	0.0106686173989487	$3.95733792957209 \times 10^{-17}$
0.50	0.0208449637367454	$9.36750677027476 \times 10^{-17}$
0.60	0.0360500384491903	$1.47451495458029 \times 10^{-16}$
0.70	0.0573386085076266	$1.87350135405495 \times 10^{-16}$
0.80	0.0858347605493895	$6.93889390390723 \times 10^{-18}$
0.90	0.122790915116630	$1.11022302462516 \times 10^{-16}$

TABLE 7. Values of $u(t)$ by TDTM when $u_0 = u_1 = 0$

t	$u(t)$	$\lvert E(t, u, u, u'')\rvert$
1.0	0.169681440907945	$2.22044604925031 \times 10^{-16}$
1.1	0.228347179733036	0.000000000000000
1.2	0.301216880594462	$4.44089209850063 \times 10^{-16}$
1.3	0.391649493935926	$5.55111512312578 \times 10^{-16}$
1.4	0.504475329551635	0.000000000000000
1.5	0.646880664334779	$8.88178419700125 \times 10^{-16}$
1.6	0.829916046968329	$3.49942297361849 \times 10^{-13}$
1.7	1.07119996745379	$4.91455764972670 \times 10^{-11}$
1.8	1.40005782727668	$5.30569010948057 \times 10^{-9}$
1.9	1.86799567198019	$4.53317788640106 \times 10^{-7}$
2.0	2.57195440696007	0.0000316344756043918

We used the TDTM to study the solutions of the initial value problem (2.3-2.4) with $t_0 = 0$. We considered the four cases $u_0 = 1$, $u_1 = 0$, $u_0 = 0$, $u_1 = 1$, $u_0 = u_1 = 1$, and $u_0 = u_1 = 0$. While the solution of the first two cases have been studied by other methods, the last two case were not considered before. We made a comparison between our results and the existing results in the literature. We used the deviation to test the accuracy of our results and we believe that the deviation is a good tool for testing the accuracy of numerical solutions of differential equations.

The method is very easy to apply and very effective and it can be applied for other Painlevé equations. Moreover continuous analytic continuation can be used to improve the results at points far from the initial point. Also it is interesting to use a combination of TDTM and Padé approximation to study solutions of initial value problems of Painlevé equations. We wish to confirm that there are no known conflicts of interest associated with this publication and there has been no significant financial support for this work that could have influenced its outcome.

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