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S_α-CONNECTEDNESS IN TOPOLOGICAL SPACES

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ABSTRACT. In this paper, connectedness of a class of S_α-open sets in a topological space X is introduced. The connectedness of this class on X, called S_α-connectedness, turns out to be equivalent to connectedness of X when X is locally indiscrete or with finite α-topology. The S_α-continuous and S_α-irresolute mappings are defined and their relationship with other mappings such as continuous mappings and semi-continuous mappings are discussed. An intermediate value theorem is obtained. The hyperconnected spaces constitute a subclass of the class of S_α-connected spaces.

1. Introduction

The study of connectedness via various generalized open sets is not a new idea in topological spaces. Njastad [10] introduced the α-open sets and investigated the topological structure on the class of these sets; the α-open sets form a topology. The classes of semi-open sets [7] , β-open sets [1] , α_β-open sets [14], P_β-open sets [13] and S_α-open sets [17] were introduced. The classes of β-open, α-open and semi-open sets contain the class of open sets. Based on these classes, the concepts of β-connectedness [9, 12, 4], α-connectedness [4], semi-connectedness [11] and α_β-connectedness [14] were

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introduced, respectively. It is already known that connectedness of a topological space X is equivalent to α-connectedness of X. Here a connectedness based on the class of S_α-open sets called S_α-connectedness is introduced in a topological space X, which turns out to be stronger than the connectedness of the original topology, but it turns out to be equivalent to connectedness in case of finite α-topologies on X. The class of S_α-connected spaces contains the class of hyperconnected spaces and also contains the class of hyperconnected modulo an ideal spaces [15]. It is shown that in the class of locally indiscrete spaces, the classes of open sets, S_α-open sets and semi-open sets coincide. Through this, it is shown that $\mathbb{R}$ is not S_α-connected. One might be interesting to study the connectedness of smaller class than the class of semi-open sets under which $\mathbb{R}$ is not connected. It motivates to study S_α-connectedness.

This paper is organised as follows. In Section-2, the basic properties of S_α-open sets are developed, S_α-closure and its properties are obtained and the notion of a S_α-continuous function is defined. In Section-3, the notion of S_α-connectedness is introduced and its relationship with various other weaker and stronger forms of connectedness is investigated. It is shown that in a locally indiscrete space S_α-connectedness is equivalent to connectedness. Several characterizations of S_α-connected spaces are obtained. It is shown that the class of hyperconnected spaces is a subclass of S_α-connected spaces. Section-4 contains the properties of S_α-connected sets. In Section-5, we introduced the notion of S_α-irresolute function and studied the behaviour of S_α-connected spaces with respect to several type of mappings. Section-6 covers the concept of S_α-component. It is shown that S_α-component of a space containing a point is contained in the component containing that point.

2. Preliminaries

Let (X, τ) or X be a topological space or a space. We will denote by $Cl(A)$ and $Int(A)$ the closure of A and the interior of A, for a subset A of X, respectively.

Definition 2.1. A subset A of a topological space X is said to be

(1) α-open [10] if $A \subseteq Int(Cl(Int(A)))$
(2) β-open [1] if $A \subseteq Cl(Int(Cl(A)))$
(3) semi-open [7] if $A \subseteq Cl(Int(A)))$.

The complement of a α-open (β-open, semi-open) set is said to be α-closed (resp. β-closed, semi-closed).

Definition 2.2. (1) A semi-open subset A of a topological space X is said to be S_α-open [17] if for each $x \in A$ there exists a α-closed set F such that $x \in F \subseteq A$.

(2) A α-open subset A of a topological space X is said to be α_β-open [14] if for each $x \in A$ there exists a β-closed set F such that $x \in F \subseteq A$.

A subset B of topological space X is S_α-closed (α_β-closed) if $X \setminus A$ is S_α-open (resp., α_β-open) in X.

The family of all α-open (β-open, S_α-open, semi-open, α-closed, semi-closed, β-closed, S_α-closed) subsets of X is denoted by $\alpha O(X)(resp., \beta O(X), S_\alpha O(X), SO(X), \alpha C(X), SC(X), \beta C(X), S_\alpha C(X))$.

We have the following inclusions: $\tau \subseteq \alpha O(X) \subseteq SO(X) \subseteq \beta O(X)$ and $S_\alpha O(X) \subseteq SO(X) \subseteq \beta O(X)$.

Definition 2.3. [17] A point $x \in X$ is said to be an S_α-interior point of $A \subseteq X$ if there exists an S_α-open set U containing x such that $x \in U \subseteq A$. The set of all S_α-interior points of A is said to be S_α-interior of A and it is denoted by $S_\alpha Int(A)$.

Lemma 2.1. *The interior of a nonempty semi-open and hence S_α-open set in a space is nonempty.*

Proof. Follows directly from the definition of a semi-open set. □

Definition 2.4. [17] Intersection of all S_α-closed sets containing F is called the S_α-closure of F and it is denoted by $S_\alpha Cl(F)$.

It may be noted that S_α-open sets are obtained from semi-open sets but this collection neither contains the collection of open sets nor it is contained in the collection of open sets. Thus, the study of S_α-open sets is meaningful.

Example 2.1. *Let* $X = \{a, b, c\}, \tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$. *Then* $SO(X) = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$ *and* $S_\alpha O(X) = \{\emptyset, \{a, c\}, \{b, c\}X\}$. *Here* $\{a\} \in \tau$ *but* $\{a\} \notin S_\alpha O(X)$ *and* $\{b, c\} \in S_\alpha O(X)$ *and* $\{b, c\} \notin \tau$

Lemma 2.2. *If A is dense in a space X, then $S_\alpha Cl(A) = X$.*

Proof. Suppose that $S_\alpha Cl(A) = F \subset X$. Then F is semi-closed since F is S_α-closed. Therefore, $Int(Cl(A)) \subseteq Int(Cl(F)) \subseteq F$. □

Theorem 2.1. *The S_α-closure of a dense subset of a connected space is connected.*

Proof. Follows from Lemma 2.2. □

Definition 2.5. A topological space X is said to be

(1) locally indiscrete [6] if every open subset of X is closed.
(2) hyperconnected [6] if every nonempty open subset of X is dense in X.

Theorem 2.2. *If X is a hyperconnected space, then the S_α-closure of any non-empty open set is X.*

Proof. Follows from Lemma 2.2. □

Definition 2.6. Let X and Y be two topological spaces. A function f from X to Y is S_α-continuous at a point $x \in X$ if for each open set V in Y containing $f(x)$, there exists an S_α- open set U in X containing x such that $f(U) \subseteq V$. If f is S_α-continuous at every point x of X, then it is called S_α-continuous on X.

Theorem 2.3. *Let X and Y be two topological spaces. A function f from X to Y is S_α-continuous if and only if the inverse image of every open set in Y is S_α-open in X.*

Proof. Let V be any open set in Y. If $f^{-1}(V) = \emptyset$, then it is obviously S_α-open. If $f^{-1}(V) \neq \emptyset$, then for any $x \in f^{-1}(V), f(x) \in V$. Since f is S_α-continuous, there exists an S_α open set U in X containing x such that $f(U) \subseteq V$. Then $f^{-1}(V)$, being the union of S_α-open sets, is S_α-open in X. The converse follows from the definition. □

Definition 2.7. Let X and Y be two topological spaces. A function f from X to Y is semi-continuous [7](α-continuous [5, 8], α_β-continuous [14]) at a point $x \in X$ if for each open set V in Y containing $f(x)$, there exists a semi-open(resp. α-open, α_β-open) set U in X containing x such that $f(U) \subseteq V$. If f is semi-continuous (α-continuous, α_β-continuous) at every point x of X, then it is called semi-continuous (resp., α-continuous, α_β-continuous) on X.

3. S_α-Connected Space

Definition 3.1. Two nonempty subsets A and B of a topological space X are said to be

(1) S_α-separated if $A \cap S_\alpha Cl(B) = \emptyset = S_\alpha Cl(A) \cap B$.

(2) α-separated [4] if $A \cap \alpha Cl(B) = \emptyset = \alpha Cl(A) \cap B$.

(3) semi-separated [11] if $A \cap SCl(B) = \emptyset = SCl(A) \cap B$.

(4) β-separated [4, 9, 12] if $A \cap \beta Cl(B) = \emptyset = \beta Cl(A) \cap B$.

(5) α_β-separated [14] if $A \cap \alpha_\beta Cl(B) = \emptyset = \alpha_\beta Cl(A) \cap B$.

It is obvious that two S_α-separated sets are disjoint. If A and B are two S_α-separated sets in X with $\emptyset \neq C \subset A$ and $\emptyset \neq D \subset B$. Then C and D are also S_α-separated sets in X.

The following example constructs S_α- separated sets:

Example 3.1. *In Example 2.1, let $A = \{a\}$ and $B = \{b\}$.Then $S_\alpha Cl(A) = A$ and $S_\alpha Cl(B) = B$. Therefore, $A \cap S_\alpha Cl(B) = S_\alpha Cl(A) \cap B = A \cap B = \emptyset$. Thus, A and B are S_α-separated sets.*

Definition 3.2. A subset S of a topological space X is said to be

(1) S_α-connected in X if S is not the union of two S_α-separated sets in X.
(2) α-connected [4] in X if S is not the union of two α-separated sets in X.
(3) semi-connected [11] in X if S is not the union of two semi-separated sets in X.
(4) β-connected [4, 9, 12] in X if S is not the union of two β-separated sets in X.
(5) α_β-connected [14] in X if S is not the union of two α_β-separated sets in X.

Example 3.2. (1) *In Example 2.1, X cannot be expressed as the union of two S_α-separated sets in X. Thus, X is S_α-connected.*
(2) *An infinite space with cofinite topology is S_α-connected.*

Theorem 3.1. *A topological space X is S_α-connected if and only if X cannot be expressed as the union of two disjoint nonempty S_α-open subsets of X.*

Proof. Let X be S_α-connected, and A and B be two disjoint nonempty S_α-open subsets of X such that $X = A \cup B$. Then A and B are S_α-closed in X. Thus, $A \cap S_\alpha Cl(B) = \emptyset = S_\alpha Cl(A) \cap B$. Then X is not S_α-connected, a contradiction.

Conversely, suppose that $X = A \cup B$, $A \neq \emptyset \neq B$ and $A \cap S_\alpha Cl(B) = \emptyset = S_\alpha Cl(A) \cap B$. Then A and B are nonempty disjoint S_α-open sets, a contradiction. Thus, X is S_α-connected. □

Lemma 3.1. *A clopen subset of a topological space X is both S_α-open and S_α-closed.*

Proof. Let A be both open and closed in X. Then A is semi-open and α-closed. Then A is S_α-open. Similarily, $X \setminus A$ is also S_α-open. □

Example 3.3. *Let $\mathbb{R}$ be a space with usual topology. Then (a, b) is both S_α-open and S_α-closed set but it is not a clopen set.*

Theorem 3.2. *For a topological space X, the following are equivalent:*

(1) *X is S_α-connected.*
(2) *The only subsets of X which are both S_α-open and S_α-closed are X and the empty set.*
(3) *There is no nonconstant onto S_α-continuous function from X to a discrete space which contains more than one point.*

Proof. (1)⇒(2). Follows from Theorem 3.1.
(2)⇒(3). Let Y be a discrete space with more than one point, and $f : X \to Y$ be onto S_α-continuous function. Let $y \in Y$ and $A = \{y\}$. Since $f : X \to Y$ is S_α-continuous and onto, by Theorem 2.3, $f^{-1}(A)$ is nonempty, S_α-open and S_α-closed subset in X. Since $f^{-1}(A)$ is nonempty, $f^{-1}(A) = X$. That is, f is constant.
(3)⇒(1). Suppose that X is not S_α-connected. If $X = A \cup B$, where A and B are nonempty subsets of X such that $S_\alpha Cl(A) \cap B = \emptyset$ and $S_\alpha Cl(B) \cap A = \emptyset$. Then A and B both are S_α-open sets in X. Assume that $Y = \{0, 1\}$ with discrete topology. We define a map f from X to Y by $f(x) = 0$ if $x \in A$ and $f(x) = 1$ if $x \in B$. Then f is nonconstant S_α-continuous and onto mapping, a contradiction to (3). □

Lemma 3.2. [17] *If X is hyperconnected, then $S_\alpha O(X) \cap S_\alpha C(X) = \{\emptyset, X\}$.*

Theorem 3.3. *If a space X is hyperconnected, then it is S_α-connected.*

Proof. Follows from Lemma 3.2 and Theorem 3.2. □

Remark 1. In Theorem 3.3, we have essentially proved that if a topological space (X, τ) is hyperconnected, then the generalized topological space (X, μ), where, $\mu = S_\alpha O(X)$ is μ-connected in the sense of Császár [2] and Tyagi et al. [16]. However, if we take any arbitrary generalized topology μ on X which is finer than $S_\alpha O(X)$ or incomparable with it, then the generalized topological space (X, μ) need not be μ-connected even if $\tau \subseteq \mu$. For example, let X be a countable infinite set with cofinite topology τ. Then for a fixed element $a \in X$, $\mu = \tau \cup \{\{a\}\}$ is a generalized topology on X. Now (X, τ) is hyperconnected but (X, μ) is not μ-connected since $\{a\}$ is μ-clopen.

In contrast to connectedness of topological spaces, if a topology τ_1 is finer than the topology τ_2, then S_α-connectedness of (X, τ_2) does not imply the S_α-connectedness of (X, τ_1).

Example 3.4. *Let $X = \{a, b, c\}$ and $\tau_1 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, X\}$ and $\tau_2 = \{\emptyset, \{a, b\}, X\}$. Then $\tau_2 \subseteq \tau_1$. Now in $(X, \tau_1), SO(X) = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}, \{a, c\}, X\}$ and $S_\alpha O(X) = \{\emptyset, \{b\}, \{b, c\}, \{a, c\}, X\}$ and in $(X, \tau_2), SO(X) = \{\emptyset, \{a, b\}, X\}$ and $S_\alpha O(X) = \{\emptyset, X\}$. So (X, τ_2) is S_α-connected but (X, τ_1) is not S_α-connected as $\{b\}$ and $\{a, c\}$ are S_α-separated sets or S_α-separation of X in τ_1.*

Theorem 3.4. [14] *A topological space X is α_β-connected if and only if X is connected.*

Theorem 3.5. *If a space X is S_α-connected, then it is connected.*

Proof. Let X be S_α-connected. Then the only subsets of X which are both S_α-open and S_α-closed in X are $\emptyset$ and X. Suppose that X is not connected, then there is a nonempty proper subset A of X which is both open and closed. By Lemma 3.1, A is both S_α-open and S_α-closed in X, a contradiction. □

Example 3.5. *Let $\mathbb{R}$ be a space with usual topology. Then $\mathbb{R}$ is connected but the sets (a, b) and $\mathbb{R} \setminus (a, b)$ constitute a S_α-separation on X.*

Theorem 3.6. [4] *A topological space X is connected if and only if X is α-connected.*

Theorem 3.7. [14] *A topological space X is α_β-connected if and only if X is α-connected.*

Theorem 3.8. *If a space X is semi-connected, then it is S_α-connected.*

Proof. Follows from the fact that $S_\alpha O(X) \subseteq SO(X)$. □

Theorem 3.9. [17] *If a space X is T_1 or locally indiscrete, then $S_\alpha O(X) = SO(X)$.*

Corollary 3.1. *A T_1-space or locally indiscrete space X is semi-connected if and only if X is S_α-connected.*

Theorem 3.10. *If a space X is locally indiscrete, then $S_\alpha O(X) = SO(X) = \tau$.*

Proof. It is sufficient to show that every semi-open set is open in X. Let A be a semi-open set in X. Then $A \subseteq Cl(Int(A))$. Since X is locally indiscrete, $Cl(Int(A)) = Int(A)$. □

Corollary 3.2. *A locally indiscrete space X is connected if and only if X is S_α-connected.*

Here we consider finite α-topology, that is, a topology with finite number of α-open sets.

Theorem 3.11. *Let X be a space with finite α-topology. Then X is α-connected if and only if it is S_α-connected.*

Proof. Suppose that X is not S_α-connected. Then there are S_α-separated sets A and B such that $X = A \cup B$. Then A and B are α-closed, a contradiction. □

Corollary 3.3. *Let X be a space with finite α-topology. Then X is connected if and only if it is S_α-connected.*

Remark 2. In Corollary 3.3, we have proved that μ-connectedness and connectedness are equivalent in case of finite α-topology, where $\mu = S_\alpha O(X)$. However, if we take any arbitrary generalized topology μ on X which is finer than $S_\alpha O(X)$ or incomparable with it, then the generalized topological space (X, μ) need not be μ-connected even if $\tau \subseteq \mu$. For example, Let (X, τ) be an indisrete topological space and $\mu = \{\emptyset, A, X - A, X\}$. Then (X, τ) is connected but (X, μ) is not μ-connected since A is μ-clopen. Here $\alpha O(X, \tau) = \{\emptyset, X\}$ and τ is finite α-topology.

Theorem 3.12. [10] *The α-sets with respect to a given topology are exactly those sets which may be written as a difference between an open set and a nowhere dense set.*

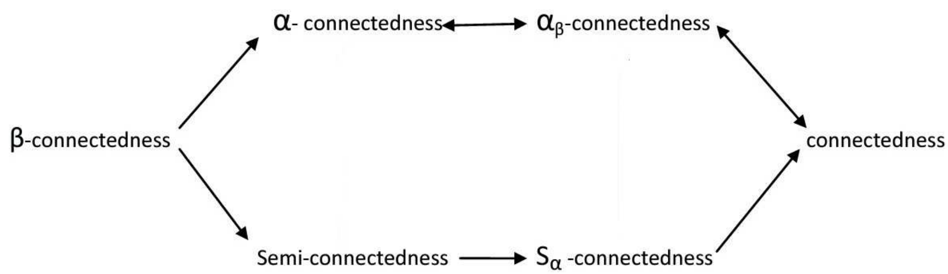
Theorem 3.13. *If every closed set in a space X contains some nonempty open set, then α-topology of X concides with the original topology.*

Proof. Follows from Theorem 3.12. □

Theorem 3.14. *Let X be a space with finite topology and every closed set in X contains some nonempty open set. Then X is S_α-connected if and only if it is connected.*

Proof. Follows from Theorem 3.13 and Corollary 3.3. □

We give a moderate figure for relationships of various strong forms of connectedness.



Theorem 3.15. *Generalization of Intermediate value theorem: Let* $f: X \to \mathbb{R}$ *be a* S_α*-continuous map from a* S_α*-connected space* X *to the real line* $\mathbb{R}$*. If* x *and* y *are two points of* X *such that* $a = f(x)$ *and* $b = f(y)$*, then every real number* r *between* a *and* b *is attained at a point in* X*.*

Proof. Suppose that there is no point $c \in X$, such that $f(c) = r$. Then $A = (-\infty, r)$ and (r, ∞) are disjoint open sets in $\mathbb{R}$. Since f is S_α-continuous, $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint S_α-open sets in X and $X = f^{-1}(A) \cup f^{-1}(B)$, a contradiction. □

4. Properties of S_α-connected sets

The S_α-closure of a subset A in a space X may be distinct from the closure of A. Thus, the results in the following section can not be inferred from the known corrosponding results for connectedness, however the proofs are parallel.

Theorem 4.1. *If* A *is a* S_α*-connected set of a topological space* X *and* U, V *are* S_α*-separated sets of* X *such that* $A \subseteq U \cup V$*, then either* $A \subseteq U$ *or* $A \subseteq V$*.*

Proof. Since $A = (A \cap U) \cup (A \cap V)$, we have $(A \cap U) \cap S_\alpha Cl(A \cap V) \subseteq U \cup S_\alpha Cl(V) = \emptyset$. If $A \cap U$ and $A \cap V$ are nonempty, then A is not S_α-connected, a contradiction. Therefore, either $A \subseteq U$ or $A \subseteq V$. □

Theorem 4.2. *If A is S_α-connected set of a topological space X and $A \subseteq N \subseteq S_\alpha Cl(A)$, then N is S_α-connected.*

Proof. Assume that N is not S_α-connected set. Then there exist S_α-separated sets U and V such that $N = U \cup V$.By Theorem 4.1, either $A \subseteq U$ or $A \subseteq V$. If $A \subseteq U$, then $S_\alpha Cl(A) \cap V = \emptyset$, a contradiction. The proof is now complete. □

Corollary 4.1. *If A is a S_α-connected subset of a topological space X, then $S_\alpha Cl(A)$ is S_α-connected.*

Theorem 4.3. *Let A and B be subsets of a topological space X. If A and B are S_α-connected and not S_α-separated, then $A \cup B$ is S_α-connected.*

Proof. Suppose that $A \cup B$ is not S_α-connected. Then there are S_α-separated sets C and D in X such that $A \cup B = C \cup D$. By Theorem 4.1, either $A \subseteq C$ or $A \subseteq D$ and $B \subseteq C$ or $B \subseteq D$. If $A \subseteq C$ and $B \subseteq C$, then $(A \cup B) \subseteq C$ and $D = \emptyset$, a contradiction. If $A \subseteq C$ and $B \subseteq D$, then A and B are S_α-separated sets in X, a contradiction. □

Theorem 4.4. *If $\{B_\gamma; \gamma \in \Gamma\}$ is a nonempty family of S_α-connected subsets of a topological space X such that $\bigcap B_\gamma \neq \emptyset$, then $\bigcup B_\gamma$ is S_α-connected.*

Proof. Suppose that $N = \bigcup B_\gamma$ and N is not S_α-connected. Then $N = U \cup V$, where U and V are S_α-separated sets in X. Since $\bigcap B_\gamma \neq \emptyset$, there is a point x in $\bigcap B_\gamma$. Since $x \in N$, either $x \in U$ or $x \in V$. Suppose that $x \in U$. Since $x \in \bigcap B_\gamma$, B_γ and U intersect for each γ. By Theorem 4.1, B_γ must be in U for each $\gamma \in \Gamma$. Then $N \subseteq U$, a contradiction. The proof now follows. □

Theorem 4.5. *If $\{A_n; n \in \mathbb{N}\}$ is an infinite sequence of S_α-connected subsets of a topological space X and $A_n \cap A_{n+1} \neq \emptyset$ for each $n \in \mathbb{N}$, then $\bigcup A_n$ is S_α-connected.*

Proof. The proof follows by induction and Theorem 4.4. □

5. S_α-Connectedness and Mappings

Definition 5.1. Let X and Y be two topological spaces. A function f from X to Y is said to be S_α-irresolute if the inverse image of every S_α-open set in Y under f is S_α-open in X.

Theorem 5.1. *Let f be a S_α-irresolute function from space X onto a space Y. If X is S_α-connected, then Y is S_α-connected.*

Proof. Suppose that Y is not S_α-connected. Then there is a nonempty proper subset A of Y which is both S_α-open and S_α-closed. Then inverse image of A under f is both S_α-open and S_α-closed in X, a contradiction. □

Corollary 5.1. *A S_α-irresolute function maps S_α-connected set onto connected set.*

Theorem 5.2. *Let f be a S_α-continuous function from a space X onto a space Y. If X is S_α-connected, then Y is connected.*

Proof. Suppose Y is not connected. Then there is A nonempty proper subset of X which is both open and closed. Then the inverse image of A under f is both S_α-open and S_α-closed in X, a contradiction. □

Though the concept of continuity, S_α-continuity and S_α-irresolute are independent of each other but they behave similarly in case of S_α-connectedness, that is, these functions map a S_α-connected set onto a connected set.

Theorem 5.3. *If (X, τ) is locally indiscrete space and Y is any space. Then for any function $f : X \to Y$ the following statements are equivalent:*

(1) *f is continuous.*
(2) *f is α-continuous.*
(3) *f is semi-continuous.*

(4) *f is α_β-continuous.*

(5) *f is S_α-continuous.*

Proof. Follows from the fact that in a locally indiscrete space (X, τ), $\alpha_\beta O(X) = \alpha O(X) = \tau = SO(X) = S_\alpha O(X)$. □

Theorem 5.4. *Let X be a locally indiscrete S_α-connected space. Then it has indiscrete topology.*

Corollary 5.2. *A locally indiscrete space X is S_α-connected space if and only if it is hyperconnected.*

Obviously, continuity implies semi-continuity.

Theorem 5.5. *If X is a T_1-space, then every semi-continuous function from X to a space Y is S_α-continuous.*

Proof. Follows from Theorem 3.9. □

Theorem 5.6. *If X is a T_1 space, then every S_α-irresolute function from X to any space is S_α-continuous and also semi-continuous.*

Proof. Since X is $T_1, \tau \subseteq S_\alpha O(X) = SO(X)$. The proof is now immediate. □

Theorem 5.7. *Let f be a function from a space X to a space Y. If f is S_α-continuous, then it is semi-continuous.*

Proof. Follows from the fact that $S_\alpha O(X) \subseteq SO(X)$. □

Definition 5.2. A bijective function f from (X, τ) to (Y, μ) is said to be S_α-homeomorphism if f and f^{-1} both are S_α-irresolutes.

Definition 5.3. [3] A bijective function f from a space X to a space Y is said to be semi-homeomorphism if

(1) f is semi-irresolutes (i.e. $f^{-1}(V)$ is semi-open in X for each semi-open V in Y)

(2) f is pre-semi-open map (i.e. images of semi-open sets are semi-open).

Theorem 5.8. *A S_α-homeomorphism preserves S_α-connectedness.*

Theorem 5.9. [3] *Every homeomorphism is a semi-homeomorphism.*

Theorem 5.10. *Every homeomorphism is S_α-homeomorphism.*

Proof. Let f be any homeomorphism from X to Y. It is sufficient to show that f is S_α-irresolute. Suppose that $V \in S_\alpha O(Y)$ and $f^{-1}(V) = U$. Then by Theorem 5.9, U is semi-open in X, since V is semi-open. Let $x \in U$ be arbitrary. Then $f(x) \in f(U) = V$. Since $V \in S_\alpha O(Y)$, there is an α-closed set W in Y such that $f(x) \in W \subseteq V$. It implies that $x \in f^{-1}(W) \subseteq f^{-1}(V)$ and $f^{-1}(W)$ is α-closed, since f is a homeomorphism. □

Theorem 5.11. *A homeomorphism preserves S_α-connectedness.*

Proof. Follows from Theorem 5.8 and Theorem 5.10. □

Theorem 5.12. *If X is S_α-connected space, then $X \times \{a\}$ is also S_α-connected.*

Proof. Obviously, X is homeomorphic to $X \times \{a\}$. Then by Theorem 5.11, $X \times \{a\}$ is S_α-connected. □

Theorem 5.13. *If X and Y are two S_α-connected spaces, then $X \times Y$ is also S_α-connected.*

Proof. For any point (a, b) in the product $X \times Y$, by Theorem 4.3 and Theorem 5.12, each of the subspace $X \times \{b\} \cup \{x\} \times Y$ is S_α-connected since it is the union of two S_α-connected subspaces with a point in common. Then by Theorem 4.4, $X \times Y$ is S_α-connected. □

Theorem 5.14. *Let X_β, $\beta \in A$, be a family of spaces. If $\prod X_\beta$ is S_α-connected, then each X_β is connected.*

Proof. Let $\prod X_\beta$ is S_α-connected. Then $\prod X_\beta$ is connected. Since the projection $p_\gamma : \prod X_\beta \to X_\gamma$ is a continuous map. So each X_β is connected. □

The following example shows that the converse of Theorem 5.14 is not true.

Example 5.1. *Let $X = \mathbb{R}$ with usual topology on it. Then $\mathbb{R}$ is connected but $\prod \mathbb{R}_\beta$ is not S_α-connected.*

Theorem 5.15. *Let X_β, $\beta \in A$, be a family of spaces. If each X_β is S_α-connected, then $\prod X_\beta$ is connected.*

Proof. The proof follows from the fact that S_α-connectedness implies connectedness and each X_β is connected if and only if $\prod X_\beta$ is connected. □

Example 5.2. *Let $X = \mathbb{R}$ with usual topology on it. Then $\prod \mathbb{R}_\beta$ is connected but each $\mathbb{R}_\beta$ is not S_α-connected.*

Theorem 5.16. *If $f : X \to Y$ is α-homeomorphism and semi-homeomorphism, then f is S_α-homeomorphism.*

Proof. Let $V \in S_\alpha O(Y)$. Since f is a semi-homeomorphism, $f^{-1}(V)$ is semi-open in X. There exist a α-closed set F in Y such that $f(x) \in F \subseteq V$. Then $x \in f^{-1}(F) \subseteq f^{-1}(V)$. Since f is α-homeomorphism, $f^{-1}(F)$ is α-closed set in X. Thus, $f^{-1}(V) \in S_\alpha O(X)$. Now let $U \in S_\alpha O(X)$. Since f is semi-homeomorphism, $f(U)$ is semi-open in Y. There exist a α-closed set E in X such that $x \in E \subseteq U$. Then $f(x) \in f(E) \subseteq f(U)$. Since f is α-homeomorphism, $f(E)$ is α-closed set in Y. Thus, $f(V) \in S_\alpha O(Y)$. □

Corollary 5.3. *If $f : X \to Y$ is α-homeomorphism and semi-homeomorphism, then f preserves S_α-connectedness.*

6. S_α-COMPONENTS

Definition 6.1. Let x be any element of a space X. The S_α-component containing x, $C_{S_\alpha}(x)$, is the union of all S_α-connected subsets of X which contain x.

By Theorem 4.4, the component $C_{S_\alpha}(x)$ is S_α-connected and hence connected. It follows from its definition that $C_{S_\alpha}(x)$ is not properly contained in any S_α-connected subset of X. Thus, $C_{S_\alpha}(x)$ is a maximal S_α-connected subset of X.

Lemma 6.1. *If A is S_α-component of X containing x, then it is contained in component of X containing x.*

Theorem 6.1. *Let X be a space. Then:*

(1) *Each S_α-component of X is S_α-closed.*
(2) *Each S_α-connected subset of X is contained in a S_α-component of X.*
(3) *The set of all S_α-components of X forms a partition of X.*

Proof. (1) If $C_{S_\alpha}(x)$ is a S_α-component containing x in X, then $C_{S_\alpha}(x)$ is S_α-connected. So $S_\alpha Cl(C_{S_\alpha}(x))$ is also S_α-connected. By the maximality of $C_{S_\alpha}(x)$, we have $C_{S_\alpha}(x) = S_\alpha Cl(C_{S_\alpha}(x))$. Thus, $C_{S_\alpha}(x)$ is S_α-closed in X.

(2) If A is a nonempty S_α-connected subset of X, then $A \subset C_{S_\alpha}(a)$ for each a in A.

(3) For $x \in X, \{x\}$ is S_α-connected. Then there is a S_α-component $U \subseteq X$ containing x. So X will be contained in the union of S_α-components. Let C_1 and C_2 be two distinct S_α-components such that $C_1 \cap C_2 \neq \emptyset$. Then $C_1 \cup C_2$ is S_α-connected, which contradicts the fact that C_1 and C_2 are S_α-components. Therefore C_1 and C_2 are disjoint. Thus, the S_α-components constitute a partition of X.

□

Theorem 6.2. *If X has finite number of components, then each component is both S_α-open and S_α-closed.*

Proof. If a space X has finite number of components, then each component is both open and closed and hence by Lemma 3.1, both S_α-open and S_α-closed. □

Theorem 6.3. *If X has finite number of S_α-components, then each S_α-component is both S_α-open and S_α-closed.*

Theorem 6.4. [17] *If the family of semi-open subsets of a topological space X forms topology on X, then the family of S_α-open sets also forms topology on X.*

Theorem 6.5. *Let the family of semi-open subsets of a topological space X be a topology on X. If X has finite number of S_α-components, then each component is both S_α-open and S_α-closed.*

Proof. If the space X has finite number of S_α-components, then each S_α-component is the complement of the union of other S_α-components and hence by Theorem 6.1 and Theorem 6.4, both S_α-open and S_α-closed. □

Theorem 6.6. *If $f : X \to Y$ is S_α-continuous or S_α-irresolute and $C(x)$ is the component containing x in X, then $f(C_{S_\alpha}(x)) \subseteq C(f(x))$.*

Proof. Follows from Corollary 5.1 and Theorem 5.2. □

Corollary 6.1. *If f is S_α-homeomorphism, then $f(C_{S_\alpha}(x)) = C_{S_\alpha}(f(x))$.*

Corollary 6.2. *If f is homeomorphism, then $f(C_{S_\alpha}(x)) = C_{S_\alpha}(f(x))$.*

Theorem 6.7. *Let X be a space with finite α-topology on it and f be S_α-homeomorphism. Then $f(C(x)) = C(f(x))$*

Theorem 6.8. *A S_α-connected, S_α-open and S_α-closed subset A of a space X is a S_α-component of X.*

Proof. If possible, suppose that A is not a S_α-component of X. Then $A \subset B$ and B is S_α-component of X. By Theorem 6.1, B is S_α-closed. Therefore, $B \setminus A$ is S_α-closed. Then A and $B \setminus A$ constitute a S_α-separation of B, a contradiction. □

Theorem 6.9. *A S_α-connected, S_α-open and S_α-closed subset A of a space X with finite α-topology, is a component of X.*

Definition 6.2. A space X is called locally S_α-connected at $x \in X$ if for each S_α-open set U containing x, there is a S_α-connected S_α-open set V such that $x \in V \subseteq U$. The space X is locally S_α-connected if it is locally S_α-connected at each of its points.

Theorem 6.10. *A space X is locally S_α-connected if and only if the S_α-components of each S_α-open subset of X are S_α-open.*

Proof. Suppose that X is locally S_α-connected. Let U be an S_α-open subset of X and C be a S_α-component of U. If $x \in C$, then there is a S_α-connected S_α-open set $V \subseteq X$ such that $x \in V \subseteq U$. Since C is a S_α-component of U and V is a S_α-connected subset of U containing x, $V \subseteq C$. Thus, C is a S_α-open set. Conversely, let $U \subseteq X$ be a S_α-open set, and $x \in U$. By our hypothesis, the S_α-component V of U containing x is S_α-open, so X is locally S_α-connected at x. □

Theorem 6.11. *Let $f : X \to Y$ be a S_α-irresolute, S_α-closed surjection. If X is locally S_α-connected, then Y is locally S_α-connected.*

Proof. Suppose that X is locally S_α-connected. Let C be a S_α-component of $U \in S_\alpha O(Y)$, and let $x \in f^{-1}(C)$. Then there exists a S_α-connected S_α-open set V in X such that $x \in V \subseteq f^{-1}(U)$, since X is locally S_α-connected and $f^{-1}(U)$ is S_α-open in X. It follows that $f(x) \in f(V) \subseteq C$ for $f(V)$ is S_α-connected. So, by Theorem 6.10, $x \in V \subseteq f^{-1}(C)$, and $f^{-1}(C) \in S_\alpha O(X)$. Since f is S_α-closed surjection, $Y \setminus C = f(X \setminus f^{-1}(C))$ is S_α-closed. □

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