Matrix Transformations on the Set ces\((p, q)\)

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Abstract. The main purpose of this paper is to characterize the matrices of the classes \((\text{ces}(p, q), cs)\) and \((\text{ces}(p, q), bs)\), where \(cs\) is the space of convergent series and \(bs\) is the space of bounded series.

1. Introduction

Let \(\omega\) be the space of all (real or complex) sequences, and \(X, Y\) are two subsets of \(\omega\). If \(A = (a_{n,k})_{n,k=1,2,3,...}\) be an infinite matrix of complex numbers we say that \(A\) defines a matrix transformation form \(X\) into \(Y\) and denoted it by \(A \in (X, Y)\), if for every sequence \(x = (x_k) \in X\) the sequence \(A(x) = A_n(x)\) is in \(Y\), where

\[A_n(x) = \sum_{k=1}^{\infty} a_{n,k}x_k\quad \text{for all } n,\]

provided the series on the right is convergent.

In [4] Lim investigated the sequence space \(\text{ces}(p)\). In [2] Khan and Rahman defined and studied the sequence space \(\text{ces}(p, q)\). If \(p = (p_r)\), then for \(\inf p_r > 0\)

\[\text{ces}(p, q) = \left\{ x \in \omega : \sum_{r=0}^{\infty} \left( \frac{1}{Q_{2^r}} \sum_{r} q_k |x_k| \right)^{p_r} < \infty \right\},\]

where \(q = (q_k)\) is a sequence of positive real numbers, \(Q_{2^r} = \sum_{r} q_k\) and \(\sum_{r}\) denotes a sum over the ranges \(2^r \leq k < 2^{r+1}\). They showed \(\text{ces}(p, q)\) is a complete paranorm.
space paranormed by
\[ g(x) = \left( \sum_{r=0}^{\infty} \left( \frac{1}{Q_{2r}} \sum_{r} q_{k} |x_{k}| \right)^{p_{r}} \right)^{\frac{1}{p_{r}}} \]
provided \( H = \sup_{r} p_{r} < \infty \) and \( M = \max \{1, H\} \).

In [5] Stieglitz and Tietz defined convergent and bounded series as

\[ cs = \left\{ x : \left( \sum_{i=1}^{n} x_{i} \right) \in c \right\}, \]
\[ c_{0}s = \left\{ x : \left( \sum_{i=1}^{n} x_{i} \right) \in c_{0} \right\}, \]
\[ bs = \left\{ x : \left( \sum_{i=1}^{n} x_{i} \right) \in \ell_{\infty} \right\}, \]
where \( c, c_{0} \) and \( \ell_{\infty} \) are the sequence spaces of all convergent, null and bounded sequences respectively.

We state the following inequality (see [3]) which will be used later. For any integer \( E > 1 \) and any two complex numbers \( a \) and \( b \) we have

\[ |ab| \leq E \left( |a|^t E^{-t} + |b|^p \right), \tag{1.1} \]
where \( p > 1 \) and \( \frac{1}{p} + \frac{1}{t} = 1. \)

2. MATRIX TRANSFORMATION ON \( ces(p, q) \)

The following notations are used throughout for all integers \( n \geq 1 \), we write

\[ t_{n}(Ax) = \sum_{i=1}^{n} A_{i}(x) = \sum_{k=1}^{\infty} b_{nk} x_{k}, \]
where \( b_{nk} = \sum_{i=1}^{n} a_{ik} \).

Now we prove:

**Theorem 2.1.** Let \( 1 < p_{r} \leq \sup_{r} p_{r} < \infty \). Then \( A \in (ces(p, q), cs) \) if and only if
(i) there exists an integer $E > 1$ such that

$$T = \sup_n (U_n) < \infty,$$

where

$$U_n = \sum_{r=0}^{\infty} \left( Q_{2^r} \max_r (q_k^{-1} | b_{nk} |) \right)^{t_r} E^{-t_r},$$

$$\frac{1}{p_r} + \frac{1}{t_r} = 1 \quad r = 0, 1, 2, \ldots$$

(ii) $\lim_{n \to \infty} b_{nk} = \alpha_k$ for all $k$.

**Proof.**  **Necessity.** Suppose that $A \in (\text{ces}(p, q), cs)$. Then $t_n(Ax)$ exists for each $n$ and $x \in \text{ces}(p, q)$. If we put $\sigma_n(x) = t_n(Ax)$, then $(\sigma_n)_n$ is a sequence of continuous real functions on $\text{ces}(p, q)$. Also $\text{ces}(p, q)$ is complete and further $\sup_n |t_n(Ax)| < \infty$ on $\text{ces}(p, q)$. Now arguing with uniform boundedness principle (see Khan and Rahman [2], Th.3) we have condition (i). Condition (ii) is obtained by taking $x = e_k \in \text{ces}(p, q)$, where $e_k$ is a sequence with 1 at the $k$-th place and zeros elsewhere.

**Sufficiency.** Suppose conditions (i) and (ii) hold. Then the conditions imply

$$\sum_{r=0}^{\infty} \left( Q_{2^r} \max_r (q_k^{-1} | \alpha_k |) \right)^{t_r} E^{-t_r} = \lim_{n \to \infty} (U_n) \leq \sup_n (U_n) < \infty.$$

Thus the series $\sum_{k=1}^{\infty} b_{nk} x_k$ and $\sum_{k=1}^{\infty} \alpha_k x_k$ converge for each $n$ and $x \in \text{ces}(p, q)$. Put $t_{nk} = b_{nk} - \alpha_k$. Then

$$\sum_{k=1}^{\infty} b_{nk} x_k = \sum_{k=1}^{\infty} t_{nk} x_k + \sum_{k=1}^{\infty} \alpha_k x_k .$$

By (ii) for $k_0 \in Z^+$, where $Z^+$ is the set of positive integers, we have

$$\lim_{n \to \infty} \sum_{k \leq 2^{k_0}} t_{nk} x_k = 0 .$$

Moreover, for each $x \in \text{ces}(p, q)$ and $\epsilon > 0$ we can choose an integer $k_0 \in Z^+$ such that

$$g_{k_0}(x) = \sum_{r=k_0}^{\infty} \left( \frac{1}{Q_{2^r}} \sum_r q_r |x_k| \right)^{p_r} < \epsilon^M.$$
Now put $B_r(n) = \max_r \left( q_k^{-1} |t_{nk}| \right) = \max_r \left( q_k^{-1} |b_{nk} - \alpha_k| \right)$. Then by ([2], Th.2) and by inequality (1.1) we have

$$\sum_{k=2^{k_0}}^{\infty} |t_{nk}| |x_k| / \left( g_{k_0}(x) \right)^{1/M} \leq \sum_{r=k_0}^{\infty} \max_r \left( q_k^{-1} |t_{nk}| \right) \left( \sum_r q_k |x_k| \right) / \left[ g_{k_0}(x) \right]^{1/M}$$

$$= \sum_{r=k_0}^{\infty} \left( Q_{2^r} B_r(n) \frac{1}{Q_{2^r}} \sum_r q_k |x_k| \right) / \left[ g_{k_0}(x) \right]^{1/M}$$

$$\leq E \left[ \sum_{r=k_0}^{\infty} \left( Q_{2^r} B_r(n) \right)^{t_r} E^{-t_r} \right] + \sum_{r=k_0}^{\infty} \left( \frac{1}{Q_{2^r}} \sum_r q_k |x_k| \right)^{p_r} / \left[ g_{k_0}(x) \right]^{p_r/M}$$

$$\leq E \left[ \sum_{r=k_0}^{\infty} \left( Q_{2^r} B_r(n) \right)^{t_r} E^{-t_r} + 1 \right]$$

Thus,

$$\sum_{k=2^{k_0}}^{\infty} |t_{nk}| |x_k| \leq E \sum_{r=k_0}^{\infty} \left( Q_{2^r} B_r(n) \right)^{t_r} E^{-t_r} [g_{k_0}(x)]^{1/M}$$

$$< E(2U_n + 1) \epsilon ,$$

where $\sum_{r=k_0}^{\infty} \left( Q_{2^r} B_r(n) \right)^{t_r} E^{-t_r} \leq 2U_n < \infty$ for all $n$.

It follows immediately that

$$\lim_{n \to \infty} \sum_{k=1}^{\infty} b_{nk} x_k = \sum_{k=1}^{\infty} \alpha_k x_k$$

and this completes the proof.

**Corollary 2.1.** [1] Let $1 < p_r \leq \sup_r p_r < \infty$. Then $A \in (ces(p), cs)$ if and only if

(i) there exists an integer $E > 1$ such that

$$T = \sup_n \left( U_n \right) < \infty ,$$
where

\[ U_n = \sum_{r=0}^{\infty} \left( 2^r \max_{k} |b_{nk}| \right)^{t_r} E^{-t_r}, \]

\[ \frac{1}{p_r} + \frac{1}{t_r} = 1 \quad r = 0, 1, 2, \ldots \]

(ii) \( \lim_{n \to \infty} b_{nk} = \alpha_k \) for all \( k \).

**Proof.** If we take \( q_n = 1 \) for all \( n \) in the above theorem then we obtain the results.

**Corollary 2.2.** Let \( 1 < p_r \leq \sup_r p_r < \infty \). Then \( A \in (\text{ces}(p), c_0s) \) if and only if

the condition (i) of Corollary 2.1 holds, and \( \lim_{n \to \infty} b_{nk} = 0 \).

**Theorem 2.2.** Let \( 1 < p_r \leq \sup_r p_r < \infty \). Then \( A \in (\text{ces}(p, q), bs) \) if and only if

(i) there exists an integer \( E > 1 \) such that

\[ T = \sup_n (U_n) < \infty, \]

where

\[ U_n = \sum_{r=0}^{\infty} \left( Q_{2^r} \max_{k} (q_k^{-1} |b_{nk}|) \right)^{t_r} E^{-t_r}, \]

\[ \frac{1}{p_r} + \frac{1}{t_r} = 1 \quad r = 0, 1, 2, \ldots \]

**Proof.** Necessity follows by using similar argument as in Theorem 2.1. For sufficiency, suppose that the condition (i) holds and that \( x \in \text{ces}(p, q) \). Then by inequality (1.1) we have
\[ |t_n(Ax)| = \left| \sum_{k=1}^{\infty} b_{nk}x_k \right| \leq \sum_{k=1}^{\infty} |b_{nk}x_k| \]
\[ \leq \sum_{r=0}^{\infty} \sum_{r} |b_{nk}| |x_k| \]
\[ \leq \sum_{r=0}^{\infty} \max_{r} (q_k^{-1}|b_{nk}|) \sum_{r} q_k |x_k| \]
\[ = \sum_{r=0}^{\infty} Q_{2r} \max_{r} (q_k^{-1}|b_{nk}|) \frac{1}{Q_{2r}} \sum_{r} q_k |x_k| \]
\[ \leq E \left[ \sum_{r=0}^{\infty} \left( Q_{2r} \max_{r} (q_k^{-1}|b_{nk}|) \right)^{t_r} E^{-t_r} + \sum_{r=0}^{\infty} \left( \frac{1}{Q_{2r}} \sum_{r} q_k |x_k| \right)^{p_r} \right] \]
\[ \leq E \left( T + \sum_{r=0}^{\infty} \left( \frac{1}{Q_{2r}} \sum_{r} q_k |x_k| \right)^{p_r} \right). \]

We conclude \( \sup_n |t_n(Ax)| < \infty \).

Therefore, \( A \in (\text{ces}(p, q), bs) \) and this completes the proof.

**Corollary 2.3.** [1] Let \( 1 < p_r \leq \sup_r p_r < \infty \). Then \( A \in (\text{ces}(p), bs) \) if and only if there exists an integer \( E > 1 \) such that

\[ T = \sup_n (U_n) < \infty , \]
where

\[ U_n = \sum_{r=0}^{\infty} \left( 2^r \max_{r} |b_{nk}| \right)^{t_r} E^{-t_r} , \]
\[ \frac{1}{p_r} + \frac{1}{t_r} = 1 , \quad r = 0, 1, 2, \ldots \]

**Proof.** When \( q_n = 1 \) for all \( n \) in the above Theorem 2.2 then we get the result.
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REFERENCES


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