ON A BEST EXTENSION OF A HALF-DISCRETE HILBERT-TYPE INEQUALITY

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Abstract. By using the way of weight functions and the technique of real analysis, a best extension of a half-discrete Hilbert-type inequality with one-pair conjugate exponents and two interval variables is given. The equivalent forms, the operator expressions and the reverses are considered.

1. Introduction

Suppose that $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $f(\geq 0) \in L^p(0, \infty)$, $g(\geq 0) \in L^q(0, \infty)$, $||f||_p = \{\int_0^\infty f^p(x)dx\}^{\frac{1}{p}} > 0$, $||g||_q > 0$. Then we have the following famous Hardy-Hilbert’s integral inequality (cf. [1]):

\begin{equation}
\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}dxdy < \frac{\pi}{\sin(\pi/p)} ||f||_p||g||_q,
\end{equation}

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible. If $a_m, b_n \geq 0, a = \{a_m\}_{m=1}^\infty \in l^p, b = \{b_n\}_{n=1}^\infty \in l^q, ||a||_p = \{\sum_{m=1}^\infty a_m^p\}^{\frac{1}{p}} > 0, ||b||_q > 0$, then we still have the following discrete Hardy-Hilbert’s inequality with the same best constant factor $\frac{\pi}{\sin(\pi/p)}$:

\begin{equation}
\sum_{m=1}^\infty \sum_{n=1}^\infty \frac{a_mb_n}{m+n} < \frac{\pi}{\sin(\pi/p)} ||a||_p||b||_q,
\end{equation}

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Inequalities (1.1) and (1.2) are important in analysis and its applications (cf. [2], [3], [4]). In 1998, by introducing an independent parameter $\lambda \in (0, 1]$, Yang [5] gave an extension of (1.1) (for $p = q = 2$). Recently, by using the way of weight functions, Yang [6] gave some best extensions of (1.1) and (1.2) as follows: For $r > 1$, $\frac{1}{r} + \frac{1}{s} = 1$, we have

$$f(u_{n}) = \left(\int_{0}^{\infty} |f(x)|^{p}dx\right)^{\frac{1}{p}}$$

$$n_{0} = 1$$

where, $B(u, v) = \int_{0}^{\infty} \frac{1}{(1 + t)^{u+v}} dt(u, v > 0)$ is the Beta function and $\phi(x) = x^{p(1-\frac{1}{s})-1}$, $\phi(x) = x^{q(1-\frac{1}{s})-1}$, $0 < ||f||_{p, \phi} := \left\{\int_{0}^{\infty} \phi(x)|f(x)|^{p}dx\right\}^{\frac{1}{p}} < \infty$, $0 < ||g||_{q, \psi} < \infty$, $0 < ||a||_{p, \phi} := \left\{\sum_{n=1}^{\infty} \phi(n)|a_{n}|n\right\}^{\frac{1}{p}} < \infty$ and $0 < ||b||_{q, \psi} < \infty$. Some Hilbert-type inequalities about the other measurable kernels are provided in [7]-[14].

About the case of half-discrete Hilbert-type inequalities with the non-homogeneous kernels, Hardy et al. provided some results in Theorem 351 of [1]. But they did not prove that the the constant factors in the inequalities are the best possible. And Yang [15] gave a result with the kernel $1_{n_{0}+x}$ similar to $\frac{1}{n_{0}+x}$ by introducing an interval variable as follows: If $u(t)$ is a differentiable strictly increasing function in $(n_{0} - 1, \infty)(n_{0} \in \mathbb{N})$, such that $u((n_{0} - 1)^{+}) = 0$ and $u(\infty) = \infty$, $\lambda > 0$,

$$(u(t))^{\frac{2-\lambda}{2}}u'(t))(t \in (n_{0} - 1, \infty))$$

is decreasing, and

$$f(x), a_{n} \geq 0, 0 < \int_{n_{0}-1}^{\infty} \frac{(u(x))^{1-\lambda}}{u(x)} f^{2}(x) dx < \infty, 0 < \sum_{n=n_{0}}^{\infty} \frac{(u(n))^{1-\lambda}}{u(n)} a_{n}^{2} < \infty,$$

then

$$\int_{n_{0}-1}^{\infty} f(x) \sum_{n=n_{0}}^{\infty} \frac{a_{n}}{(1 + u(n)u(x))^{\lambda}} dx$$

$$< B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) \left\{\int_{n_{0}-1}^{\infty} \frac{(u(x))^{1-\lambda}}{u'(x)} f^{2}(x) dx \sum_{n=n_{0}}^{\infty} \frac{(u(n))^{1-\lambda}}{u'(n)} a_{n}^{2}\right\}^{\frac{1}{2}},$$

where the constant factor $B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right)$ is the best possible.
In this paper, by using the way of weight functions and the technique of real analysis, a best extension of (1.5) with one-pair conjugate exponents and two interval variables is given. The equivalent forms, the operator expressions and some reverses are considered.

2. Some Lemmas

**Lemma 2.1.** If \( \lambda > 0, u(x)(x \in (b, c)), v(x)(x \in (n_0 - 1, \infty), n_0 \in \mathbb{N}) \) are strictly increasing differentiable functions and \([v(x)]^{\frac{1}{2}} - v'_{x} \) is decreasing with \( u(b^+) = v((n_0 - 1)^+) = 0, u(c^-) = v(\infty) = \infty \), define two weight functions as follows

\[
\omega(n) : = [v(n)]^{\frac{1}{2}} \int_{b}^{c} \frac{u'(x)}{(1 + v(n)u(x))^{\lambda}} [u(x)]^{\frac{1}{2} - 1} dx, n \geq n_0 (n \in \mathbb{N}),
\]

\[
\varpi(x) : = [u(x)]^{\frac{1}{2}} \sum_{n=n_0}^{\infty} \frac{v'(n)}{(1 + v(n)u(x))^{\lambda}} [v(n)]^{\frac{1}{2} - 1}, x \in (b, c).
\]

If we define the function \( \theta_{\lambda}(x) \) as follows, then we have the following inequality:

\[
\begin{aligned}
0 &< B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right)(1 - \theta_{\lambda}(x)) < \varpi(x) < \omega(n) = B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right), \\
\theta_{\lambda}(x) &:= \frac{1}{B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right)} \int_{0}^{\infty} \frac{u(x)v(n_0)}{(1 + u(x)v(y))^{\lambda}} t^{\frac{1}{2} - 1} dt = O([u(x)]^{\frac{1}{2}}), x \in (b, c).
\end{aligned}
\]

**Proof.** Setting \( t = v(n_0)u(x) \) in (2.1), we find

\[
\omega(n) = \int_{0}^{\infty} \frac{1}{(t + 1)^{\lambda}} t^{\frac{1}{2} - 1} dt = B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right).
\]

For any fixed \( x \in (b, c) \), in view of the fact that the function

\[
\frac{[v(y)]^{\frac{1}{2} - 1}v'(y)}{(1 + u(x)v(y))^{\lambda}} (y \in (n_0 - 1, \infty))
\]

is strictly decreasing, we find

\[
\begin{aligned}
\varpi(x) &< [u(x)]^{\frac{1}{2}} \int_{n_0 - 1}^{\infty} \frac{1}{(1 + u(x)v(y))^{\lambda}} [v(y)]^{\frac{1}{2} - 1} v'(y) dy \\
&= \frac{1}{(t + 1)^{\lambda}} t^{\frac{1}{2} - 1} dt = B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) = \omega(n).
\end{aligned}
\]
Moreover,
\[
\varpi(x) > [u(x)]^{\frac{1}{2}} \int_{n_0}^{\infty} \frac{1}{(1 + u(x)v(y))^{\lambda}} [v(y)]^{\frac{1}{2} - 1} v'(y) dy
\]
\[
t = u(x) \varpi(y) \int_{u(x)v(n_0)}^{\infty} \frac{t^{\frac{1}{2} - 1}}{(t + 1)^\lambda} dt = B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right)[1 - \theta_\lambda(x)].
\]
Clearly \( \theta_\lambda(x) > 0 \) and
\[
\theta_\lambda(x) < \frac{1}{B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right)} \int_{0}^{u(x)v(n_0)} t^{\frac{1}{2} - 1} dt = \frac{2}{\lambda B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right)} (u(x)v(n_0))^{\frac{1}{2}}.
\]
Hence, we have (2.3) and (2.4).

□

**Lemma 2.2.** Let the assumptions of Lemma 2.1 be fulfilled and additionally,
\[ p > 0(p \neq 1), \frac{1}{p} + \frac{1}{q} = 1, a_n \geq 0, n \geq n_0 (n \in \mathbb{N}), f(x) \text{ is a non-negative measurable function in } (b,c). \text{ Then}
\]
\( (i) \) for \( p > 1 \), we have the following inequalities:
\[
J_1 : = \left\{ \sum_{n=n_0}^{\infty} \frac{v'(n)}{v(n)^{1-\frac{2p}{q}}} \left[ \int_{b}^{c} \frac{f(x)}{(1 + v(n)u(x))^{\lambda}} dx \right]^{p} \right\}^{\frac{1}{p}} \leq \left[ B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) \right]^{\frac{1}{q} \frac{1}{q}} \left\{ \int_{b}^{c} \varpi(x) \frac{[u(x)]^{p(1-\frac{2}{q})-1}}{[v'(x)]^{p-1}} f^p(x) dx \right\}^{\frac{1}{p}},
\]
\[
(2.5)
\]
and
\[
L_1 : = \left\{ \int_{b}^{c} \varpi(x)^{1-q} u'(x) \left[ \sum_{n=n_0}^{\infty} \frac{a_n}{(1 + u(x)v(n))^{\lambda}} \right] \left[ \sum_{n=n_0}^{\infty} \frac{a_n}{(1 + u(x)v(n))^{\lambda}} \right]^{\frac{1}{q}} dx \right\}^{\frac{1}{q}} \leq \left\{ B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) \sum_{n=n_0}^{\infty} \frac{[v(n)]^{q(1-\frac{2}{q})-1}}{[v'(n)]^{q-1} a_n^q} \right\}^{\frac{1}{q}};
\]
\[
(2.6)
\]
\( (ii) \) for \( 0 < p < 1 \), we have the reverses of (2.5) and (2.6).
Proof. (1) By Hölder’s inequality (cf. [16]) and (2.3), we have

\[
\left[ \int_b^c \frac{f(x)}{(1 + v(n)u(x))^\lambda} \, dx \right]^p \\
= \left\{ \int_b^c \frac{1}{(1 + v(n)u(x))^\lambda} \left[ \frac{u(x)}{v(n)} \right]^{(1 - \frac{1}{q})/p} \left[ u'(x) \right]^{1/q} \right. \left[ \frac{v(n)}{u(x)} \right]^{(1 - \frac{1}{q})/p} \left[ v'(n) \right]^{1/q} \left[ f(x) \right] \right. \\
\times \left[ \frac{v(n)}{u(x)} \right]^{(1 - \frac{1}{q})/p} \left[ u'(x) \right]^{1/q} \left[ v'(n) \right]^{1/q} \left. \, dx \right\}^p \\
\leq \int_b^c \frac{v'(n)}{(1 + v(n)u(x))^\lambda} \left[ \frac{u(x)}{v(n)} \right]^{(1 - \frac{1}{q})/p} \left[ u'(x) \right]^{p-1} f^p(x) \, dx \\
\times \left\{ \int_b^c \frac{u'(x)}{(1 + v(n)u(x))^\lambda} \left[ \frac{v(n)}{u(x)} \right]^{(1 - \frac{1}{q})/p} \left[ v'(n) \right]^{p-1} \, dx \right\}^{p-1} \\
= \frac{[B(\frac{\lambda}{2}, \frac{\lambda}{2})]}{[v(n)]^\frac{\lambda}{p - 1}} v'(n) \left[ \frac{v(n)}{u(x)} \right]^{(1 - \frac{1}{q})/p} \left[ u'(x) \right]^{p-1} \, dx.
\]

Then by Lebesgue term by term integration theorem (cf. [17]), we have

\[
J_1 \leq \left[ B(\frac{\lambda}{2}, \frac{\lambda}{2}) \right]^\frac{1}{q} \left\{ \sum_{n=n_0}^\infty \frac{v'(n)f^p(x)}{(1 + v(n)u(x))^\lambda} \left[ \frac{u(x)}{v(n)} \right]^{(1 - \frac{1}{q})/p} \left[ u'(x) \right]^{p-1} \, dx \right\}^{\frac{1}{p}} \\
= \left[ B(\frac{\lambda}{2}, \frac{\lambda}{2}) \right]^\frac{1}{q} \left\{ \int_b^c \sum_{n=n_0}^\infty \frac{v'(n)f^p(x)}{(1 + v(n)u(x))^\lambda} \left[ \frac{u(x)}{v(n)} \right]^{(1 - \frac{1}{q})/p} \left[ u'(x) \right]^{p-1} \, dx \right\}^{\frac{1}{p}} \\
= \left[ B(\frac{\lambda}{2}, \frac{\lambda}{2}) \right]^\frac{1}{q} \left\{ \int_b^c \sum_{n=n_0}^\infty \frac{v'(n)f^p(x)}{[u'(x)]^{p-1}} \left[ \frac{u(x)}{v(n)} \right]^{(1 - \frac{1}{q})/p} \left[ u'(x) \right]^{p-1} \, dx \right\}^{\frac{1}{p}},
\]

and (2.5) follows. Still by Hölder’s inequality, we have

\[
\left[ \sum_{n=n_0}^\infty \frac{a_n}{(1 + u(x)v(n))^\lambda} \right]^q \\
= \left\{ \sum_{n=n_0}^\infty \frac{1}{(1 + u(x)v(n))^\lambda} \left[ \frac{u(x)}{v(n)} \right]^{(1 - \frac{1}{q})/p} \left[ u'(x) \right]^{1/q} \left[ v(n) \right]^{(1 - \frac{1}{q})/p} \left[ v'(n) \right]^{1/q} \right. \left[ a_n \right] \left[ v'(n) \right]^{1/q} \left[ u'(x) \right]^{1/q} \left[ v(n) \right]^{(1 - \frac{1}{q})/p} \left[ u(x) \right]^{(1 - \frac{1}{q})/p} \left[ v(n) \right]^{(1 - \frac{1}{q})/p} \left[ v'(n) \right]^{1/q} \right. \left[ a_n \right] \left. \right\}^{q}
\]
Then we have

\[
\sum_{n=n_0}^\infty \frac{1}{(1 + u(x)v(n))^\lambda} \frac{[u(x)]^{(1 - \frac{1}{2})} (p - 1)}{[v(n)]^{1 - \frac{1}{2}}} \frac{v'(n)}{|u'(x)|^{p - 1}} \right)^{q - 1}
\]

\[
\times \sum_{n=n_0}^\infty \frac{1}{(1 + u(x)v(n))^\lambda} \frac{[v(n)]^{(1 - \frac{1}{2})(q - 1)}}{|u(x)|^{1 - \frac{1}{2}}} \frac{u'(x)}{|v'(n)|^{q - 1}} a_n^q
\]

\[
= \frac{[u(x)]^{1 - \frac{1}{2}}}{[w(x)]^{1 - q} u'(x)} \sum_{n=n_0}^\infty \frac{[u(x)]^{\frac{3}{2} - 1} u'(x)[v(n)]^{\frac{3}{2}} [v(n)]^{q(1 - \frac{1}{2}) - 1}}{(1 + u(x)v(n))^\lambda} \frac{v'(n)}{|v'(n)|^{q - 1}} a_n^q.
\]

Then we have

\[
L_1 \leq \left\{ \int_b^c \left\{ \sum_{n=n_0}^\infty \frac{[u(x)]^{\frac{3}{2} - 1} u'(x)[v(n)]^{\frac{3}{2}} [v(n)]^{q(1 - \frac{1}{2}) - 1}}{(1 + u(x)v(n))^\lambda} \frac{v'(n)}{|v'(n)|^{q - 1}} a_n^q \right\} dx \right\}^{\frac{1}{q}}
\]

\[
= \left\{ \sum_{n=n_0}^\infty \frac{[v(n)]^{\frac{3}{2}}}{[u(x)]^{\frac{3}{2} - 1} u'(x)} \int_b^c \frac{[u(x)]^{\frac{3}{2} - 1} u'(x) [v(n)]^{\frac{3}{2}} [v(n)]^{q(1 - \frac{1}{2}) - 1}}{(1 + u(x)v(n))^\lambda} \frac{v'(n)}{|v'(n)|^{q - 1}} a_n^q \right\}^{\frac{1}{q}}
\]

\[
\leq \left\{ \sum_{n=n_0}^\infty \frac{\omega(n)}{[v(n)]^{q(1 - \frac{1}{2}) - 1}} \frac{a_n^q}{[v'(n)]^{q - 1}} \right\}^{\frac{1}{q}},
\]

and then in view of (2.3), since \( \omega(n) = B(\frac{1}{2}, \frac{1}{2}) \), inequality (2.6) follows.

(ii) By the reverse Holder’s inequality (cf. [16]) and the same way, for \( q < 0 \), we have the reverses of (2.5) and (2.6).

\[\square\]

### 3. Main Results

Setting \( \Phi(x) := \frac{[u(x)]^{p(1 - \frac{1}{2}) - 1}}{|w'(x)|^p}, \Phi(x) := (1 - \theta_\lambda(x)) \Phi(x)(x \in (b, c)), \)

\( \Psi(n) := [v(n)]^{q(1 - \frac{1}{2}) - 1}(n \in \mathbb{N}, n \geq n_0), \)

we have \( [\Phi(x)]^{1 - q} = \frac{u'(x)}{|w(x)|^{1 - \frac{2q}{p}}}, [\Psi(n)]^{1 - p} = \frac{v'(n)}{|v(n)|^{1 - \frac{2q}{p}}}. \)

**Theorem 3.1.** Let the assumptions of Lemma 2.1 be fulfilled and additionally, \( p > 1, \frac{1}{p} + \frac{1}{q} = 1, f(x) \geq 0(x \in (b, c)), a_n \geq 0, n \geq n_0(n \in \mathbb{N}), \)
\( f \in L_{p,\Phi}(b, c), a = \{a_n\}_{n=n_0}^\infty \in l_{q,\Psi}, 0 < ||f||_{p,\Phi} = \{\int_b^c \Phi(x) f^p(x) dx\}^{\frac{1}{p}} < \infty \) and

\[
0 < ||a||_{q,\Psi} = \{\sum_{n=n_0}^{\infty} \Psi(n) a_n^q\}^{\frac{1}{q}} < \infty.
\]

Then we have the following equivalent inequalities:

\[
I := \sum_{n=n_0}^{\infty} \int_b^c \frac{a_n f(x) dx}{(1 + v(n) u(x))^\lambda} = \int_b^c \sum_{n=n_0}^{\infty} \frac{a_n f(x) dx}{(1 + u(x)v(n))^\lambda}
\]

\[
< B(\frac{\lambda}{2}, \frac{\lambda}{2}) ||f||_{p,\Phi} ||a||_{q,\Psi}, \tag{3.1}
\]

\[
J := \left\{ \sum_{n=n_0}^{\infty} [\Psi(n)]^{1-q} \left[ \int_b^c \frac{f(x) dx}{(1 + v(n) u(x))^\lambda} \right]^p \right\}^{\frac{1}{p}}
\]

\[
< B(\frac{\lambda}{2}, \frac{\lambda}{2}) ||f||_{p,\Phi}, \tag{3.2}
\]

and

\[
L := \left\{ \int_b^c [\Phi(x)]^{1-q} \left[ \sum_{n=n_0}^{\infty} \frac{a_n}{\Phi(n)^q} \right]^q dx \right\}^{\frac{1}{q}}
\]

\[
< B(\frac{\lambda}{2}, \frac{\lambda}{2}) ||a||_{q,\Psi}, \tag{3.3}
\]

where the same constant factor \( B(\frac{\lambda}{2}, \frac{\lambda}{2}) \) in the above inequalities is the best possible.

**Proof.** By Lebesgue term by term integration theorem (cf. [17]), there are two expressions for \( I \) in (3.1). In view of (2.3) and (2.5), for \( \varpi(x) < B(\frac{\lambda}{r}, \frac{\lambda}{s}) \), we have (3.2). By Hölder’s inequality, we have

\[
I = \sum_{n=n_0}^{\infty} [\Psi^{-\frac{1}{p}}(n) \int_b^c \frac{f(x) dx}{(1 + v(n) u(x))^\lambda}] [\Psi^{-\frac{1}{q}}(n) a_n] \leq J ||a||_{q,\Psi}. \tag{3.4}
\]

Then by (3.2), we have (3.1). On the other-hand, assuming that (3.1) is valid, setting

\[
a_n := [\Psi(n)]^{1-p} \left[ \int_b^c \frac{f(x) dx}{(1 + v(n) u(x))^{\lambda}} \right]^{p-1}, n \geq n_0,
\]

we can choose the best constant factor.
then $J^{p-1} = ||a||_{q,\Psi}$. By (2.5), we find $J < \infty$. If $J = 0$, then (3.2) is naturally valid; if $J > 0$, then by (3.1), we have

$$||a||^q_{q,\Psi} = J^p = I < B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right)||f||_{p,\Phi}||a||_{q,\Psi},$$

$$||a||^{q-1}_{q,\Psi} = J < B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right)||f||_{p,\Phi},$$

and we have (3.2), which is equivalent to (3.1).

In view of (2.3) and (2.6), for $[\varpi(x)]^{1-q} > [B(\frac{\lambda}{2}, \frac{\lambda}{2})]^{1-q}$, we have (3.3). By Hölder’s inequality, we find

$$I = \int_b^c \Phi^{\frac{1}{p}}(x)f(x)||\Phi^{\frac{1}{q}}(x)\sum_{n=n_0}^{\infty} \frac{a_n}{(1+u(x)v(n))^\lambda}||dx \leq ||f||_{p,\Phi}L. \tag{3.5}$$

Then by (3.3), we have (3.1). On the other-hand, assuming that (3.1) is valid, setting

$$f(x) := \Phi(x)^{1-q}\left[\sum_{n=n_0}^{\infty} \frac{a_n}{(1+u(x)v(n))^\lambda}\right]^{q-1}, x \in (b, c),$$

then $L^{q-1} = ||f||_{p,\Phi}$. By (2.6), we find $L < \infty$. If $L = 0$, then (3.3) is naturally valid; if $L > 0$, then by (3.1), we have

$$||f||^q_{p,\Phi} = L^q = I < B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right)||f||_{p,\Phi}||a||_{q,\Psi},$$

$$||f||^{q-1}_{p,\Phi} = L < B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right)||a||_{q,\Psi},$$

and we have (3.3) which is equivalent to (3.1).

Hence, inequalities (3.1), (3.2) and (3.3) are equivalent.

There exists an unified constant $d \in (b, c)$, satisfying $u(d) = 1$. For $0 < \varepsilon < \frac{q\lambda}{2}$, setting $\tilde{f}(x) = [u(x)]^{\frac{1}{p}+\frac{\varepsilon}{2}-1}u'(x), x \in (b, d); \tilde{f}(x) = 0, x \in [d, c), \tilde{a_n} = [v(n)]^{\frac{1}{q}-\frac{\varepsilon}{2}-1}v'(n), n \geq n_0$, if there exists a positive number $k(\leq B(\frac{\lambda}{2}, \frac{\lambda}{2}))$, such that (3.1) is still valid
as we replace $B(\frac{1}{2}, \frac{1}{2})$ by $k$, then in particular, we have

$$
\int_1^c \sum_{n=n_0}^{\infty} \frac{\tilde{a}_n f(x)dx}{(1+u(x)v(n))^\lambda} < k||f||_{p,\Phi}||\tilde{a}||_{q,\Psi}
$$

$$
= k\frac{1}{\varepsilon} \int_1^c \sum_{n=n_0}^{\infty} [v(n)]^{\varepsilon-1} v'(n) dx
$$

\leq k\frac{1}{\varepsilon} \int_1^c \sum_{n=n_0}^{\infty} [v(y)]^{\varepsilon-1} v'(y) dy

(3.6)

$$
= k\frac{1}{\varepsilon} \sum_{n=n_0}^{\infty} [v(n)]^{\varepsilon-1} v'(n) + [v(n)]^{-\varepsilon} v'(n) dy
$$

In view of the decreasing property of $\frac{1}{(1+u(x)v(y))^{\lambda}} [v(y)]^{\varepsilon-1} v'(y)$, we find

$$
\int_1^c \sum_{n=n_0}^{\infty} [v(n)]^{\varepsilon-1} v'(n) dx
$$

$$
\geq \int_1^c \sum_{n=n_0}^{\infty} [v(y)]^{\varepsilon-1} v'(y) dy
$$

$$
= \frac{1}{\varepsilon} B(\frac{\lambda}{2} - \frac{\varepsilon}{q}, \frac{\lambda}{2} + \frac{\varepsilon}{q}) - A(x),
$$

where

$$
A(x) := \int_1^c [u(x)]^{\varepsilon-1} u'(x) \left[ \int_0^{u(x)v(n_0)} \frac{t^{\frac{1}{2} - \frac{\varepsilon}{q} - 1}}{(1+t)^{\lambda}} dt \right] dx.
$$

Since we find

$$
0 < A(x) < \int_1^c [u(x)]^{\varepsilon-1} u'(x) \left[ \int_0^{u(x)v(n_0)} t^{\frac{1}{2} - \frac{\varepsilon}{q} - 1} dt \right] dx
$$

$$
= \frac{[v(n_0)]^{\frac{1}{2} - \frac{\varepsilon}{q}}}{(\frac{1}{2} - \frac{\varepsilon}{q})(\frac{1}{2} + \frac{\varepsilon}{p})}.
$$
Hilbert’s operator $T$ is an unified representation of $O(1)(\varepsilon \to 0^+)$. By (3.6) and (3.7), we have

$$B(\frac{\lambda}{2} - \frac{\varepsilon}{q}, \frac{\lambda}{2} + \frac{\varepsilon}{q}) - \varepsilon O(1) < k\{\varepsilon v(n_0)v'(n_0) + [v(n_0)]^{-\varepsilon}\}^{\frac{1}{q}},$$

and then $B(\frac{\lambda}{2}, \frac{\lambda}{2}) \leq k(\varepsilon \to 0^+)$. Hence $k = B(\frac{\lambda}{2}, \frac{\lambda}{2})$ is the best value of (3.1).

We confirm that the constant factor $B(\frac{\lambda}{2}, \frac{\lambda}{2})$ in (3.2) ((3.3)) is the best possible, otherwise we can came to a contradiction by (3.4) ((3.5)) that the constant factor in (3.1) is not the best possible.

Remark 1. Set two weight normal spaces as follows:

$$L_{p, \Phi}(b,c) = \{f|||f|||_{p, \Phi} < \infty, l_{q, \Psi} = \{a|||a|||_{q, \Psi} < \infty\}.$$

(i) Define a half-discrete Hilbert’s operator $T : L_{p, \Phi}(b,c) \to L_{p, \Psi}$ as follows: For $f \in L_{p, \Phi}(b,c)$, there exists an unified representation $Tf \in L_{p, \Psi}$, satisfying $Tf(n) = \int_{\phi}^{\epsilon} f(x) (1+u(n))^{-\varepsilon} dx$, $n \geq n_0$. Then by (3.1), it follows $||Tf||_{p, \Psi} < B(\frac{\lambda}{2}, \frac{\lambda}{2})||f||_{p, \Phi}$ and $T$ is bounded with $||T|| \leq B(\frac{\lambda}{2}, \frac{\lambda}{2})$. Since the constant factor in (3.2) is the best possible, we have $||T|| = B(\frac{\lambda}{2}, \frac{\lambda}{2})$.

(ii) Define a half-discrete Hilbert’s operator $\tilde{T} : l_{q, \Psi} \to L_{q, \Psi}$ as follows: For $a \in l_{q, \Psi}$, there exists an unified representation $\tilde{T}a \in L_{q, \Psi}$, satisfying

$$\tilde{T}a(x) = \sum_{n=n_0}^{\infty} a_n \frac{\epsilon(x(n))}{(1+u(n))^{-\varepsilon}}, x \in (b,c).$$

Then by (3.2), it follows $||\tilde{T}a||_{q, \Psi} < B(\frac{\lambda}{2}, \frac{\lambda}{2})||a||_{q, \Psi}$ and $\tilde{T}$ is bounded with $||\tilde{T}|| \leq B(\frac{\lambda}{2}, \frac{\lambda}{2})$. Since the constant factor in (3.3) is the best possible, we have $||\tilde{T}|| = B(\frac{\lambda}{2}, \frac{\lambda}{2}) = ||T||$.

In the following theorem, for $0 < p < 1$, we still use the formal symbols of $||f||_{p, \Phi}$ and $||a||_{q, \Psi}$ et al.

**Theorem 3.2.** Let the assumptions of Lemma 2.1 be fulfilled and additionally, $0 < p < 1, \frac{1}{p} + \frac{1}{\xi} = 1, f(x) \geq 0(x \in (b,c)), a_n \geq 0(n \geq n_0, n \in \mathbb{N}),$ $0 < ||f||_{p, \Phi} = \{\int_{b}^{\epsilon} (1 - \theta_{\lambda}(x)) \Phi(x) f^p(x) dx\}^{\frac{1}{p}} < \infty$ and
0 < \|a\|_{q, \Psi} = \left\{ \sum_{n=n_0}^{\infty} \Psi(n)a_n^q \right\}^{\frac{1}{q}} < \infty. Then we have the following equivalent inequalities:

\[ I = \sum_{n=n_0}^{\infty} \int_{b}^{c} \frac{a_n f(x) dx}{(1 + v(n)u(x))^\lambda} = \int_{b}^{c} \sum_{n=n_0}^{\infty} \frac{a_n f(x) dx}{(1 + u(x)v(n))^\lambda} \]

\[ \geq B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) \|f\|_{p, \tilde{\Phi}} \|a\|_{q, \Psi}, \]  

(3.8)

\[ J = \left\{ \sum_{n=n_0}^{\infty} [\Psi(n)]^{1-p} \left[ \int_{b}^{c} \frac{f(x) dx}{(1 + v(n)u(x))^\lambda} \right]^p \right\}^{\frac{1}{p}} \]

\[ \geq B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) \|f\|_{p, \tilde{\Phi}}, \]  

(3.9)

and

\[ \tilde{L} = \left\{ \int_{b}^{c} [\tilde{\Phi}(x)]^{1-q} \left[ \sum_{n=n_0}^{\infty} \frac{a_n}{(1 + u(x)v(n))^\lambda} \right]^q dx \right\}^{\frac{1}{q}} \]

\[ \geq B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) \|a\|_{q, \Psi}. \]  

(3.10)

Moreover, if there exists a constant \( \delta_0 > 0 \), such that for any \( \delta \in [0, \delta_0) \), \([v(y)]^{\frac{\lambda}{2} + \delta - 1} v'(y)\) is decreasing in \((n_0 - 1, \infty)\), then the same constant factor \( B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) \) in the above inequalities is the best possible.

Proof. In view of (2.3) and the reverse of (2.5), for \( \varpi(x) > B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right)(1 - \theta_\lambda(x)) \), we have (3.9). By the reverse Hölder’s inequality, we have

\[ I = \sum_{n=n_0}^{\infty} \left[ \Psi^{\frac{1}{q}}(n) \int_{b}^{c} \frac{f(x) dx}{(1 + v(n)u(x))^\lambda} \right] \left[ \Psi^{\frac{1}{q}}(n)a_n \right] \geq J \|a\|_{q, \Psi}. \]

(3.11)

Then by (3.9), we have (3.8). On the other-hand, assuming that (3.8) is valid, setting \( a_n \) as Theorem 1, then \( J^{p-1} = \|a\|_{q, \Psi} \). By the reverse of (2.5), we find \( J > 0 \). If \( J = \infty \),
then (3.9) is naturally valid; if \( J < \infty \), then by (3.8), we have

\[
||a||^{q}_{q,\Psi} = J^p = I > B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right)||f||_{p,\tilde{\Phi}}||a||_{q,\Psi},
\]

\[
||a||^{q-1}_{q,\Psi} = J > B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right)||f||_{p,\tilde{\Phi}},
\]

and we have (3.9) which is equivalent to (3.8).

In view of (2.3) and the reverse of (2.6), for \([\varpi(x)]^{1-q} > [B(\frac{\lambda}{r}, \frac{\lambda}{s})(1-\theta_{\lambda}(x))]^{1-q}(q < 0)\), we have (3.10). By the reverse Hölder’s inequality, we have

\[(3.12) \quad I = \int_{b}^{c} \tilde{f}(x)dx \geq ||f||_{p,\tilde{\Phi}} \tilde{L}.
\]

Then by (3.10), we have (3.8). On the other-hand, assuming that (3.8) is valid, setting

\[
f(x) := [\tilde{\Phi}(x)]^{1-q}\left[\sum_{n=n_0}^{\infty} \frac{a_n}{(1+u(x)v(n))^\lambda}\right]^{q-1}, \quad x \in (b,c),
\]

then \( \tilde{L}^{q-1} = ||f||_{p,\tilde{\Phi}} \). By the reverse of (2.6), we find \( \tilde{L} > 0 \). If \( \tilde{L} = \infty \), then (3.10) is naturally valid; if \( \tilde{L} < \infty \), then by (3.8), we have

\[
||f||^p_{p,\tilde{\Phi}} = \tilde{L}^q = I > B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right)||f||_{p,\tilde{\Phi}}||a||_{q,\Psi},
\]

\[
||f||^{p-1}_{p,\tilde{\Phi}} = \tilde{L} > B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right)||a||_{q,\Psi},
\]

and we have (3.10) which is equivalent to (3.8).

Hence inequalities (3.8), (3.9) and (3.10) are equivalent.

For \( 0 < \varepsilon < \min\left\{ \frac{|a|}{2}, |q|\right\} \), setting \( \tilde{f}(x) = [u(x)]^{\frac{1}{2}+\varepsilon} u'(x), x \in (b,d) \); \( \tilde{f}(x) = 0, x \in [d,c) \), \( \tilde{a}_n = [v(n)]^{\frac{1}{2}+\varepsilon-1} v'(n) \), \( n \geq n_0 \), if there exists a positive number \( k(B(\frac{\lambda}{2}, \frac{\lambda}{2})) \), such that (3.8) is still valid as we replace \( B(\frac{\lambda}{2}, \frac{\lambda}{2}) \) by \( k \), then in particular, for \( q < 0 \), we have

\[
\tilde{I} := \int_{b}^{c} \sum_{n=n_0}^{\infty} \frac{\tilde{a}_n \tilde{f}(x)dx}{(1+u(x)v(n))^{\lambda}} > k||f||_{p,\tilde{\Phi}}||\tilde{a}||_{q,\Psi}
\]
\[ k \left\{ \int_b^d (1 - O([u(x)]^2)) |u(x)|^{-\varepsilon} u'(x) dx \right\}^{\frac{1}{p}} \left\{ \sum_{n=n_0}^{\infty} |v(n)|^{-\varepsilon} v'(n) \right\}^{\frac{1}{q}} \]

\[ = k \left\{ \frac{1}{\varepsilon} - O(1) \right\}^{\frac{1}{p}} \left\{ \sum_{n=n_0}^{\infty} |v(n)|^{-\varepsilon} v'(n) \right\}^{\frac{1}{q}} \]

\[ \geq k \left\{ \frac{1}{\varepsilon} - O(1) \right\}^{\frac{1}{p}} \left\{ \sum_{n=n_0}^{\infty} |v(n)|^{-\varepsilon} v'(n) \right\}^{\frac{1}{q}} \]

\[ = \frac{k}{\varepsilon} \left\{ 1 - \varepsilon O(1) \right\}^{\frac{1}{p}} \left\{ \varepsilon |v(n_0)|^{-\varepsilon} v'(n_0) + |v(n_0)|^{-\varepsilon} \right\}^{\frac{1}{q}}. \]

In view of the decreasing property of \( \frac{|v(y)|^{\frac{2}{p} - \frac{q}{q} - 1} v'(y)}{(1 + u(x)v(y))^\lambda} \), setting \( t = u(x)v(y) \), we find

\[ \tilde{I} = \int_b^d |u(x)|^{\frac{2}{q} + \frac{q}{q} - 1} u'(x) \sum_{n=n_0}^{\infty} \frac{|v(n)|^{\frac{2}{q} - \frac{q}{q} - 1} v'(n)}{(1 + u(x)v(n))^\lambda} dx \]

\[ \leq \int_b^d |u(x)|^{\frac{2}{q} + \frac{q}{q} - 1} u'(x) \left[ \int_{n_0-1}^{\infty} \frac{|v(y)|^{\frac{2}{q} - \frac{q}{q} - 1} v'(y)}{(1 + u(x)v(y))^\lambda} dy \right] dx \]

\[ = \int_b^d |u(x)|^{\lambda} u'(x) dx \int_0^{\infty} \frac{t^{\frac{2}{q} - \frac{q}{q} - 1}}{(1 + t)^\lambda} dt \]

\[ = \frac{1}{\varepsilon} B\left( \frac{\lambda}{2} - \frac{\varepsilon}{q}, \frac{\lambda}{2} + \frac{\varepsilon}{q} \right). \]

By (3.13) and (3.14), we have

\[ B\left( \frac{\lambda}{2} - \frac{\varepsilon}{q}, \frac{\lambda}{2} + \frac{\varepsilon}{q} \right) > k \left\{ 1 - \varepsilon O(1) \right\}^{\frac{1}{p}} \left\{ \varepsilon |v(n_0)|^{-\varepsilon} v'(n_0) + |v(n_0)|^{-\varepsilon} \right\}^{\frac{1}{q}}, \]

and then \( B\left( \frac{\lambda}{2}, \frac{\lambda}{2} \right) \geq k (\varepsilon \to 0^+) \). Hence \( k = B\left( \frac{\lambda}{2}, \frac{\lambda}{2} \right) \) is the best value of (3.8).

We confirm that the constant factor \( B\left( \frac{\lambda}{2}, \frac{\lambda}{2} \right) \) in (3.9) ((3.10)) is the best possible, otherwise we can came to a contradiction by (3.11) ((3.12)) that the constant factor in (3.8) is not the best possible.

\[ \square \]

Remark 2. (i) If \( \alpha > 0, u(x) = x^\alpha, b = 0, c = \infty, v(n) = n^\alpha, n_0 = 1 \), then for \( 0 < \alpha \lambda \leq 2 \), \( |v(x)|^{\frac{2}{q} - 1} v'(x) = \alpha x^{\frac{\alpha}{2} - 1} \) is decreasing. In particular, for \( \alpha = 1, 0 < \lambda \leq 2, u(x) = x(x \in (0, \infty)), v(n) = n(n \in \mathbb{N}) \) in (3.1), (3.2) and (3.3), we
have the following equivalent inequalities:

\[
I = \sum_{n=1}^{\infty} \int_0^\infty \frac{a_n f(x) dx}{(1 + nx)^\lambda} = \int_0^\infty \sum_{n=1}^{\infty} \frac{a_n f(x) dx}{(1 + nx)^\lambda}
\]

(3.15)

\[
< B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) ||f||_{p,\phi} ||a||_{q,\psi},
\]

(3.16)

\[
\left\{ \sum_{n=1}^{\infty} n^{\frac{\lambda}{2} - 1} \left[ \int_0^\infty \frac{f(x)}{(1 + nx)^\lambda} dx \right]^p \right\}^{\frac{1}{p}} < B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) ||f||_{p,\phi},
\]

and

(3.17)

\[
\left\{ \int_0^\infty x^{\frac{\lambda}{2} - 1} \left[ \sum_{n=1}^{\infty} \frac{a_n}{(1 + nx)^\lambda} \right]^q dx \right\}^{\frac{1}{q}} < B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) ||a||_{q,\psi}.
\]

(ii) For \( p = q = 2, b = n_0 - 1 = 0, c = \infty, v(x) = u(x) \) in (3.1), we have (1.5). Hence, (3.1) is a best extension of (1.5).

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References


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