BAER GAMMA RINGS WITH INVOLUTIONS

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ABSTRACT. The concept of involution in Γ-rings is introduced and with the help of involutions, we obtain some characterizations of Baer Γ-rings.

1. INTRODUCTION

As a generalization of rings, the concept of Γ-rings was first introduced by N. Nobusawa [6]. After words Barnes [1] generalized the notion of Nobusawa’s Γ-rings and gave a new definition of a Γ-ring. Now a days, Γ-rings means the Γ-rings in the sense of Barnes [1] where other Γ-rings are known as N -rings i.e., gamma rings in the sense of Nobusawa. Many Mathematicians worked on Γ-rings and obtained some fruitful results that are a generalization of many classical ring theories. In the Book “Rings with operators” Kaplansky [3] worked on Baer rings and obtained various results relating to involution and Baer rings. Paul and Sabur [9] worked on Lie and Jordan structures in simple Γ-rings and generalized some results of classical rings into Γ-rings. Paul and Sabur [10] also worked on Baer Gamma rings and obtained some characterizations of this Γ-ring.

In this paper, we introduce the notion of an involution in Γ-rings and generalize

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some results of classical Baer rings into gamma Baer rings with the help of the new concept of an involution. In [10], an example of a Baer gamma ring is given

2. Preliminaries

**Definition 2.1. Gamma Ring:** Let $M$ and $\Gamma$ be two additive abelian groups. Suppose that there is a mapping from $M \times \Gamma \times M \to M$ (sending $(x, \alpha, y)$ into $x\alpha y$) such that

(i) $(x + y)\alpha z = x\alpha z + y\alpha z, x(\alpha + \beta)z = x\alpha z + x\beta z, x\alpha(y + z) = x\alpha y + x\alpha z$

(ii) $(x\alpha y)\beta z = x\alpha(y\beta z)$, where $x, y, z \in M$ and $\alpha, \beta \in \Gamma$. Then $M$ is called a $\Gamma$-ring in the sense of Barnes [1].

**Definition 2.2. Sub $\Gamma$-ring:** Let $M$ be a $\Gamma$-ring. A non-empty subset $S$ of a $\Gamma$-ring $M$ is a sub $\Gamma$-ring of $M$ if $a, b \in S$, then $a - b \in S$ and $a\gamma b \in S, \forall \gamma \in \Gamma$.

**Definition 2.3. Ideal of $\Gamma$-rings:** A subset $A$ of the $\Gamma$-ring $M$ is a left (right) ideal of $M$ if $A$ is an additive subgroup of $M$ and $M\Gamma A = \{c\alpha a : c \in M, \alpha \in \Gamma, a \in A\}(A\Gamma M)$ is contained in $A$. If $A$ is both a left and a right ideal of $M$, then we say that $A$ is an ideal or two-sided ideal of $M$. If $A$ and $B$ are both left (respectively right or two-sided) ideals of $M$, then $A + B = \{a + b : a \in A, b \in B\}$ is clearly a left (respectively right or two-sided) ideal, called the sum of $A$ and $B$. We can say every finite sum of left (respectively right or two-sided) ideal of a $\Gamma$-ring is also a left (respectively right or two-sided) ideal.

It is clear that the intersection of any number of left (respectively right or two-sided) ideal of $M$ is also a left (respectively right or two-sided) ideal of $M$. If $A$ is a left ideal of $M$, $B$ is a right ideal of $M$ and $S$ is any non-empty subset of $M$, then the set, $A\Gamma S = \{\sum_{i=1}^{n} a_i \gamma s_i : a_i \in A, \gamma \in \Gamma, s_i \in S, n \text{ is a positive integer}\}$ is a left ideal of $M$ and $S\Gamma B$ is a right ideal of $M$. $A\Gamma B$ is a two-sided ideal of $M$. If $a \in M$, then the principal ideal generated by $a$ denoted by $< a >$ is the intersection
of all ideals containing $a$ and is the set of all finite sum of elements of the form $na + xoa + a\beta y + u\gamma a\mu v$, where $n$ is an integer, $x, y, u, v$ are elements of $M$ and $\alpha, \beta, \gamma, \mu$ are elements of $\Gamma$. This is the smallest ideal generated by $a$. Let $a \in M$. The smallest left (right) ideal generated by $a$ is called the principal left (right) ideal $< a | (| a >)$. 

**Definition 2.4. Unity element of a $\Gamma$-ring:** Let $M$ be a $\Gamma$-ring. $M$ is called a $\Gamma$-ring with unity if there exists an element $e \in M$ such that $a\gamma e = e\gamma a = a$ for all $a \in M$ and some $\gamma \in \Gamma$. We shall frequently denote $e$ by 1 and when $M$ is a $\Gamma$-ring with unity, we shall often write $1 \in M$. Note that not all $\Gamma$-rings have an unity. When a $\Gamma$-ring has an unity, then the unity is unique.

**Definition 2.5. Nilpotent element:** Let $M$ be a $\Gamma$-ring. An element $x$ of $M$ is called nilpotent if for some $\gamma \in \Gamma$, there exists a positive integer $n = n(\gamma)$ such that $(x\gamma)^nx = (x\gamma x\gamma...\gamma x\gamma)x = 0$.

**Definition 2.6. Nil ideal:** An ideal $A$ of a $\Gamma$-ring $M$ is a nil ideal if every element of $A$ is nilpotent that is, for all $x \in A$ and some $\gamma \in \Gamma$, $(x\gamma)^nx = (x\gamma x\gamma...\gamma x\gamma)x = 0$, where $n$ depends on the particular element $x$ of $A$.

**Definition 2.7. Nilpotent ideal:** An ideal $A$ of a $\Gamma$-ring $M$ is called nilpotent if $(A\gamma)^nA = (A\gamma A\gamma...\gamma A\gamma)A = 0$, where $n$ is the least positive integer.

**Definition 2.8. Annihilator of a subset of a $\Gamma$-ring:** Let $M$ be a $\Gamma$-ring. Let $S$ be a subset of $M$. Then the left annihilator $l(S)$ of $S$ is defined by $L(S) = \{m \in M : m\gamma S = 0 \text{ for every } \gamma \in \Gamma\}$, whereas the right annihilator $r(S)$ is defined by $R(S) = \{m \in M : S\gamma m = 0 \text{ for every } \gamma \in \Gamma\}$.

**Definition 2.9. Idempotent element:** Let $M$ be a $\Gamma$-ring. An element $e$ of $M$ is called idempotent if $e\gamma e = e \neq 0$ for some $\gamma \in \Gamma$. 
Definition 2.10. Centre of a $\Gamma$-ring: Let $M$ be $\Gamma$-ring. The centre of $M$, written as $Z$, is the set of those elements in $M$ that commute with every element in $M$, that is, $Z = \{m \in M : m\gamma x = x\gamma m \text{ for all } x \in M \text{ and } \gamma \in \Gamma\}$.

Definition 2.11. $\Gamma M$-homomorphism: Let $M$ be a $\Gamma$-ring. Let $A$ and $B$ be the left ideals of $M$. A $\Gamma M$-homomorphism is a function $\phi : A \to B$ such that

(i) $\phi(x + y) = \phi(x) + \phi(y)$ for all $x, y \in A$
(ii) $(m\gamma x) = m\gamma (x)$ for all $x \in A, m \in M$ and $\gamma \in \Gamma$.

In case, $A$ and $B$ are right ideals, then (i) and (ii) become

(i) $\phi(x + y) = \phi(x) + \phi(y)$ for all $x, y \in A$
(ii) $\phi(x\gamma m) = \phi(x)\gamma m$ for all $x \in A, m \in M$ and $\gamma \in \Gamma$.

Definition 2.12. $\Gamma M$-isomorphism: Let $M$ be a $\Gamma$-ring. Let $A$ and $B$ be two left ideals of $M$. Let $\phi : A \to B$ be a $\Gamma M$-homomorphism from $A$ into $B$. We call $\phi$, a $\Gamma M$-isomorphism, if $\phi$ is one-one and onto. We say that $A$ and $B$ are $\Gamma M$-isomorphic and we write $A \cong B$.

Definition 2.13. Baer $\Gamma$-ring: A $\Gamma$-ring $M$ is called a Baer $\Gamma$-ring if the right annihilator of every non-empty subset of $M$ is generated by an idempotent element of $M$.

3. Baer Gamma Rings with Involution

Definition 3.1. Let $M$ be a $\Gamma$-ring. A mapping $I : M \to M$ is called an involution if

(i) $I(a + b) = I(a) + I(b)$,
(ii) $I(ab) = I(b)\alpha I(a)$ and
(iii) $I^2(a) = a$, for all $a, b \in M, \alpha \in \Gamma$.

Example 3.1. Let $R$ be an associative ring with 1 having an involution $^*$. Let $M = M1.2(R)$ and $\Gamma = \{ \begin{pmatrix} n_1 \cdot 1 \\ n_2 \cdot 1 \end{pmatrix} : n_1, n_2 \in \mathbb{Z} \}$. Then $M$ is a $\Gamma$-ring. Define $I : M \to M$ by $I((a, b)) = (a^*, b^*)$. Then it is clear that $I$ is an involution on $M$. 
We know that if $e$ is the idempotent elements of a $\Gamma$-ring $M$, then $M\Gamma e$ and $e\Gamma M$ are respectively left ideal and right ideal of $M$, which is shown in [7].

**Theorem 3.1.** Let $e$ and $f$ be idempotents in a $\Gamma$-ring $M$. The following are equivalent:

(1) $e \Gamma M, f \Gamma M$ are $\Gamma M$-isomorphism

(2) $M \Gamma e, M \Gamma f$ are $\Gamma M$-isomorphism

(3) There exist elements $x \in e \Gamma M \Gamma f, y \in f \Gamma M \Gamma e$ with $x\alpha y = e, y\alpha x = f, \alpha \in \Gamma$.

**Proof.** Since condition (3) is left-right symmetric it will suffice to identify (2) and (3). (3) implies (2). Map $M \Gamma e$ to $M \Gamma f$ by right multiplication by $x$, $M \Gamma f$ to $M \Gamma e$ by right multiplication by $y$. The product both ways to clearly the identity. (2) implies (3). Let $\phi$ be the map from $M \Gamma e$ to $M \Gamma f$ and set $\text{EL} \phi(e) = x$. Then since $\phi$ is a $\Gamma M$-homomorphism map we have $\phi(a) = \phi(a\alpha e) = a\alpha \phi(e) = a\alpha x$ for any in $a \in M \Gamma e$ and $\alpha \in \Gamma$ i.e., $\phi$ is a right multiplication by $x$. In particular $x = \phi(e) = e\alpha x$ and $x \in e \Gamma M \Gamma f$. Similarly the map from $M \Gamma f \Gamma$ to $M \Gamma e$ is right multiplication by an element $y \in f \Gamma M \Gamma e$. Evidently $x\alpha y = e, y\alpha x = f, \alpha \in \Gamma$. □

**Definition 3.2.** Idempotents $e, f$ in a $\Gamma$-ring $M$ are equivalent, written $e \sim f$ if they satisfy (and hence all) of the conditions in theorem 3.3. Note that a Baer $\Gamma$-ring is finite if and only whenever $e \sim 1$, then $e = 1$.

**Definition 3.3.** An element of a $\Gamma$-ring $M$ with involution $I$ is called self-adjoint if $I(x) = I$. A projection is a self-adjoint idempotent. A subset $S$ is self-adjoint if $x \in S$ implies $I(x) \in S$. A Baer $\Gamma$-ring with involution $I$ is a $\Gamma$-ring with involution $I$ such that for any subset $S, R(S) = e \Gamma M$ with $e$ a projection.

By applying the involution we get that in a Baer $\Gamma$-ring with involution $I$ the left annihilator of any subset is like wise generated by a projection. In particular, a Baer $\Gamma$-ring with involution $I$ is a Baer $\Gamma$-ring. The projection of $e$ generating $e \Gamma M$ is unique. For if $e \Gamma M = f \Gamma M$ with $e$ and $f$ projections, we find $e = f\alpha e, \alpha \in$
Γ, \( f = e\alpha f = I(e\alpha f) = I(f)\alpha I(e) = f\alpha e = I(f\alpha e) = I(e) = e. \) Because of this uniqueness, we can call \( e \) the right-annihilating projection of the subset \( S \) of \( M \).

Even more useful is \( g = 1 - e \) which we shall call the right projection of \( S \). Then \( s\alpha g = s\alpha(1 - e) = s\alpha 1 - s\alpha e = s \) for all \( s \in S \) and \( \alpha \in \Gamma \) is the smallest such projection. If \( f \) is an idempotent in a Baer \( \Gamma \)-ring with involution \( I \) and \( e \) is its right projection, we readily see that \( e \sim f \).

**Theorem 3.2.** Let \( e \) and \( f \) be idempotents in a Baer \( \Gamma \)-ring with involution \( I, f \in e\Gamma M\Gamma e. \) Let \( g \) and \( h \) be the right projections of \( e \) and \( f \). Then \( e - f \sim g - h. \)

**Proof.** Noting that \( g \geq h, e\gamma g = e, g\gamma e = g, f\gamma h = f, h\gamma f = f, \gamma \in \Gamma \) , we verify directly that \( e - e\gamma h \) and \( g - g\gamma f \) implement an equivalence of \( e - g \) and \( f - h. \)

**Definition 3.4.** For projection \( e, f \) in a \( \Gamma \)-ring with involution \( I \) write \( e \leq f \) in case \( e\gamma f = e, \gamma \in \Gamma \) (which is equivalent to \( e = f\gamma e \)). One readily sees that this relation makes the projections into a partially ordered set.

**Theorem 3.3.** The projections in a Baer \( \Gamma \)-ring with involution \( I \) form a complete lattice.

**Proof.** Given a family \( \{e_i\} \) of projections, let \( e \) be their right projection. One readily sees that \( e \) is the least upper bound (LUB) of \( e_i \)'s. Dually, there is a greatest lower bound (GLB). Hence the theorem is proved.

**Definition 3.5.** Let \( M \) be a Baer \( \Gamma \)-ring with involution \( I \). \( B \) is a sub \( \Gamma \)-ring of \( M \). We say that \( B \) is a Baer sub \( \Gamma \)-ring with involution \( I \) of \( M \) if

1. \( B \) is a self adjoint sub \( \Gamma \)-ring
2. If \( S \subset B \) and \( e \) is the right annihilating projection of \( S(iM), \) then \( e \in B. \)

If \( B \) is a Baer sub \( \Gamma \)-ring with involution \( I \), then \( B \) is itself obviously a Baer \( \Gamma \)-ring with involution \( I \). Its unity element is the same as that of \( M \) (take the annihilator of
0). The lattice of projections in $B$ is a complete sub lattice of that of $M$. If $M$ is a Baer $\Gamma$-ring with involution $I$ and $e$ is a projection in $M$, the projections of $e\Gamma Me$ are the projections of $f \in M$ with $f \leq e$. It follows easily that $e\Gamma Me$ is a Baer $\Gamma$-ring with involution $I$ and that a family of projections in $e\Gamma Me$ has the same LUB whatever computed in $e\Gamma Me$ or in $M$.

**Theorem 3.4.** Let $M$ be a Baer $\Gamma$-ring with involution $I$ and $S$ be a self-adjoint subset of $M$. Let $T$ be the commuting $\Gamma$-ring of $S$. Then $T$ is a Baer sub $\Gamma$-ring with involution $I$ of $M$.

**Proof.** Since $S$ is self-adjoint, the sub $\Gamma$-ring $T$ is also self-adjoint. Given $V \subset T$, write $R(V) = e\Gamma M$ (this is the annihilator in $M$ of course). We must show that $e$ lies in $T$. Thus given , we have to prove $e\gamma s = s\gamma e, \gamma \in \Gamma$. Given $s \in S$, we have $s\gamma v = v\gamma s$ and $v\gamma e = 0$, then $v\gamma(1 - e)\gamma s\gamma e = v\gamma s\gamma e - v\gamma e\gamma s\gamma e = v\gamma s\gamma e - 0 = v\gamma s\gamma e = s\gamma v\gamma e = s\gamma 0 = 0$. Since $v$ is arbitrary in $V, (1 - e)\gamma s\gamma e \in e\Gamma M$. Hence $(1 - e)\gamma s\gamma e = 0$. Thus $s\gamma e = e\gamma s\gamma e$. Apply involution $I$, we have $I(sve) = I(e\gamma s\gamma e)$. This implies that $I(e)\gamma I(s) = I(e)\gamma I(s)\gamma I(e)$. So $e\gamma s = e\gamma s\gamma e$. Hence $s\gamma e = e\gamma s$.

**Corollary 3.1.** The center of a Baer $\Gamma$-ring with involution $I$ is a Baer sub $\Gamma$-ring with involution $I$.

**Theorem 3.5.** In a Baer $\Gamma$-ring $M$ with involution $I. x\alpha I(x) = 0$ implies $x = 0, x \in M, \alpha \in \Gamma$.

**Proof.** Let $e$ be the right annihilating projection of $x$. Then $x\alpha e = 0$. Now $I(x\alpha e) = I(0) = 0$. This implies that $I(e)\alpha I(x) = 0$. So $e\alpha I(x) = 0$. Since $x\alpha I(x) = 0$, we have $I(x) \in e\Gamma M, I(x) = e\alpha I(x) = 0$. Now $x = I^2(x) = I(I(x)) = I(0) = 0$ It follows that a Baer $\Gamma$-ring with involution $I$ has no nil left or right ideals. For let $A$ be a nil right ideal in a Baer $\Gamma$-ring with involution $I$. If $x \in A$, then $y = x\alpha I(x) \in A$. If
(yα)n y is the smallest power of y that is 0, let z = (yα)n−1 y. Then zαI(z) = zαz = 0 whence z = 0 by theorem 3.12. Hence x = 0 . The argument for a nil left ideal is analogous. □

A fortiori, a Baer Γ-ring with involution I has no nilpotent ideals. We turn now to the consideration of equivalence of projection in a Baer Γ-ring with involution I.

**Theorem 3.6.** Let M be a Baer Γ-ring with involution I, an element of M such that xαI(x) is a projection of e for α ∈ Γ. Then I(x)αx is also a projection of f. We have x ∈ eΓMΓf, I(x) ∈ fΓMΓe and thus e ∼ f.

**Proof.** Set y = eαx − x. Then

\[
yαI(y) = (eαx − x)αI(eαx − x) = ((e − 1)αx)αI(eαx − x) = ((e − 1)αx)α(I(x)αI(e) − I(x)) = ((e − 1)αx)α(I(x)αe − I(x)) = (e − 1)αxαI(x)α(e − 1) = (e − 1)αeα(e − 1) = (eαe − 1αe)α(e − 1) = (e − e)α(e − 1) = 0α(e − 1) = 0α(e − 1) = 0
\]

By Theorem 3.12, y = 0. So eαx − x = 0. Thus eαx = x. If f = I(x)αx then I(f) = I(I(x)αx) = I(x)αI2(x) = I(x)αx = f and fαf = I(x)αxαI(x)αx =
$I(x)e\alpha x = I(x)\alpha x = f$. Now $f\alpha I(x) = I(x)\alpha x I(x) = I(x)\alpha e = I(x)$. Thus $x\alpha I(x)\alpha x = e\alpha x = x, I(x) \in f\Gamma M e$. Hence $e \sim f$. \hfill \Box$

**Definition 3.6.** An element $x$ in a $\Gamma$-ring $M$ with involution $I$ is called a partial isometry if $x\alpha I(x)$ and $I(x)\alpha x, \alpha \in \Gamma$ are projections.

**Definition 3.7.** In a $\Gamma$-ring $M$ with involution $I$, projections $e, f$ are called $I$-equivalent, written $e \sim_I f$, if there exists a partial isometry $x \in e \Gamma M f$ with $x\alpha I(x) = e, I(x)\alpha x = f$.

It is easy verified that $\sim_I$ is an equivalence relation and that $e \sim_I f$ implies $e \sim f$. Note that if $M$ is a Baer $\Gamma$-ring with involution $I$ the condition $x \in e \Gamma M f$ in the definition of $I$-equivalence is redundant (Theorem 3.13). We now wish to make some comparisons between Baer $\Gamma$-rings and Baer $\Gamma$-rings with involution $I$. We begin by exhibiting a condition that can convert a Baer $\Gamma$-ring into a Baer $\Gamma$-ring with involution $I$.

**Theorem 3.7.** Let $M$ be a $\Gamma$-ring with involution $I$ and suppose that for every $x$ in $M, 1 + I(x)\alpha x, \alpha \in \Gamma$ is invertible in $M$. Then for any idempotent $f$ in $M$ there exists a projection $e$ such that $f\Gamma M = e\Gamma M$.

**Proof.** Let $x = I(f) - f$. Then $I(x) = I(I(f) - f) = f - I(f)$. Therefore $I(x)\alpha x = (f - I(f))\alpha (I(f) - f)$. So $1 + I(x)\alpha x = 1 + (f - I(f))\alpha (I(f) - f)$. Since $1 + I(x)\alpha x$ is invertible in $M, 1 + (f - I(f))\alpha (I(f) - f)$ is invertible in $M$. Take $z = 1 + (f - I(f))\alpha (I(f) - f)$. Then, $z$ is invertible, say $t = z^{-1}$. Also we have, $z = I(z)$,
then $t = I(t)$. Therefore

$$faz = f\alpha(1 - f - I(f) + f\alpha I(f) + I(f)\alpha f) = f\alpha 1 - f\alpha f - f\alpha I(f) + f\alpha I(f)\alpha f = f - f - f\alpha I(f) + f\alpha I(f) + f\alpha I(f)\alpha f = f\alpha I(f)\alpha f$$

Similarly $z\alpha f = f\alpha I(f)\alpha f$. It follows that $t$ commutes with $f$. We have also seen that $t$ commutes with $I(f)$. Now we choose $e = f\alpha I(f)\alpha t$. Then $I(e) = I(f\alpha I(f)\alpha t) = I(t)\alpha I^2(f)\alpha I(f) = t\alpha f\alpha I(f) = f\alpha t\alpha I(f) = f\alpha I(f)\alpha t = e$. Also

$$eae = f\alpha I(f)\alpha t\alpha f\alpha I(f)\alpha t = t\alpha f\alpha I(f)\alpha f\alpha I(f)\alpha t = t\alpha(f\alpha I(f)\alpha f)\alpha I(f)\alpha t = t\alpha z\alpha f\alpha I(f)\alpha t = (t\alpha z)\alpha(f\alpha I(f)\alpha t) = 1ae = e$$

Thus $e$ is a projection. Evidently $fae = e$ whence $e\Gamma M \subset f\Gamma M$. Again

$$eaf = f\alpha I(f)\alpha f\alpha t = faz\alpha t = f\alpha(z\alpha t) = f\alpha 1 = f$$

Therefore $f\Gamma M \subset e\Gamma M$. Hence $f\Gamma M = e\Gamma M$. □

**Corollary 3.2.** Let $M$ be a Baer $\Gamma$-ring with an involution $I$ and suppose that $1 + I(x)\alpha x, \alpha \in \Gamma$ is invertible for every $x$ in $M$. Then $M$ is a Baer $\Gamma$-ring with involution $I$. 
Next we give a condition which identifies the two versions of equivalence.

**Theorem 3.8.** Let $M$ be a $\Gamma$-ring with involution $I$. Assume that for any $y \in M$ there exists a self-adjoint $z \in M$ which commutes with everything that commutes with $I(y)\alpha y$ and satisfies $z\alpha z = I(y)\alpha y, \alpha \in \Gamma$. Then equivalent projections in $M$ are $I$-equivalent.

**Proof.** Let the projections $e, f$ be equivalent via $x, y, x \in e \\Gamma M \Gamma f, y \in f \\Gamma M \Gamma f, x\alpha y = e, y\alpha x = f$. Choose $z$ (relative to $y$) are permitted by the hypothesis. We have $x\alpha I(x)\alpha I(y)\alpha y = x\alpha I(y\alpha x)\alpha y = x\alpha I(f)\alpha y = x\alpha y = e$ Since $e$ is self-adjoint, $e = I(e)$. Then $e = I(e) = I(x\alpha I(x)\alpha I(y)\alpha y) = I(y)\alpha I^2(y)\alpha I^2(x)\alpha I(x) = I(y)\alpha y\alpha x\alpha I(x)\alpha I(x)$ Therefore $x\alpha I(x)\alpha I(y)\alpha y = I(y)\alpha y\alpha x\alpha I(x)$. Thus $x\alpha I(x)$ commutes with $y\alpha I(y)$ and hence also with $z$. Now we have $I(y)\alpha yae = I(y)\alpha y$. Then $I(I(y)\alpha yae) = I(I(y)\alpha yae)$. So, $I(e)\alpha I(y)\alpha I^2(y) = I(y)\alpha I^2(y)$. Thus $e\alpha I(y)\alpha y = I(y)\alpha(y)$. Hence $e\alpha I(y)\alpha y = I(y)\alpha(y)ae$. The element $w = e\alpha zax \in e \\Gamma M \Gamma f$ implements the desired $I$-equivalence of $e$ and $f$. Now

$$I(w)\alpha w = I(e\alpha zax)\alpha(e\alpha zax)$$

$$= I(x)\alpha I(z)\alpha I(e)\alpha eaxz$$

$$= I(x)\alpha zaeaeaz$$

$$= I(x)\alpha zaeaz$$

$$= I(x)\alpha azazx$$

$$= I(x)\alpha azx$$

$$= I(x)\alpha I(y)\alpha yax$$

$$= I(yax)\alpha(yax)$$

$$= I(f)\alpha f = f.$$
Again we have

\[ w\alpha I(w) = e\alpha z\alpha x I(e\alpha z\alpha x) \]
\[ = e\alpha z\alpha x I(x)\alpha I(z)\alpha I(e) \]
\[ = e\alpha z\alpha x I(x)\alpha z\alpha e \]
\[ = e\alpha (x\alpha I(x))\alpha Z\alpha z\alpha e \]
\[ = e\alpha x\alpha I(x)\alpha Z\alpha z\alpha e \]
\[ = x\alpha I(x)\alpha I(y)\alpha y\alpha e \]
\[ = x\alpha I(y\alpha x)\alpha y\alpha e \]
\[ = x\alpha I(f)\alpha y \]
\[ = x\alpha f\alpha y \]
\[ = x\alpha y = e. \]

Hence \( e \) and \( f \) are \( I \)-equivalent. \( \square \)

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