GENERALIZED CLOSED SETS IN IDEAL $M$-SPACES

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ABSTRACT. Dontchev et al. [2] introduced and investigated the notion of $I$-$g$-closed sets in ideal topological spaces as a modification of $g$-closed sets due to Levine [5]. The concept of ideal $m$-spaces was introduced by Al-Omari and Noiri [1]. In this paper, we introduce and study the concept of generalized closed ($Ig^*$-closed) sets in an ideal $m$-space.

1. Introduction

The notion of ideal topological spaces was first studied by Kuratowski [4]. Jankovic and Hamlett [3] obtained the further properties of ideal topological spaces. In 1970, Levine [5] initiated the investigations of generalized closed ($g$-closed) sets in topological spaces. As a modification of $g$-closed sets, Dontchev et al. [2] introduced the notion of $I$-$g$-closed sets in an ideal topological space $(X, \tau, I)$, where $\tau$ is a topology and $I$ is an ideal.

Popa and Noiri [7] called a subfamily $m$ of the power set $\mathcal{P}(X)$ of a nonempty set $X$ a minimal structure, if $\emptyset, X \in m$. Recently, Ozbakir and Yildirim [6] have defined the minimal local function $A^*_m$ in an ideal minimal space $(X, m, I)$. As an analogous

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notion to \( \mathcal{I}_{g}\)-closed sets in \((X, \tau, \mathcal{I})\), they defined and studied \( m\mathcal{I}_{g}\)-closed sets in \((X, m, \mathcal{I})\).

Quite recently, the present authors called a subcollection \( \mathcal{M} \) of \( \mathcal{P}(X) \) a minimal structure on \( X \) if (1) \( \emptyset, X \in \mathcal{M} \) and (2) \( \mathcal{M} \) is closed under finite intersections. They defined the local function \( A_\ast \) in an ideal minimal space \((X, \mathcal{M}, \mathcal{I})\). Then \( \text{Cl}_{\ast}(A) = A \cup A_\ast \) is a Kuratowski closure operator which generates a new topology \( \mathcal{M}_{\ast} \) containing the minimal structure \( \mathcal{M} \). In this paper, by using the local function \( A_\ast \) we introduce and investigate the notion of \( \mathcal{I}_{g}\)-closed sets in \((X, \mathcal{M}, \mathcal{I})\). In the last section, we introduce the notion of \( T_\ast \)-spaces and investigate the relationship between \( T_\ast \)-spaces and \( T_{1\frac{1}{2}} \)-spaces.

2. Preliminaries

Let \((X, \tau)\) be a topological space with no separation properties assumed. For a subset \( A \) of a topological space \((X, \tau)\), \( \text{Cl}(A) \) and \( \text{Int}(A) \) denote the closure and the interior of \( A \) in \((X, \tau)\), respectively. An ideal \( \mathcal{I} \) on a topological space \((X, \tau)\) is a non-empty collection of subsets of \( X \) which satisfies the following properties:

1. \( A \in \mathcal{I} \) and \( B \subseteq A \) implies that \( B \in \mathcal{I} \).
2. \( A \in \mathcal{I} \) and \( B \in \mathcal{I} \) implies \( A \cup B \in \mathcal{I} \).

An ideal topological space is a topological space \((X, \tau)\) with an ideal \( \mathcal{I} \) on \( X \) and is denoted by \((X, \tau, \mathcal{I})\). For a subset \( A \subseteq X \), \( A_\ast(\mathcal{I}, \tau) = \{ x \in X : A \cap U \notin \mathcal{I} \) for every open set \( U \) containing \( x \} \) is called the local function of \( A \) with respect to \( \mathcal{I} \) and \( \tau \) (see [3, 4]) and is simply denoted by \( A_\ast \) instead of \( A_\ast(\mathcal{I}, \tau) \).

**Definition 2.1.** [1] A subfamily \( \mathcal{M} \) of the power set \( \mathcal{P}(X) \) of a nonempty set \( X \) is called an \( m \)-structure on \( X \) if \( \mathcal{M} \) satisfies the following conditions:

1. \( \mathcal{M} \) contains \( \emptyset \) and \( X \),
(2) $\mathcal{M}$ is closed under the finite intersection.

The pair $(X, \mathcal{M})$ is called an $m$-space. An $m$-space $(X, \mathcal{M})$ with an ideal $\mathcal{I}$ on $X$ is called an ideal $m$-space and is denoted by $(X, \mathcal{M}, \mathcal{I})$.

A. Al-Omari and T. Noiri [1] introduced the following definitions and results

**Definition 2.2.** A set $A \in \mathcal{P}(X)$ is called an $m$-open set if $A \in \mathcal{M}$.

$B \in \mathcal{P}(X)$ is called an $m$-closed set if $X - B \in \mathcal{M}$. We set $m\text{Int}(A) = \bigcup\{U : U \subseteq A, U \in \mathcal{M}\}$ and $m\text{Cl}(A) = \bigcap\{F : A \subseteq F, X - F \in \mathcal{M}\}$.

**Definition 2.3.** Let $(X, \mathcal{M}, \mathcal{I})$ be an ideal $m$-space. For a subset $A$ of $X$, we define the following set: $A^*(\mathcal{I}, \mathcal{M}) = \{x \in X : A \cap U \notin \mathcal{I} \text{ for every } U \in \mathcal{M}(x)\}$, where $\mathcal{M}(x) = \{U \in \mathcal{M} : x \in U\}$. In this case there is no confusion $A^*(\mathcal{I}, \mathcal{M})$ is briefly denoted by $A_*$ and is called the $\mathcal{M}$-local function of $A$ with respect to $\mathcal{I}$ and $\mathcal{M}$.

**Lemma 2.1.** Let $(X, \mathcal{M}, \mathcal{I})$ be an ideal $m$-space and $A, B$ any subsets of $X$. Then the following properties hold:

1. $(\emptyset)_* = \emptyset$,
2. $(A_*)_* \subset A_*$,
3. $A_* \cup B_* = (A \cup B)_*$.

**Definition 2.4.** Let $(X, \mathcal{M}, \mathcal{I})$ be an ideal $m$-space. For any subset $A$ of $X$, we put $\text{Cl}_*(A) = A \cup A_*$. Then the operator $\text{Cl}_*$ is a Kuratowski closure operator. The topology generated by $\text{Cl}_*$ is denoted by $\mathcal{M}_*$, that is $\mathcal{M}_* = \{U \subseteq X : \text{Cl}_*(X - U) = X - U\}$. The closure and the interior of $A$ with respect to $\mathcal{M}_*$ are denoted by $\text{Cl}_*(A)$ and $\text{Int}_*(A)$, respectively.

**Theorem 2.1.** Let $(X, \mathcal{M}, \mathcal{I})$ be an ideal $m$-space. Then $\mathcal{M}_*$ is a topology containing the minimal structure $\mathcal{M}$. 
Lemma 2.2. Let \((X, M)\) be an \(m\)-space, \(I\) and \(J\) be ideals on \(X\), and let \(A, B\) be subsets of \(X\). Then the following properties hold:

1. If \(A \subseteq B\), then \(A_* \subseteq B_*\).
2. If \(I \subseteq J\), then \(A_* (I) \supseteq A_* (J)\).
3. \(A_* = mCl(A) \subseteq mCl(A)\)
4. If \(A \subseteq A_*\), then \(A_* = mCl(A) = mCl(A)\).
5. If \(A \in I\), then \(A_* = \emptyset\).

3. \(I_{g^*}\)-closed sets

In this section 3 we investigate the class of generalized \(m\)-closed sets in an ideal \(m\)-space.

Definition 3.1. A subset \(A\) of an ideal \(m\)-space \((X, M, I)\) is said to be \(I_{g^*}\)-closed (resp. \(mg\)-closed) if \(A_* \subseteq U\) (resp. \(mCl(A) \subseteq U\)) whenever \(A \subseteq U\) and \(U \in M\). The complement of an \(I_{g^*}\)-closed (resp. \(mg\)-closed) set is said to be \(I_{g^*}\)-open (resp. \(mg\)-open).

Definition 3.2. [5] Let \((X, \tau)\) be a topological space. A subset \(A\) of \(X\) is called a \(g\)-closed set if \(Cl(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is open.

Definition 3.3. [2] Let \((X, \tau, I)\) be an ideal topological space. A subset \(A\) of \(X\) is called an \(I\)-\(g\)-closed set if \(A_* \subseteq U\) whenever \(A \subseteq U\) and \(U \in \tau\). The complement of an \(I\)-\(g\)-closed set is said to be \(I\)-\(g\)-open.

Remark 1. Let \((X, \tau)\) be a topological space and \(I\) be an ideal on \(X\). If we take the \(m\)-structure \(M = \tau\), then \(I_{g^*}\)-closed (resp. \(mg\)-closed) sets coincide with \(I\)-\(g\)-closed (resp. \(g\)-closed) sets.
**Proposition 3.1.** Let $(X, \mathcal{M}, \mathcal{I})$ be an ideal $m$-space. Then the following properties are hold:

1. Every $m$-closed set is $mg$-closed.
2. Every $mg$-closed set is $I_g^*$-closed.

**Proposition 3.2.** The union of two $I_g^*$-closed sets in an ideal $m$-space $(X, \mathcal{M}, \mathcal{I})$ is $I_g^*$-closed.

**Proof.** Let $A$, $B$ be two $I_g^*$-closed sets, and $A \cup B \subseteq U$, where $U \in \mathcal{M}$. Since $A$ and $B$ are $I_g^*$-closed sets, then $A_\ast \subseteq U$ and $B_\ast \subseteq U$. Hence by Lemma 2.1, $A_\ast \cup B_\ast = (A \cup B)_\ast \subseteq U$ and hence $A \cup B$ is $I_g^*$-closed. □

**Definition 3.4.** A subset $A$ of an ideal $m$-space $(X, \mathcal{M}, \mathcal{I})$ is said to be $M_\ast$-closed (resp. $M_\ast$-dense in itself, $M_\ast$-perfect) if $A_\ast \subseteq A$ (resp. $A \subseteq A_\ast$, $A_\ast = A$).

**Proposition 3.3.** Let $(X, \mathcal{M}, \mathcal{I})$ be an ideal $m$-space and $A$ be a subset of $X$. If $A$ is $I_g^*$-closed and $m$-open, then $A$ is $M_\ast$-closed.

**Proposition 3.4.** Let $(X, \mathcal{M}, \mathcal{I})$ be an ideal $m$-space. Then every subset of $X$ is $I_g^*$-closed if and only if every $m$-open set is $M_\ast$-closed.

**Proof.** Suppose every subset of $X$ is $I_g^*$-closed. If $U$ is $m$-open, then it is $I_g^*$-closed and hence $U_\ast \subseteq U$. Hence $U$ is $M_\ast$-closed. Conversely, suppose that every $m$-open set is $M_\ast$-closed. If $A$ is any subset of $X$ and $U$ is an $m$-open set such that $A \subseteq U$, then $A_\ast \subseteq U_\ast \subseteq Cl_\ast(U) = U$ and hence $A$ is $I_g^*$-closed. □
Theorem 3.1. Let \((X, \mathcal{M}, \mathcal{I})\) be an ideal m-space. For a subset \(A\) of \(X\), the following properties are hold:

1. \(A\) is \(\mathcal{I}_g\)-closed if and only if \(\text{Cl}_s(A) \subseteq U\) whenever \(A \subseteq U\) and \(U \in \mathcal{M}\).
2. If \(A\) is \(\mathcal{I}_g\)-closed, then the following equivalent properties hold:
   - (a) \(\text{Cl}_s(A) - A\) contains no a nonempty m-closed set.
   - (b) \(A_\ast - A\) contains no a nonempty m-closed set.

Proof. (1) Suppose that \(A\) is \(\mathcal{I}_g\)-closed. Then \(A_\ast \subseteq U\) whenever \(A \subseteq U\) and \(U \in \mathcal{M}\) and hence \(\text{Cl}_s(A) = A \cup A_\ast \subseteq U\) whenever \(A \subseteq U\) and \(U \in \mathcal{M}\). The converse is obvious.

(2) Suppose \(F \subseteq \text{Cl}_s(A) - A\) and \(F\) is m-closed. Since \(F \subseteq X - A\), \(A \subseteq X - F\) and \(X - F \in \mathcal{M}\). Since \(A\) is \(\mathcal{I}_g\)-closed, \(\text{Cl}_s(A) \subseteq X - F\) and \(F \subseteq X - \text{Cl}_s(A)\). Therefore, \(F \subseteq \text{Cl}_s(A) \cap (X - \text{Cl}_s(A)) = \emptyset\). Thus, (a) is proved.

(a) \(\Leftrightarrow\) (b): This follows from the fact that \(\text{Cl}_s(A) - A = A_\ast - A\). \(\Box\)

Corollary 3.1. For a subset of an ideal m-space \((X, \mathcal{M}, \mathcal{I})\), the following diagram holds:

\[
\begin{array}{ccc}
m\text{-closed} & \longrightarrow & \mathcal{M}_\ast\text{-closed} \\
\downarrow & & \downarrow \\
mg\text{-closed} & \longrightarrow & \mathcal{I}_g\ast\text{-closed}
\end{array}
\]

None of these implications in Corollary 3.1 is reversible as shown by the below examples.

Example 3.1. Let \(X = \{a, b, c\}\), \(\mathcal{M} = \{\emptyset, X, \{a\}, \{b\}, \{b, c\}\}\), and \(\mathcal{I} = \{\emptyset, \{a\}\}\). Then \(A = \{a, b\}\) is an mg-closed set but it is not \(\mathcal{M}_\ast\)-closed.

Example 3.2. Let \(X = \{a, b, c, d\}\), \(\mathcal{M} = \{\emptyset, X, \{a, c\}, \{d\}\}\), and \(\mathcal{I} = \{\emptyset, \{a\}\}\). Then \(A = \{a\}\) is an \(\mathcal{M}_\ast\)-closed set but it is not mg-closed.
Remark 2. (1) By Lemma 2.2, since $I_* = \emptyset$, for every $I \in \mathcal{I}$, $I$ is $\mathcal{I}_g^*$-closed for every $I \in \mathcal{I}$.

(2) By Lemma 2.1, since $(A_*)_* \subseteq A_*$, it follows that $A_*$ is always $\mathcal{I}_g^*$-closed for every subset $A$ of $X$.

Corollary 3.2. Let $(X, \mathcal{M}, \mathcal{I})$ be an ideal $m$-space and $A$ be an $\mathcal{I}_g^*$-closed set. Then the following properties are equivalent:

1. $A$ is an $\mathcal{M}_s$-closed set;
2. $\text{Cl}_s(A) - A$ is an $m$-closed set;
3. $A_* - A$ is an $m$-closed set.

Proof. (1) $\Rightarrow$ (2): If $A$ is $\mathcal{M}_s$-closed, then $\text{Cl}_s(A) = A \cup A_* = A$ and hence $\text{Cl}_s(A) - A = \emptyset$ is $m$-closed.

(2) $\Rightarrow$ (3): This follows from the fact that $\text{Cl}_s(A) - A = A_* - A$.

(3) $\Rightarrow$ (1): Let $A_* - A$ be $m$-closed. Since $A$ is $\mathcal{I}_g^*$-closed, by Theorem 3.1, $A_* - A = \emptyset$ and hence $A_* \subseteq A$. Therefore $\text{Cl}_s(A) = A \cup A_* = A$ and $A$ is $\mathcal{M}_s$-closed.

Corollary 3.3. Let $(X, \mathcal{M}, \mathcal{I})$ be an ideal $m$-space and $A$ be a subset of $X$. Then $A$ is $\mathcal{M}_s$-closed if and only if $A_* - A$ is $m$-closed and $A$ is $\mathcal{I}_g^*$-closed.

Proof. Let $A$ be an $\mathcal{M}_s$-closed set. Then $\text{Cl}_s(A) = A_* \cup A = A$ and $A_* \subseteq A$. Since $A_* - A = \emptyset$, then $A_* - A$ is an $m$-closed set. By Corollary 3.1, every $\mathcal{M}_s$-closed set is $\mathcal{I}_g^*$-closed and hence $A$ is $\mathcal{I}_g^*$-closed.

Conversely. Let $A_* - A$ be $m$-closed and $A$ is $\mathcal{I}_g^*$-closed. Then by Corollary 3.2, $A$ is $\mathcal{M}_s$-closed.

Theorem 3.2. Let $(X, \mathcal{M}, \mathcal{I})$ be an ideal $m$-space. If $A$ is $\mathcal{M}_s$-dense in itself and $\mathcal{I}_g^*$-closed in $X$, then $A$ is $mg$-closed.
Proof. Suppose $A$ is an $M_*$-dense in itself and $I^*_g$-closed subset of $X$. If $U \in M$ and $A \subseteq U$, then by Theorem 3.1, $Cl_*(A) = A_* \cup A = A_* \subseteq U$. Since $A$ is $M_*$-dense in itself, by Lemma 2.2 $mCl(A) = A_* \subseteq U$ and hence $A$ is $mg$-closed. \hfill $\square$

**Theorem 3.3.** Let $(X, M, I)$ be an ideal $m$-space and $A, B$ be subsets of $X$. If $A \subseteq B \subseteq Cl_*(A)$ and $A$ is $I^*_g$-closed, then $B$ is $I^*_g$-closed.

Proof. Let $B \subseteq U$ and $U \in M$. Since $A \subseteq B \subseteq U$ and $A$ is $I^*_g$-closed, then by Theorem 3.1, $Cl_*(A) \subseteq U$ and hence $Cl_*(B) \subseteq Cl_*(Cl_*(A)) = Cl_*(A) \subseteq U$. Therefore, by Theorem 3.1, $B$ is $I^*_g$-closed. \hfill $\square$

**Corollary 3.4.** Let $(X, M, I)$ be an ideal $m$-space and $A, B$ be subsets of $X$. If $A \subseteq B \subseteq A_*$ and $A$ is $I^*_g$-closed, then $A$ and $B$ are $mg$-closed.

Proof. Let $A \subseteq B \subseteq A_*$. Then by Lemmas 2.1 and 2.2, we have $A_* \subseteq B_* \subseteq (A_*)_* \subseteq A_*$ and hence $A_* = B_*$. Therefore, $A$ and $B$ are $M_*$-dense in itself. Since $A \subseteq B \subseteq A_* \subseteq Cl_*(A)$, then by Theorem 3.3, $B$ is $I^*_g$-closed. Therefore, by Theorem 3.2, $A$ and $B$ are $mg$-closed. \hfill $\square$

**Corollary 3.5.** Let $(X, M, I)$ be an ideal $m$-space and $I = \emptyset$. Then $A$ is $I^*_g$-closed if and only if $A$ is $mg$-closed.

Proof. The proof follows from the fact that for $I = \emptyset$, $A \subseteq mCl(A) = A_*$ and hence every subset of $X$ is $M_*$-dense in itself. Therefore, by Theorem 3.2 every $I^*_g$-closed set is $mg$-closed. \hfill $\square$

The following theorem gives a characterization of $I^*_g$-open sets.

**Theorem 3.4.** Let $(X, M, I)$ be an ideal $m$-space and $A$ be a subset of $X$. Then $A$ is $I^*_g$-open if and only if $F \subseteq Int_*(A)$ whenever $F$ is $m$-closed and $F \subseteq A$. 

Proof. Suppose $A$ is $\mathcal{I}_g^*$-open. If $F$ is $m$-closed and $F \subseteq A$, then $X - A \subseteq X - F$ and so $Cl_*(X - A) \subseteq X - F$. Therefore, $F \subseteq Int_*(A)$. Conversely, suppose the condition holds. Let $U \in \mathcal{M}$ such that $X - A \subseteq U$. Then $X - U \subseteq A$ and so $X - U \subseteq Int_*(A)$ which implies that $Cl_*(X - A) \subseteq U$. Therefore, $X - A$ is $\mathcal{I}_g^*$-closed and so $A$ is $\mathcal{I}_g^*$-open. \hfill \square

**Theorem 3.5.** Let $(X, \mathcal{M}, \mathcal{I})$ be an ideal $m$-space and $A, B$ be subsets of $X$. If $A$ is $\mathcal{I}_g^*$-open and $Int_*(A) \subseteq B \subseteq A$, then $B$ is $\mathcal{I}_g^*$-open.

Proof. This is an immediate consequence of Theorems 3.3 and 3.4. \hfill \square

**Theorem 3.6.** Let $(X, \mathcal{M}, \mathcal{I})$ be an ideal $m$-space and $A$ be a subset of $X$. Then for the following statements, (1) implies (2) and (2) is equivalent to (3).

1. $A$ is $\mathcal{I}_g^*$-closed.
2. $A \cup (X - A_*)$ is $\mathcal{I}_g^*$-closed.
3. $A_* - A$ is $\mathcal{I}_g^*$-open.

Proof. (1) $\Rightarrow$ (2): Suppose $A$ is $\mathcal{I}_g^*$-closed. If $U \in \mathcal{M}$ and $(A \cup (X - A_*)) \subseteq U$, then $X - U \subseteq X - (A \cup (X - A_*)) = A_* - A$. Since $A$ is $\mathcal{I}_g^*$-closed, by Theorem 3.1, it follows that $X - U = \emptyset$ and hence $X = U$. Since $X$ is the only $m$-open set containing $A \cup (X - A_*), A \cup (X - A_*)$ is $\mathcal{I}_g^*$-closed.

(2) $\iff$ (3): This follows from the fact that $A \cup (X - A_*) = X - (A_* - A). \hfill \square

**Definition 3.5.** Let $(X, \mathcal{M}, \mathcal{I})$ be an ideal $m$-space and $A, B$ be subsets of $X$ such that $B \subseteq A$. Then

1. The family $\{U \subseteq A : U = V \cap A \text{ for some } V \in \mathcal{M}\}$ is an $\mathcal{M}$-structure on $A$ and is denoted by $\mathcal{M}_A$.
2. The family $\{I \subseteq A : I \in \mathcal{I}\}$ is an ideal on $A$ and is denoted by $\mathcal{I}_A$. 
(3) For the ideal $m$-space $(A, \mathcal{M}_A, \mathcal{I}_A)$, the local function $B_{s(A)}$ is defined as follows: $B_{s(A)} = \{ x \in A : B \cap U \notin \mathcal{I}_A \text{ for any } U \in \mathcal{M}_A(x) \}$, where $\mathcal{M}_A(x) = \{ U \in \mathcal{M}_A : x \in U \}$.

**Lemma 3.1.** Let $(X, \mathcal{M}, \mathcal{I})$ be an ideal $m$-space and $B \subseteq A \subseteq X$. Then $B_{s(A)} = B_s \cap A$ holds.

**Proof.** First we prove $B_{s(A)} \subseteq B_s \cap A$. Let $x \notin B_s \cap A$. We consider the following two cases:

**Case 1.** $x \notin A$. Since $B_{s(A)} \subseteq A$, then $x \notin B_{s(A)}$.

**Case 2.** $x \in A$. In this case $x \notin B_s$. There exists a set $V \in \mathcal{M}$ such that $x \in V$ and $V \cap B \in \mathcal{I}$. Since $x \in A$, we have a set $A \cap V \in \mathcal{M}_A$ such that $x \in A \cap V$ and $(B \cap V) \cap A \in \mathcal{I}_A$. Consequently $x \notin B_{s(A)}$.

Secondly, we prove $B_s \cap A \subseteq B_{s(A)}$. Let $x \notin B_{s(A)}$. Then, there exists $V \in \mathcal{M}$ such that $x \in V \cap A \in \mathcal{M}_A$ and $(V \cap A) \cap B \in \mathcal{I}_A$. Since $B \subseteq A$, then $V \cap B \in \mathcal{I}_A \subseteq \mathcal{I}$, thus $V \cap B \in \mathcal{I}$ for some $V \in \mathcal{M}$ containing $x$. This shows that $x \notin B_s$. Therefore, we obtain $x \notin B_s \cap A$. $\square$

**Theorem 3.7.** Let $(X, \mathcal{M}, \mathcal{I})$ be an ideal $m$-space. Let $B \subseteq A \subseteq X$, where $A$ is an $\mathcal{I}_g$-closed and $m$-open set. Then $B$ is $\mathcal{I}_g$-closed in $(A, \mathcal{M}_A, \mathcal{I}_A)$ if and only if $B$ is $\mathcal{I}_g$-closed in $(X, \mathcal{M}, \mathcal{I})$.

**Proof.** We first note that since $B \subseteq A$ and $A$ is both $\mathcal{I}_g$-closed and $m$-open, then $A_s \subseteq A$ and thus $B_s \subseteq A_s \subseteq A$. By Lemma 3.1, $A \cap B_s = B_{s(A)}$ and we have $B_s = B_{s(A)} \subseteq A$.

**Necessity.** Suppose that $B$ is $\mathcal{I}_g$-closed in $A$. If $U$ is an $m$-open subset of $X$ such that $B \subseteq U$, then $B = B \cap A \subseteq U \cap A$, where $U \cap A$ is $m$-open in $A$. Since $B$ is $\mathcal{I}_g$-closed in $A$, $B_s = B_{s(A)} \subseteq U \cap A \subseteq U$. Therefore $B$ is $\mathcal{I}_g$-closed in $X$. 

Sufficiency. Suppose that $B$ is $I_g^*$-closed in $X$. Let $U$ be an $m$-open subset of $A$ such that $B \subseteq U$. Then $U = V \cap A$ for some $m$-open subset $V$ of $X$. Since $B \subseteq V$ and $B$ is $I_g^*$-closed in $X$, $B_* \subseteq V$. Thus $B_*(A) = B_* \cap A \subseteq V \cap A = U$. Therefore $B$ is $I_g^*$-closed in $A$.

\[ \square \]

4. $T_\ast$-spaces

Proposition 4.1. Let $(X, \mathcal{M}, \mathcal{I})$ be an ideal $m$-space. For $x \in X$, the set $X - \{x\}$ is $I_g^*$-closed or $m$-open.

Proof. Suppose $X - \{x\}$ is not $m$-open. Then $X$ is the only $m$-open set containing $X - \{x\}$. This implies that $(X - \{x\})_* \subseteq X$. Hence $X - \{x\}$ is $I_g^*$-closed.

Definition 4.1. An ideal $m$-space $(X, \mathcal{M}, \mathcal{I})$ is called a $T_\ast$-space if every $I_g^*$-closed set in $(X, \mathcal{M}, \mathcal{I})$ is $\mathcal{M}_\ast$-closed.

Theorem 4.1. Let $(X, \mathcal{M}, \mathcal{I})$ be an ideal $m$-space. Then the following properties are equivalent:

(1) $X$ is a $T_\ast$-space.

(2) Every singleton of $X$ is either $m$-closed or $\mathcal{M}_\ast$-open.

Proof. (1) $\Rightarrow$ (2): Let $x \in X$. If $\{x\}$ is not $m$-closed. Then $X - \{x\}$ is not $m$-open and hence by Proposition 4.1 $X - \{x\}$ is $I_g^*$-closed. Since $(X, \mathcal{M}, \mathcal{I})$ is a $T_\ast$-space, $X - \{x\}$ is $\mathcal{M}_\ast$-closed and thus $\{x\}$ is $\mathcal{M}_\ast$-open.

(2) $\Rightarrow$ (1): Let $A$ be an $I_g^*$-closed subset of $(X, \mathcal{M}, \mathcal{I})$ and $x \in A_*$. We show that $x \in A$.

Case 1. If $\{x\}$ is $m$-closed and $x \notin A$, then $A \subseteq X - \{x\} \in \mathcal{M}$. Since $A$ is $I_g^*$-closed, $A_* \subseteq X - \{x\}$. This is contrary to $x \in A_*$. Hence $x \in A$. 


Case 2. If \( \{x\} \) is \( \mathcal{M}_* \)-open, since \( x \in A_* \subseteq Cl_*(A) \), then \( \{x\} \cap A \neq \emptyset \). Hence \( x \in A \). Thus in both cases we have \( x \in A \). Therefore, \( A_* \subseteq A \) and hence \( A \) is \( \mathcal{M}_* \)-closed. This shows that \( X \) is a \( T_* \)-space. \( \square \)

We recall that a topological space \((X, \tau)\) is called a \( T^{\frac{1}{2}}_2 \)-space \([5]\) if every \( g \)-closed set of \( X \) is closed in \( X \).

**Proposition 4.2.** If an ideal \( m \)-space \((X, \mathcal{M}, \mathcal{I})\) is a \( T_* \)-space, then the topological space \((X, \mathcal{M}_*)\) is a \( T^{\frac{1}{2}}_2 \)-space.

**Proof.** Let \( A \) be any \( g \)-closed set of \((X, \mathcal{M}_*)\). Suppose that \( A \subseteq U \) and \( U \in \mathcal{M} \). Then \( U \in \mathcal{M}_* \) and hence \( Cl_*(A) \subseteq U \). Therefore, \( A \) is \( \mathcal{I}_g \)-closed and by the hypothesis \( A \) is \( \mathcal{M}_* \)-closed. This shows that \((X, \mathcal{M}_*)\) is a \( T^{\frac{1}{2}}_2 \)-space. \( \square \)

**Definition 4.2.** \([2]\) An ideal topological space \((X, \tau, \mathcal{I})\) is called a \( T\mathcal{I}_2 \)-space if every \( \mathcal{I} \)-\( g \)-closed set of \( X \) is \( \tau^* \)-closed.

**Corollary 4.1.** Let \((X, \tau, \mathcal{I})\) be an ideal topological space. Then the following implications hold:

\[
(X, \tau) \text{ is } T^{\frac{1}{2}}_2 \quad \longrightarrow \quad (X, \tau, \mathcal{I}) \text{ is } T\mathcal{I}_2 \quad \longrightarrow \quad (X, \tau^*) \text{ is } T^{\frac{1}{2}}_2
\]

**Proof.** The first implication follows from Corollary 3.4 of \([2]\). By putting \( \tau = \mathcal{M} \) in Proposition 4.2, we obtain the second implication. \( \square \)

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