EDGE MAXIMAL GRAPHS CONTAINING NO SPECIFIC WHEELS

M.S.A. BATAINEH (1), M.M.M. JARADAT (2) AND A.M.M. JARADAT (3)

ABSTRACT. Let \( k \geq 4 \) be a positive integer. Let \( G(n; W_k) \) denote the class of graphs on \( n \) vertices containing no wheel \( W_k \) as a subgraph. In this paper, we study the following: (1) Edge maximal graphs containing no odd wheels. Furthermore, we characterize the extremal graphs. (2) The edge maximal graph containing no even wheels. (3) The edge maximal graph containing no specific even wheels.

1. Introduction.

Unless otherwise specified a graph \( G \) is finite, undirected, and has no loops or multiple edges. We denote the vertex set of \( G \) by \( V(G) \), the edge set of \( G \) by \( E(G) \) and the number of edges of \( G \) by \( \mathcal{E}(G) \). The cycle on \( n \) vertices is denoted by \( C_n \). A wheel graph \( W_n, n \geq 4 \) is defined to be a cycle \( C_{n-1} \) to which we add a new vertex that is adjacent each vertex of \( C_{n-1} \). Let \( G \) be a graph, and \( u \in V(G) \). The degree of a vertex \( u \) in \( G \), denoted by \( d_G(u) \), is the number of edges of \( G \) incident to \( u \). \( \Delta(G) \) stands for the maximum degree in \( G \). The neighbor set of a vertex \( u \) of \( G \) is the set of adjacent vertices to \( u \), and denoted by \( N_G(u) \). For vertex disjoint subgraphs

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For a positive integer $n$ and a set of graphs $\mathcal{F}$, let $\mathcal{G}(n;\mathcal{F})$ denote the class of non-bipartite $\mathcal{F}$-free graphs on $n$ vertices, and

$$f(n;\mathcal{F}) = \max\{E(G) : G \in \mathcal{G}(n;\mathcal{F})\}.$$  

An important problem in extremal graph theory is to determine the value of the function $f(n;\mathcal{F})$ and to characterize the extremal graphs of $\mathcal{G}(n;\mathcal{F})$ in which $f(n;\mathcal{F})$ is attained. This problem has been studied by many authors, see [2, 3, 5, 13]. We state the following powerful result which determines the asymptotic behavior of the maximal graphs in many situations:

**Theorem 1.1.** (Erdos-Stone-Simonovits) Let $\mathcal{F}$ be any finite set of graphs and $r$ be the minimum chromatic number of $F \in \mathcal{F}$. Then

$$f(n;\mathcal{F}) = (1 - \frac{1}{r-1}) \left(\frac{n^2}{2}\right) + o(n^2).$$

One can notice that if $r = 2$ (i.e., if any of the subgraphs of $\mathcal{F}$ is bipartite), then this theorem does not tell us much.

Moon [13] proved that, if $G$ is a graph on $n$ vertices containing no wheels, then $E(G) \leq \left\lfloor \frac{n^2}{4} \right\rfloor + \left\lfloor \frac{n+1}{4} \right\rfloor$. Furthermore, he characterized the extremal graphs. Alhayyel and et al [1] proved that (1) $f(n;W_5) = \left\lfloor \frac{n-2}{4} \right\rfloor + \left\lfloor \frac{s}{4} \right\rfloor$ for $n \geq 3$ where $s = n$ if $n \neq 4k + 2$ and $s = n - 1$ if $n = 4k + 2$ and (2) $f(n;W_6) = \left\lfloor \frac{n^2}{3} \right\rfloor$ for $n \geq 6$.

The results cited above trigger off the following question: Can we determine the exact value of $f(n;\mathcal{F})$, and characterize the extremal graphs for the following cases?:

1. $\mathcal{F} = \{W_5, W_7, \ldots, W_{2k+1}, \ldots\}$ which is the family of all odd wheels.
2. $\mathcal{F} = \{W_4, W_6, \ldots, W_{2k}, \ldots\}$ which is the family of all even wheels.
(3) $\mathcal{F} = \{W_{2k}\}$ which consists of a specific even wheel where $k \geq 2$ is a positive integer.

(4) $\mathcal{F} = \{W_{2k+1}\}$ which consists of specific even wheel where $k \geq 2$ is a positive integer.

In this paper we determine the extremal function for the class of graphs that contains no odd wheels and characterize the extremal graphs as well which is the answer for question 1. Furthermore, we determine the edge maximal graphs containing no even wheels and graphs containing no specific even wheels which answers the first part of question 2 and 3. We pose the other problems as future ideas to explore. The following theorem will be used in proving our results which can be found in [6].

**Lemma 1.2.** (Bondy) Let $G$ be a graph on $n$ vertices with $\mathcal{E}(G) > \left\lceil \frac{n^2}{4} \right\rceil$. Then $G$ contains cycles of every length $l$ for $3 \leq l \leq \left\lfloor \frac{n+3}{2} \right\rfloor$.

### 2. Edge maximal $W_k$ free graphs

We begin with some constructions which are similar to the constructions made in [13]. Let $G$ be a graph with $n \geq 5$ vertices, let $H_n$ denote the class of graphs obtained by splitting the vertices of $G$ into two sets, $P$ and $Q$, with $\left\lfloor \frac{n+1}{2} \right\rfloor$ and $\left\lfloor \frac{n}{2} \right\rfloor$, respectively. There are as many edges joining pairs of vertices in $P$ (and analogously in $Q$) as are consistent with the requirement that no two of these edges have a vertex in common. In addition, each vertex in $P$ is adjacent to each vertex in $Q$. If $n \equiv 2 \mod 4$, let $L_n$ denote the class of graphs obtained as above except that $P$ and $Q$ have $\frac{n}{2} + 1$ and $\frac{n}{2} - 1$ vertices. Let $l$ be a positive integer, then the following properties of these graph can readily be verified: (2.1) $H_n$ and $L_n$ each have $\left\lfloor \frac{n^2}{4} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor$ edges if $n \neq 4l + 2$ and $\left\lfloor \frac{n^2}{4} \right\rfloor + \left\lfloor \frac{n-1}{2} \right\rfloor$ edges if $n = 4l + 2$.

(2.2) Neither $H_n$ nor $L_n$ contain any odd wheels.
(2.3) At least one odd wheel is formed when any new edge is added to $H_n$ or $L_n$.

**Theorem 2.1.** Let

$$
\varnothing(n) = \begin{cases} 
4l^2 + 2l, & \text{if } n = 4l \\
4l^2 + 4l, & \text{if } n = 4l + 1 \\
4l^2 + 6l + 1, & \text{if } n = 4l + 2 \\
4l^2 + 8l + 3, & \text{if } n = 4l + 3.
\end{cases}
$$

Then the only graphs with $n \geq 5$ vertices and at least $\varnothing(n)$ edges which contain no odd wheels are $H_n$ or $L_n$. Furthermore, $f(n, \mathcal{F}) = \varnothing(n)$ where $\mathcal{F} = \{W_5, W_7, \ldots, W_{2k+1}, \ldots\}$ which is the family of all odd wheels.

**Proof:** We use induction on $n$ to prove the theorem:

**n = 5.** One can easily see that the only graph $G$ on 5 vertices and 8 edges containing no odd wheels is $K_4$ plus a vertex $v$ adjacent to two vertices of $K_4$ (see Figure 1). Thus, $G = H_5$.

![Figure 1](image.png)

**Figure 1.** Represents the situation when $n = 5$

**n = 6.** Let $G$ be a graph on 6 vertices, 11 edges and contains no odd wheels. Since $(4)(6) > (2)(11)$, by Handshaking Lemma, $G$ must have a vertex with degree less than or equal 3, say $f$. If the degree of $f$ is equal to 1 or 2, then $\mathcal{E}(G - f) = 11 - \deg(f) \geq 9$. And so, by the Case $n = 5$, $G - f$ has a $W_5$ as a subgraph. To this end, $d_G(f) = 3$. 
Let $H$ be the graph obtained by removing $f$. Using the case $n = 5$ and (2.3) one can see that $H = H_5$. Let $\{a, b, c, d, v\}$ be the vertex set of $H_5$ as in Figure 1. Note that $f$ cannot be adjacent to three vertices of $\{a, b, c, d\}$ as otherwise a $W_5$ is produced. Also, if $f$ is adjacent to $v$ and adjacent to one vertex of $\{a, b\}$, say $a$, and one vertex of $\{c, d\}$, say $c$, then $f$ is adjacent to every vertex of the 4-cycle $vcabv$. Thus, $W_5$ is produced. Hence $f$ is adjacent to $v$ and adjacent to either one of the following (1) $a$ and $b$ and so $G$ must be $H_6$ (see Figure 2.A) or (2) $c$ and $d$ and so $G$ must be $L_6$ (see Figure 2.B).

![Figure 2](image)

**Figure 2.** Represents the situation when $n = 6$

$n = 7$. Let $G$ be a graph on 7 vertices, 15 edges and contains no odd wheels. Note that (5)(7) > (2)(15). Then $G$ contains a vertex with degree at most 4, say $x$. If degree of $x$ is less than or equal to 3, then $E(G - x) = 15 - d_G(x) \geq 12$. And so, by the Case $n = 6$, $G - x$ has a $W_5$ as a subgraph. To this end, $d_G(x) = 4$. Let $H$ be the graph obtained by removing $x$. Using the case $n = 6$ and (2.3), we get that $H$ must be either $H_6$ (as in Figure 2.A) or $L_6$ (as in Figure 2.B).
As in above, \( x \) can not be adjacent to three vertices of \( \{a, b, c, d\} \). So \( x \) is adjacent to \( v \) and \( f \) and to two vertices of \( \{a, b, c, d\} \). Now we consider the case \( H = H_6 \). It is easy to see that if \( x \) is adjacent to a vertex of \( \{a, b\} \) and a vertex of \( \{c, d\} \), then \( W_5 \) is produced. Therefore, \( x \) is adjacent to either one of the following (1) both \( a \) and \( b \) or (2) both \( c \) and \( d \). Note that either (1) or (2) implies that \( G = H_7 \) (see Figure 3A and B). To end this, we consider that \( H = L_6 \). As in above, one can see that if \( x \) is adjacent to one vertex of \( \{a, b\} \) and a vertex of \( \{c, d\} \) or is adjacent to both \( c \) and \( d \), then \( W_5 \) is produced. Thus, \( x \) must be adjacent to both \( a \) and \( b \) which implies that \( G = H_7 \) (see Figure 3C).

Figure 3. Represents the situation when \( n = 7 \)

\( n = 8 \). Let \( G \) be a graph on 8 vertices and 20 edges. A similar argument to arguments in case \( n = 7 \) will show that there needs to be a vertex of degree 5 in \( G \), say \( z \). Consider \( H \) to be the graph obtained by removing \( z \). Then, by similar arguments to above, \( H = H_7 \) (we may assume it as in Figure 3A). Also, \( z \) is adjacent to \( v, f, x \) and to two vertices of \( \{a, b, c, d\} \). Furthermore, if \( z \) is adjacent to one vertex of \( \{a, b\} \) and one vertex of \( \{c, d\} \) or to both vertices \( c \) and \( d \), then \( W_5 \) is produced. Thus, \( z \) must be adjacent only to both vertices \( a \) and \( b \) and hence, \( G = H_8 \).
Now, suppose that the result holds for \( m \geq 8 \). Let \( G \) be a graph with \( m + 1 \) vertices and at least \( \emptyset(m + 1) \) edges which contains no odd wheels. First we assume that \( G \) has exactly \( \emptyset(m + 1) \) edges. Now, we show that \( G = H_{m+1} \) or \( L_{m+1} \). We consider four cases according to \( m \).

**Case 1:** \( m = 4l \). Then \( G \) has \( 4l + 1 \) vertices and \( 4l^2 + 4l \) edges. Since \((4l + 1)(2l + 2) > 2(4l^2 + 4l)\), there is a vertex \( x \) in \( G \) with degree \( d \) where \( d \leq 2l + 1 \). If \( d < 2l \), then \( E(G - x) = 4l^2 + 4l - d > 4l^2 + 2l \). And so, by the induction step, \( G - x \) has a \( W_5 \) as a subgraph. To this end, \( d \in \{2l, 2l + 1\} \). Let \( G' \) be the graph obtained by removing \( x \). Then \( G' \) has \( 4l \) vertices. We now consider the following two subcases:

**Subcase 1.1:** \( d = 2l \). Then \( G' \) has no odd wheels, \( 4l \) vertices and \( \emptyset(4l + 1) - d = \emptyset(4l + 1) - 2l = \emptyset(4l) \). It follows from the induction hypothesis and (2.1) that \( G' = H_m \) and \( d = 2l \). Hence \( G \) consists of \( H_m \) and the vertex \( x \) which adjacent to precisely \( 2l \) vertices of \( H_m \). Clearly, each of \( P \) and \( Q \) contains only \( 2l \) vertices.

Now, observe that \( x \) can not be adjacent to more than one vertex of \( P \) and to one vertex of \( Q \) simultaneously, for if \( x \) is adjacent to two vertices of \( P \), say \( p_1 \) and \( p_2 \), and to one vertex of \( Q \) say \( q_1 \), then \( q_1 \) is adjacent to every vertex of the cycle \( p_1xp_2q_2p_1 \) which forms \( W_5 \) as a subgraph of \( G \), where \( q_2 \) is the neighbor of \( q_1 \) in \( Q \), a contradiction. Analogously, \( x \) can not be adjacent to more than one vertex of \( Q \) and to one vertex of \( P \) simultaneously.

This observation and the fact that both \( P \) and \( Q \) have \( 2l \) vertices, leave the following alternatives: The vertex \( x \) is adjacent to every vertex of \( P \) and to no vertex of \( Q \) or \( x \) is adjacent to every vertex of \( Q \) and to no vertex of \( P \), which implies that \( G = H_{m+1} \).

**Subcase 1.2:** \( d = 2l + 1 \). Then \( G' \) has \( 4l \) vertices and \( 4l^2 + 4l - 2l - 1 \) edges. Since \( 4l(2l + 1) > 2(4l^2 + 2l - 1) \), as a result \( G' \) has a vertex, say \( y \), such that \( d_{G'}(y) \leq 2l \). By the same argument as in the above, we need only to consider \( d_{G'}(y) = 2l \), so we consider this case only. Let \( G'' = G' - y \). Then \( G'' \) has \( 4l - 1 \) vertices and \( 4l^2 - 1 \) edges.
edges, so by induction step, \( G'' \) must be \( H_{4l-1} \). Moreover, \( P \) contains only \( 2l \) and \( Q \) contains only \( 2l - 1 \) vertices. Now, \( G' \) consists of \( H_{4l-1} \) and the vertex \( y \) which is adjacent to precisely \( 2l \) vertices of \( H_{4l-1} \). Clearly, as in Subcase 1.1, \( y \) can not be adjacent to more than one vertex of both \( P \) and \( Q \) simultaneously. This observation and the fact that \( y \) must be adjacent to at least one vertex of \( P \), since \( Q \) contains only \( 2l - 1 \) vertices, leaves the following alternatives:

(a) The vertex \( y \) is adjacent to every vertex of \( P \) and to no vertex of \( Q \).
(b) The vertex \( y \) is adjacent to \( 2l - 1 \) vertices of \( P \) and to one vertex of \( Q \).
(c) The vertex \( y \) is adjacent to every vertex of \( Q \) and to one vertex of \( P \).

We know that \( G \) consists of \( G' \) plus the vertex \( x \) which is adjacent to precisely \( 2l + 1 \) vertices of \( G' \). So we have the following possibilities according to the alternative (a), (b) and (c):

**I** Alternative (a) implies that \( G' \) is a graph obtained by splitting the vertices of \( G' \) into two sets, \( P_1 \) and \( Q_1 \) each of which contains only \( 2l \) vertices. There are two nonadjacent vertices, say \( q_1 \) and \( q_2 \), in \( Q_1 \) (one of them is \( y \)) each of which is adjacent to no vertex of \( Q_1 \). Further, there are as many edges joining pairs of vertices in \( P_1 \) (and analogously \( Q_1 \)) as are consistent with the requirement that no two of these edges have a vertex in common. In addition, each vertex in \( P_1 \) is adjacent to each vertex in \( Q_1 \). As in the observation of Subcase 1.1, \( x \) can not be adjacent to more than one vertex of both \( P_1 \) and \( Q_1 \) simultaneously. This observation and the fact that \( x \) should be adjacent to at least one vertex of \( P \) or \( Q \), since both \( P \) and \( Q \) have \( 2l \) vertices, leaves the following possibilities:

(i) \( x \) is adjacent to \( 2l \) vertices of \( P_1 \) and to one vertex, say \( q \), of \( Q_1 \). If this is the case, then \( q \in \{q_1, q_2\} \) and so \( G = H_{m+1} \) since otherwise \( q \) is adjacent to every vertex of the cycle \( p_1q_2q^*p_1 \) where \( q^* \) is the neighbor of \( q \) in \( Q_1 \) and \( p_1, p_2 \) are two vertices of \( P_1 \) which forms \( W_5 \) as a subgraph of \( G \), a contradiction.
(ii) $x$ is adjacent to $2l$ vertices of $Q_1$ and to one vertex $p_1$ of $P_1$. If this is the case, then by interchanging $Q_1$ and $P_1$ in (i) we get the same contradiction.

(II) Alternative (b) and (2.3) imply that $G'$ is a graph obtained by splitting the vertices of $G'$ into two sets, $P_1$ and $Q_1$ each of which contains only $2l$ vertices. There are as many edges joining pairs of vertices in $P_1$ (and analogously $Q_1$) as are consistent with the requirement that no two of these edges have a vertex in common. In addition, each vertex in $P_1$ is adjacent to each vertex in $Q_1$ except that there is a vertex $q = y$ in $Q_1$ which is not adjacent to a vertex $p$ in $P_1$. As above, $x$ cannot be adjacent to more than one vertex of both $P_1$ and $Q_1$ simultaneously. Similarly, this observation and the fact that $x$ should be adjacent to at least one vertex of $P$ or $Q$, since both $P$ and $Q$ have $2l$ vertices, leaves the following possibilities:

(i) $x$ is adjacent to $2l$ vertices of $P_1$ and to one vertex $q$ of $Q_1$. If this is the case, then the vertex $q$ is adjacent to every vertex of the cycle $p^* x p_2 q^* p^*$ where $q^*$ is the neighbor of $q$ in $Q_1$ and $p^*, p_2$ are two vertices of $P_1$ which forms $W_5$ as a subgraph of $G$, so this case is impossible.

(ii) $x$ is adjacent to $2l$ vertices of $Q_1$ and to one vertex of $P_1$, then as in the above (i), $W_5$ is produced.

(III) By using observation of Subcase 1.1, the alternative (c) is impossible, and so this case is impossible.

Case 2: $m = 4l + 1$. Then $G$ has $4l + 2$ vertices and $4l^2 + 6l + 1$ edges. Since $(4l + 2)(2l + 2) > 2(4l^2 + 6l + 1)$, as in Case 1, we need only to consider that there is a vertex, say $x$, in $G$ of degree $d$ where $d \leq 2l + 1$. Let $G'$ be the graph obtained by removing $x$. Then $G'$ has $4l + 1$ vertices and contains no odd wheels. Further $\mathcal{D}(4l + 2) - d \geq \mathcal{D}(4l + 2) - (2l + 1) = \mathcal{D}(4l + 1)$. It follows from the induction hypothesis, (2.1) and a similar argument to as in Case 1, we get that $G' = H_m$ and $d = 2l + 1$. Hence $G$ consists of $H_m$ and the vertex $x$ which joins to precisely $2l + 1$ vertices of $H_m$. We now consider the possibilities. First, suppose that $G' = H_m$. 
Clearly, $P$ contains only $2l + 1$ vertices and $Q$ contains only $2l$ vertices. As in Case 1, $x$ can not be adjacent to more than one vertex of both $P$ and $Q$ simultaneously. This observation and the fact that $x$ must be adjacent to at least one vertex of $P$, since $Q$ contains $2l$ vertices, leaves the following alternatives:

a) The vertex $x$ is adjacent to every vertex of $P$ and to no vertex of $Q$.

b) The vertex $x$ is adjacent to $2l$ vertices of $P$ and to one vertex $q_1$ of $Q$.

c) The vertex $x$ is adjacent to one vertex $p_1$ of $P$ and to every vertex of $Q$.

Alternative (a) implies that $G = H_{m+1}$, by definition. Alternative (b) is impossible for the same reason as in the observation of Subcase 1.1. In alternative (c) if $p_1$ is adjacent to another vertex $p_2$ in $P$, then $p_1$ is adjacent to every vertex of the cycle $q_2 x q_1 p_2 q_2$ where $q_1, q_2$ are two vertices in $Q$ which forms $W_5$ as a subgraph. Hence $p_1$ is the only vertex that is adjacent to no other vertices of $P$. This implies $G = L_{m+1}$.

Case 3: $m = 4l + 2$. Then $G$ has $4l + 3$ vertices and $4l^2 + 8l + 3$ edges. By using Handshaking lemma, there is some vertex $x$ in $G$ of degree $d$ where $d \leq 2l + 2$. As in the argument in Case 1 in which we excluded the case $d < 2l + 2$. As in the argument in Case 1 in which we excluded the case $d < 2l - 1$, then $G$ has $W_5$. So, we consider that $d = 2l + 2$. Let $G'$ be the graph obtained by removing $x$ and its $d$ incident edges from $G$. Then by using the same argument as in Case 1 and by the inductive hypothesis $G'$ must be either $H_m$ or $L_m$. Hence $G$ consists of $H_m$ (or $L_m$) and the vertex $x$ which joins to precisely $2l + 2$ vertices of $H_m$ (or $L_m$). We now consider the possibilities. First, suppose that $G' = H_m$. Clearly, each of $P$ and $Q$ contains only $2l + 1$ vertices. As in the observation of Subcase 1.1, $x$ can not be adjacent to more than one vertex of both $P$ and $Q$ simultaneously. This observation and the fact that $x$ must be adjacent to at least one vertex in $P$ or $Q$, since both $P$ and $Q$ contain only $2l + 1$ vertices, leaves the following alternatives:

a) The vertex $x$ is adjacent to every vertex of $P$ and one vertex $q_1$ of $Q$. 
b) The vertex $x$ is adjacent to one vertex $p_1$ of $P$ and to every vertex of $Q$.

If the alternative (a) is the case, then by using the same argument as in the alternative (c) of Case 2, we can show that $q_1$ is the only vertex adjacent to no other vertices of $Q$. This implies $G = H_{m+1}$. Similarly, if alternative (b) is the case, then $G = H_{m+1}$.

Now we consider the case $G' = L_m$. By the definition of $L_m$, $P$ contains only $2l + 2$ and $Q$ contains only $2l$ vertices. As above $x$ can not be adjacent to two vertices of $P$ and two vertices of $Q$, simultaneously. This observation and the fact that $x$ must be adjacent to at least one vertex of $P$, since $Q$ contains $2l$ vertices, leave the following alternatives:

(a) $x$ is adjacent to $2l + 2$ vertices of $P$.

(b) $x$ is adjacent to $2l + 1$ vertices of $P$ and one vertex of $Q$. If alternative (a) is the case, then $G = H_{m+1}$. By observation of Subcase 1.1, alternative (b) is impossible to happen.

**Case 4:** $m = 4l - 1$. Similarly to the other cases, we see that there is a vertex $x$ of degree at most $2l + 1$. However, it is impossible for the degree to be less than or equal to $2l$ because otherwise as in Case 1, $W_5$ is a subgraph of $G$. Further, the subgraph $G'$ that is obtained by removing $x$ is $H_m$. Hence $G$ consists of $G' = H_m$ plus the vertex $x$ which joins to precisely $2l + 1$ vertices of $H_m$. Clearly $P$ contains only $2l$ vertices and $Q$ contains only $2l - 1$ vertices. As above, it is clear that $x$ can not be adjacent to more than one vertex of both $P$ and $Q$ simultaneously. This observation and the fact that $x$ must be adjacent to at least 2 vertices of $P$, and $Q$ contains $2l - 1$ vertices, leave the following alternatives: The vertex $x$ is adjacent to every vertex of $P$ and one vertex $q_1$ of $Q$. By a similar argument as in alternative (c) of Case 2, $q_1$ can only be the vertex adjacent to no other vertices of $Q$. This implies that $G = H_{m+1}$.

For the case where $G$ has $\emptyset(m + 1) + \alpha$ edges where $\alpha \geq 1$, consider $H'$ be a graph obtained from $G$ by deleting any $\alpha$ edges. Then $H'$ has $\emptyset(m + 1)$ edges. By the
above cases $H'$ is either $H_m$ or $L_m$ and so by (2.3) $G$ has an odd wheel. The proof is completed.

Now, we consider edge maximal graphs without $W_{2k}$. Let $\mathcal{W}(n)$ be the class of complete tripartite graph $K_{n_1,n_2,n_3}$ where $n = n_1 + n_2 + n_3$ is a partition of $n$ into three parts which are as equal as possible. Note that if $G \in \mathcal{W}(n)$, then $G$ is a $W_{2k}$-free graph and $\mathcal{E}(G) = \left\lfloor \frac{n^2}{3} \right\rfloor$. Thus, $f(n; W_{2k}) \geq \left\lfloor \frac{n^2}{3} \right\rfloor$. In the following theorem, we determine the edge maximal graphs containing no even wheel.

**Theorem 2.2.** Let $k \geq 2$ be a positive integer and $G$ be a graph containing no even wheel of order $2k$. Then for $n \geq 6(k - 1)$

$$\mathcal{E}(G) \leq \left\lfloor \frac{n^2}{3} \right\rfloor.$$ 

Furthermore, the bound is best possible.

**Proof:** Let $G$ be a graph containing no even wheel of order $2k$. Let $u \in V(G)$ such that $\Delta(G) = d_G(u)$, say $d_G(u) = m$ for some positive integer $m$. If $m < \left\lceil \frac{2n}{3} \right\rceil$, then

$$2\mathcal{E}(G) = \sum_{v \in V(G)} d_G(v) < \left\lceil \frac{2n}{3} \right\rceil n.$$ 

Hence, $\mathcal{E}(G) < \left\lfloor \frac{n^2}{3} \right\rfloor$. So we need to consider the case when $m \geq \left\lceil \frac{2n}{3} \right\rceil$. Let $N_G(u) = \{v_1, v_2, ..., v_m\}$ be the neighbors of $u$ in $G$. Define $H_1 = G[v_1, v_2, ..., v_m]$ and $H_2 = G - (H_1 \cup \{u\})$. Observe that $H_1$ contains no odd cycle of length $2k - 1$ as otherwise $G$ would have $W_{2k}$ as a subgraph. Thus, by Lemma 1.2, $\mathcal{E}(H_1) \leq \left\lfloor \frac{m^2}{4} \right\rfloor$. Further, $d_G(w) \leq m$ for every $w \in V(H_2)$, and hence $\mathcal{E}(H_2) + \mathcal{E}(H_1, H_2) \leq m(n - m - 1)$. 

So we have

\[\mathcal{E}(G) = \mathcal{E}(u, H_1 \cup H_2) + \mathcal{E}(H_1) + \mathcal{E}(H_2) + \mathcal{E}(H_1, H_2)\]

\[\leq m + \frac{m^2}{4} + m(n - m - 1)\]

\[= m + \frac{m^2}{4} + nm - m^2 - m\]

\[= nm - \frac{3m^2}{4}.\]

Define \(g(m) = nm - \frac{3m^2}{4}\). Observe that \(g\) has the maximum value at \(m = \frac{2n}{3}\). So we get that \(\mathcal{E}(G) \leq \left\lfloor \frac{n^2}{3} \right\rfloor\). One notes that the bound is achievable by \(G \in \mathcal{W}(n)\). The proof is completed.

**Corollary 2.1** Let \(G\) be a graph on \(n\) vertices containing no even wheels. Then

\[\mathcal{E}(G) \leq \left\lfloor \frac{n^2}{3} \right\rfloor.\]

Furthermore, the bound is best possible.

We conclude this work by posing the following problem: Determine the extremal function for the class of graphs that contains no specific odd wheel and characterize the extremal graphs as well.

**References**


(1) Department of Mathematics, Yarmouk University, Irbid-Jordan
E-mail address: bataineh71@hotmail.com

(2) Department of Mathematics, Statistics and Physics, Qatar University, Doha-Qatar
E-mail address: mmjst4@qu.edu.qa

(3) Department of Mathematics, Jadara University, Irbid-Jordan
E-mail address: abeer.jaradat@jadara.edu.jo