ON THE DIAMETER OF ZERO-DIVISOR GRAPHS OF IDEALIZATIONS WITH RESPECT TO INTEGRAL DOMAIN

MANAL AL-LABADI

Abstract. Let \( R \) be a ring with unity and let \( M \) be an \( R \)-module. Let \( R(M) \) be the idealization of the ring \( R \) by the \( R \)-module \( M \). In this paper, we give new results on the diameter of \( \Gamma(R(M)) \) when \( R \) is an integral domain.

1. Introduction

The zero divisor graph of a ring is the (simple) graph whose vertex set is the set of non-zero zero divisors, and an edge is drawn between two distinct vertices if their product is zero. The zero divisor graph of a commutative ring has been studied extensively by several authors, see [1,2,3 and 4]. Let \( R \) be a commutative ring with unity. We use the notation \( A^* \) to refer to the nonzero elements of \( A \). For two distinct vertices \( a \) and \( b \) in a graph \( \Gamma(R) \), the distance between \( a \) and \( b \), denoted by \( d(a,b) \), is the length of the shortest path connecting \( a \) and \( b \), if such a path exists, otherwise, \( d(a,b) = \infty \). The diameter of a graph \( \Gamma(R) \) is \( \text{diam}(\Gamma) = \sup \{d(a,b) : a \text{ and } b \text{ are distinct vertices of } \Gamma\} \). We will use the notation \( \text{diam}(\Gamma(R)) \) to denote the diameter of the \( \theta \) nonzero zero divisors of \( R \). Let

Key words and phrases. : Zero - divisor graphs, Idealization rings.
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Received: Nov. 16, 2009. Accepted : Feb. 8, 2010.
$M$ be an $R$-module. Consider $R(+)M = \{(a, m) : a \in R, m \in M\}$ and let $(a, m)$ and $(b, n)$ be two elements of $R(+)M$. Define $(a, m) + (b, n) = (a + b, m + n)$ and $(a, m)(b, n) = (ab, an + bm)$. Under this definition $R(+)M$ becomes a commutative ring with unity. Call this ring the idealization ring of $M$ in $R$. For more details, we refer the reader to [5].

The set of all nonzero zero divisors of a ring $R$ is denoted by $Z^*(R)$.

**Lemma 1.1.** Let $R$ be an integral domain such that $Z_2$ is an $R$-module with $\text{ann}(Z_2) = \{0\}$. Then $R \cong Z_2$.

*Proof.* Now $1_R \cdot 1 = 1$ since $Z_2$ is unitary $R$-module. So, $(1_R + 1_R) \cdot 1 = 1_R \cdot 1 + 1_R \cdot 1 = 1 + 1 = 0$. Hence $2 \cdot 1_R = 1_R + 1_R \in \text{ann}(Z_2) = \{0\}$. Hence $\text{ch}(R) = 2$. Moreover, $r \cdot 1 = 1$ for all $r \in R^*$ because $\text{ann}(Z_2) = \{0\}$. Assume that $r \in R - \{0_R, 1_R\}$. Then $(r + 1_R) \cdot 1 = 1 + 1 = 0$, hence $r = -1_R = 1_R$, a contradiction. Then $R \cong Z_2$. \hfill $\square$

**Lemma 1.2.** Let $R$ be an integral domain such that $Z_3$ is an $R$-module with $\text{ann}(Z_3) = \{0\}$. Then $R \cong Z_3$.

*Proof.* $0, 1_R \in R$ and $1_R \neq 0_R$. Then $2 \cdot 1_R \in R$. If $2 \cdot 1_R = 0_R$, then $3 \cdot 1_R \cdot 1 = 0$ because $3 \cdot 1_R \cdot 1 = (1_R + 1_R + 1_R) \cdot 1 = 1 + 1 + 1 = 0$. So, $0 = 3 \cdot 1_R \cdot 1 = (2 \cdot 1_R + 1_R) \cdot 1 = 2 \cdot 1_R + 1_R \cdot 1 = 0 + 1 = 1$, contradiction. So, $2 \cdot 1_R \neq 0_R$. If $2 \cdot 1_R = 1_R$, then $1_R = 0_R$ since $(R, +)$ is abelian group. So, $R$ has at least 3 element $0_R, 1_R$ and $2 \cdot 1_R$. Assume that there exist, $r \in R - \{0_R, 1_R, 21_R\}, r \in R$. If $r \cdot 1 = 0$, then $r \in \text{ann}(Z_3)$. Hence $r = 0_R$, contradiction. If $r \cdot 1 = 1$, then...
(r + 21R).1 = r.1 + 21R.1 = r.1 + 1 + 1 = 1 + 1 + 1 = 0. So, r + 21R = 0R since \(\text{ann}(\mathbb{Z}_3) = \{0\}\). Then \(r = -21R = 1_R\), contradiction. If \(r.1 = 2\), then \((r + 1R).1 = 0\).

So, \(r + 1R = 0_R\) i.e \(r = -1R = 21R\), contradiction. Thus \(R = \{0_R, 1_R, 21R\}\) and

\[
\begin{array}{ccc}
0_R & 1_R & 21R \\
R & R & R \\
21R & 0_R & 21R \\
1_R & 0_R & 1_R \\
\end{array}
\]

Then \(R \cong \mathbb{Z}_3\) (as a ring).

\[\square\]

**Theorem 1.3.** Let \(R\) be an integral domain and \(M \cong \mathbb{Z}_2\) be an \(R\)-module.

(i): If \(\text{ann}(\mathbb{Z}_2) = \{0\}\), then \(\text{diam}(\Gamma(R(+)\mathbb{Z}_2)) = 0\).

(ii): If \(\text{ann}(\mathbb{Z}_2) \neq \{0\}\), then \(\text{diam}(\Gamma(R(+)\mathbb{Z}_2)) = 2\).

**Proof.** (i) \(M \cong \mathbb{Z}_2\), and \(\text{ann}(\mathbb{Z}_2) = \{0\}\). Then, by Lemma 1.1, \(R \cong \mathbb{Z}_2\). Then \(Z^*(\mathbb{Z}_2(+)\mathbb{Z}_2) = \{(0, 1)\}\), so \(\text{diam}(\Gamma(R(+)\mathbb{Z}_2)) = 0\).

(ii) \(M \cong \mathbb{Z}_2\) and \(\text{ann}(\mathbb{Z}_2) \neq \{0\}\). Then there exists at least one element in \(R^*\) such that \(r.1 = 0\). \(Z^*(R(+)\mathbb{Z}_2) = \{(0, 1)\} \cup \{(r, 0), (r, 1), \ldots\}\). Any two elements in \(\{(r, 0), (r, 1), \ldots\}\) are non adjacent, but \((r, 0).(0, 1) = (0, 0)\) and \((0, 1).(r, 1) = (0, 0)\). \((r, 0)\) — \((0, 1)\) — \((r, 1)\) so, \(\text{diam}(\Gamma(R(+)\mathbb{Z}_2)) = 2\). \[\square\]

**Theorem 1.4.** Let \(R\) be an integral domain and \(M \cong \mathbb{Z}_3\) be an \(R\)-module.

(i): If \(\text{ann}(\mathbb{Z}_3) = \{0\}\), then \(\text{diam}(\Gamma(R(+)\mathbb{Z}_3)) = 1\).

(ii): If \(\text{ann}(\mathbb{Z}_3) \neq \{0\}\), then \(\text{diam}(\Gamma(R(+)\mathbb{Z}_3)) = 2\).

**Proof.** (i) \(M \cong \mathbb{Z}_3\), \(\text{ann}(\mathbb{Z}_3) = \{0\}\). Then, by Lemma 1.2, \(R \cong \mathbb{Z}_3\). So, \(Z^*(\mathbb{Z}_3(+)\mathbb{Z}_3) = \{(0, 1), (0, 2)\}\).
Thus, $\text{diam}(\Gamma(R(+)\mathbb{Z}_3)) = 1$. $\Gamma(R(+)\mathbb{Z}_3)$ is $(0,1)-(0,2)$.

(ii) $M \cong \mathbb{Z}_3$ and $\text{ann}(\mathbb{Z}_3) \neq \{0\}$. Then there exists at least one element in $R^*$ such that $r.\mathbb{Z}_3 = 0$. So, $Z^*(R(+)\mathbb{Z}_3) = \{(0,1),(0,2)\} \cup \{(r,0),(r,1),(r,2),...\}$. Any two elements in $\{(r,0),(r,1),(r,2),...\}$ are non adjacent since $r.s \neq 0$ for any $r,s \in R^*$. But $(r,0)-(0,1)-(r,2)$. Hence, $\text{diam}(\Gamma(R(+)\mathbb{Z}_2)) = 2$. □

**Theorem 1.5.** Let $R$ be an integral domain and $|M| \geq 4$, be an $R$-module.

(i): If $r.m \neq 0$, for any $r \in R^*$ and $m \in M^*$, then $\text{diam}(\Gamma(R(+)M)) = 1$.

(ii): If there exists at least one element $m \in M^*$ such that $r.m = 0$, for any $r \in R$,
then $\text{diam}(\Gamma(R(+)M)) = 2$.

(iii): If there exists at least two elements in $R^*$ such that $r_1.m = 0$, $r_2.n = 0$,
$r_1.n \neq 0$, $r_2.m \neq 0$, $m \neq n$, for $m,n \in M^*$, then $\text{diam}(\Gamma(R(+)M)) = 3$.

**Proof.** (i) If there is no element $r \in R^*$ such that $r.m = 0$, for any $m \in M^*$, then $Z^*(R(+)M) = \{(0,m) : m \in M^*\}$. Then any two elements in $\{(0,m) : m \in M^*\}$ are at distance 1.

(ii) If there exists at least one element $m \in M^*$ such that $r.m = 0$, for any $r \in R$,
then $Z^*(R(+)M) = \{(0,m) : m \in M^*\} \cup \{(r,m),... : m \in M\}$. Then any two elements in $\{(r,m),... : m \in M\}$ are non adjacent. But $(r,n).(0,m) = (0,0)$, and $(0,m).(0,n) = (0,0)$. Then $(r,n)-(0,m)-(0,n)$, $m \neq n$, and $m,n \in M^*$. Then $\text{diam}(\Gamma(R(+)M)) = 2$.

(iii) If there exists two elements $r_1, r_2 \in R^*$ such that $r_1.m = 0$, $r_2.n = 0$, $m \neq n$, and $m,n \in M^*$, then $Z^*(R(+)M) = \{(0,m) : m \in M^*\} \cup \{(r_1,m),(r_2,n),... : m \neq n, m,n \in M^*\}$. Any two elements in $\{(r_1,n),(r_2,m),... : m \neq n, m,n \in M^*\}$ are non adjacent. But $(r_1,0)-(0,m)-(0,n)-(r_2,0)$. Then $\text{diam}(\Gamma(R(+)M)) = 3$. □
Example 1.6. Consider the ring $\mathbb{Z}_5(+)\mathbb{Z}_5$. Then $\text{diam}(\Gamma(R(+)M)) = 1$. Since $\mathbb{Z}_5$ is an integral domain and $\mathbb{Z}_5$ is a $\mathbb{Z}_5$-module, and there is no element in $\mathbb{Z}_5^*$ such that $r.m = 0$, for any $m \in \mathbb{Z}_5^*$.

Example 1.7. Consider the ring $\mathbb{Z}(+)\mathbb{Z}_{18}$. Then $\text{diam}(\Gamma(R(+)M)) = 3$. Since $\mathbb{Z}$ is an integral domain and $\mathbb{Z}_{18}$ is a $\mathbb{Z}$-module, and there exists two elements $(r_1 = 2, r_2 = 9)$ in $\mathbb{Z}_5^*$ such that $2.9 = 0$, $9.2 = 0$, $2.2 \neq 0$, $9.9 \neq 0$, $9.2 \in \mathbb{Z}_{18}^*$, then $(2,0) - (0,9) - (0,2) - (9,0)$.

Example 1.8. Consider the ring $\mathbb{Z}(+)\mathbb{Z}_5$. Then $\text{diam}(\Gamma(R(+)M)) = 2$. Since $\mathbb{Z}$ is an integral domain and $\mathbb{Z}_5$ is a $\mathbb{Z}$-module, and there exists $5 \in \mathbb{Z}$ and $5.3 = 0$, $3 \in \mathbb{Z}_5$. And for all $r \in \mathbb{Z}$, $r.3 = 0$.

References


Department of Mathematics, Philadelphia University, Amman, Jordan

E-mail address: mlabadi@philadelphia.edu.jo