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ON THE STABILITY OF QUADRATIC FUNCTIONAL EQUATIONS IN PARTIALLY ORDERED BANACH SPACES : A PARTIALLY ORDERED FIXED POINT APPROACH

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Abstract. Using partially ordered fixed point method, we investigate the Hyers-Ulam-Rassias stability and superstability of quadratic functional equations on Banach spaces endowed a partial order.

1. Introduction

The problem of the stability of functional equations was originally stated by S. M. Ulam [1]. In 1941 D.H. Hyers [2] proved the stability of the additive functional equation for the special case when the groups G_1 and G_2 are Banach spaces. In 1950, T. Aoki discussed the Hyers-Ulam stability theorem in [3]. His result was further generalized and derived as a special case by Th.M. Rassias [4] in 1978 . The stability problem for functional equations have been extensively investigated by a number of mathematicians [5, 6, 7, 8, 9, 10, 11]. The quadratic function $f(x) = cx^2$ satisfies the functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{1.1}$$

and therefore the equation (1.1) is called the quadratic functional equation. The Hyers - Ulam stability theorem for the quadratic functional equation (1.1) was proved

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by F. Skof [10] for the functions $f : E_1 \to E_2$ where E_1 is a normed space and E_2 is a Banach space. The result of Skof is still true if the relevant domain E_1 is replaced by an Abelian group and this was dealt with by P.W. Cholewa [12]. S. Czerwik [13] proved the Hyers-Ulam-Rassias stability of the quadratic functional equation (1.1). This result was further generalized by Th.M. Rassias [14], C. Borelli and G.L. Forti [15].

In this paper, we will adopt the fixed point alternative of [16], to prove the Hyers-Ulam-Rassias stability and superstability of quadratic mappings on Banach spaces endowed a partial order and associated with the following generalized quadratic type functional equation

$$f(x_1 + x_2 + x_3 - x_4) + f(x_1 - x_2 - x_3 + x_4) = 2f(x_1) + 2f(x_2 + x_3 - x_4) \tag{1.2}$$

2. Stability

In this section and next section, we suppose that $(E_1, \|.\|_1)$ is a normed space endowed with a partial order $\leq_1$ with following conditions:

(i) $x, y \in E_1$ and $x \leq_1 y \quad \Rightarrow \quad rx \leq_1 ry \qquad (\forall\, r \in \mathbb{R}^+)$;

(ii) for all $x, y \in E_1$ there exists $z \in E_1$ such that z is comparable to x and y.

Also, we suppose that $(E_2, \|.\|_2)$ is a Banach space endowed with a partial order $\leq_2$ which satisfies condition (i) and

(iii) for all $x, y \in E_1$ there exists $z \in E_1$ such that z is an upper bound for $\{x, y\}$.

(iv) If $\{x_n\}$ is a nondecreasing sequence in E_2 and $x_n \to x$, then $x \geq x_n$ for all $n \in \mathbb{N}$.

As a example, we can see that the set

$$C([0,1]) := \{f : [0,1] \to \mathbb{R} \mid f \text{ is continuous}\}$$

with the following partial order

$$f,g \in C([0,1])\ ,\ f \leq g\ \Leftrightarrow\ f(x) \leq g(x)\ for\ all\ 0 \leq x \leq 1.$$

It is easy to show that for any $f,g \in C([0,1])$ the function $\max\{f,g\}$ is upper bound of f and g.

In this section, we consider $0 \times \infty = 0$. Before of our main results we need the following proposition.

Lemma 2.1. *The functional equation (1.2) is a quadratic functional equation.*

Proof. By letting $x_3 = x_4 := 0$ in (1.2) we get

$$f(x_1 + x_2) + f(x_1 - x_2) = 2f(x_1) + 2f(x_2)$$

and this shows that (1.2) is a quadratic functional equation. Also, putting $x := x_1$ and $y := x_2 + x_3 - x_4$ in (1.1) we infer the equation (1.2). □

Theorem 2.1. *Suppose $f : E_1 \to E_2$ is a function satisfies*

$$4f(x) \leq_2 f(2x) \quad ; \quad (x \in E_1) \tag{2.1}$$

and

$$\|f(x+y+z-w) + f(x-y-z+w) - 2f(x) - 2f(y+z-w)\|_2$$

$$\leq \phi(x,z) + \phi(y,w) \tag{2.2}$$

for all $x,y,z,w \in E_1$ which x is comparable to z and y is comparable to w, where $\phi : E_1 \times E_1 \to [0,\infty)$ is a function satisfies $\phi(0,0) = 0$ and with the following condition:

$$\phi(x,y) \leq 4\ L\ \phi(\frac{x}{2},\frac{y}{2}) \tag{2.3}$$

for all $x, y \in E_1$ with x comparable to y, which $L \in (0,1)$ is a constant. Then there exists a unique a quadratic mapping $H : E_1 \to E_2$ such that

$$\|H(x) - f(x)\|_2 \leq \frac{1}{1-L}\ \phi(x,x) \tag{2.4}$$

for all $x \in E_1$.

Proof. We can see that $f(0) = 0$. Putting $z := x$ and $y = w := 0$ in (2.9), we get

$$\|f(2\ x) - 4\ f(x)\|_2 \leq \phi(x,x)$$

for all $x \in E_1$. Hence

$$\|\frac{f(2\ x)}{4} - f(x)\|_2 \leq \frac{1}{4}\ \phi(x,x) \leq \phi(x,x) \tag{2.5}$$

for all $x \in E_1$. We consider the set $X := \{g |\ g : E_1 \to E_2\}$ and introduce the metric d on X by:

$$d(h,g) := \inf\{C \in \mathbb{R}^+ ; \|h(x) - g(x)\|_2 \leq\ C\ \phi(x,x)\ for\ all\ x \in E_1\}$$

for all $h\ , g \in X$. It is easy to show that (X, d) is a complete generalized metric space. We put the partial order $\leq$ on X as follows:

$$h, g \in X \qquad h \leq g\ \Leftrightarrow\ h(x) \leq_2\ g(x)\ for\ all\ x \in E_1.$$

Now, we define the mapping $J : X \to X$ by

$$J(h)(x) := \frac{1}{4}\ h(2\ x)$$

for all $x \in E_1$. For any $g, h \in X$ with $g \leq h$, we have

$$d(g,h) < C \Rightarrow \|g(x) - h(x)\|_2 \leq C\ \phi(x,x)\ for\ all\ x \in E_1$$

$$\Rightarrow \|\frac{g(2x)}{4} - \frac{h(2x)}{4}\|_2 \leq C\ \frac{\phi(2x,2x)}{4}\ for\ all\ x \in E_1$$

$$\Rightarrow \|J(g)(x) - J(h)(x)\|_2 \leq L\ C\ \phi(x,x)\ for\ all\ x \in E_1.$$

It follows that

$$d(J(g), J(h)) \leq L\ d(g,h).$$

Applying inequalities (2.8) and (2.12), we can see that $f \leq J(f)$ and $d(J(f), f) \leq 1$, also, using the condition (i) of E_1 we can show that J is a nondecreasing mapping. Now, we show that J is a continuous function. To this end, let $\{h_n\}$ be a sequence in (X, d) such that converges to $h \in X$ and let $\epsilon > 0$ be given. Then there exist $N \in \mathbb{N}$ and $C \in \mathbb{R}^+$ with $C \leq \epsilon$ such that

$$\|h_n(x) - h(x)\|_2 \leq C\ \phi(x,x)$$

for all $x \in E_1$ and all $n \geq N$. Thus we get

$$\|h_n(2x) - h(2x)\|_2 \leq C\ \phi(2x,2x)$$

for all $x \in E_1$ and all $n \geq N$. By inequality (2.10) and definition of J, we get

$$\|J(h_n)(x) - J(h)(x)\|_2 \leq L\ C\ \phi(x,x)$$

for all $x \in E_1$ and $n \geq N$. Hence,

$$d(J(h_n), J(h)) \leq L\ C < \epsilon$$

for all $n \geq N$. It follows that J is continuous. Applying Theorem **??**, we get J has a fixed point. Let $T \in X$ is a fixed point of J, then $\lim_{n\to\infty} d(J^n(f), T) = 0$. It follows that

$$H(x) = \lim_{n\to\infty} \frac{f(2^n\ x)}{4^n} \tag{2.6}$$

for all $x \in E_1$. On the other hand, it follows from (2.8) that for all $x \in E_1$, the sequence $\{\frac{f(2^n\ x)}{4^n}\}_{n=0}^{\infty}$ is a nondecreasing sequence in E_2, hence, by using the condition (iii), we find that $f(x) \leq T(x)$, for all $x \in E_1$. This shows that $f \leq T$. Now, we can see that

$$d(J(f), J(T)) \leq L\ d(f, T)$$

and hence

$$d(f, T) \leq \frac{1}{1-L}.$$

This implies the inequality (2.11). The inequality (2.10) shows that

$$4^{-n}\ \phi(2^n\ x, 2^n\ y) \leq L^n\ \phi(x, y) \tag{2.7}$$

for all $x, y \in E_1$ which x is comparable to y and for all $n \in \mathbb{N}$. Let $x, y \in E_1$ are arbitrary elements, then there exists $z \in E_1$ such that z is comparable to x and y. This implies that $2^n z$ is comparable to $2^n x$ and $2^n y$ for all $n \in \mathbb{N}$. It follows from (2.9) that

$$\begin{aligned}
&\|f(2^n(x+y)) + f(2^n(x-y)) - 2f(2^n x) - 2f(2^n y)\|_2 \\
&\qquad = \|f(2^n\ x + 2^n\ y + 2^n z - 2^n z) + f(2^n\ x - 2^n\ y - 2^n z + 2^n z) \\
&\qquad\qquad - f(2^n x) - f(2^n y + 2^n z - 2^n z)\|_2 \\
&\qquad \leq \phi(2^n\ x, 2^n\ z) + \phi(2^n\ y, 2^n\ z)
\end{aligned}$$

for all $n \in \mathbb{N}$. Since $L \in (0, 1)$ and by using (2.13) and (2.14), we find that H is a Cauchy mapping. To prove the uniqueness property of H, we suppose that T_1 is another quadratic function satisfying (2.11). It is clear that $J(T_1) = T_1$.

Fix the arbitrary element $x \in E_1$, then there exists $h(x) \in E_2$ such that $h(x) =$ *upper bound* $\{H(x), T_1(x)\}$. This shows that $h : E_1 \to E_2$ is a function comparable to H and T_1. Hence,

$$\begin{aligned} d(H, T_1) &\leq d(H, J^n(h)) + d(J^n(h), T_1) \\ &= d(J^n(H), J^n(h)) + d(J^n(h), J^n(T_1)) \\ &\leq L^{-n}\ d(H, h) + L^{-n}\ d(h, T_1) \end{aligned}$$

for all $n \in \mathbb{N}$. Therefore, $T = T_1$ and this completes the proof. □

Theorem 2.2. *Suppose $f : E_1 \to E_2$ is a function satisfies*

$$f(x) \leq_2 4\ f(\frac{x}{2}) \quad ; \quad (x \in E_1) \tag{2.8}$$

and

$$\|f(x+y+z-w) + f(x-y-z+w) - 2f(x) - 2f(y+z-w)\|_2$$

$$\leq \phi(x, z) + \phi(y, w) \tag{2.9}$$

for all $x, y, z, w \in E_1$ which x is comparable to z and y is comparable to w, where $\phi : E_1 \times E_1 \to [0, \infty)$ is a function satisfies $\phi(0, 0) = 0$ and with the following condition:

$$4\ L\ \phi(x, y) \leq \phi(2x, 2y) \tag{2.10}$$

for all $x, y \in E_1$ with x comparable to y, which $L \in (1, \infty)$ is a constant. Then there exists a unique a quadratic mapping $H : E_1 \to E_2$ such that

$$\|H(x) - f(x)\|_2 \leq \frac{1}{L-1}\ \phi(x, x) \tag{2.11}$$

for all $x \in E_1$.

Proof. It is clearly that $f(0) = 0$. Putting $z = x := \frac{x}{2}$ and $y = w := 0$ in (2.9), we get

$$\|f(x) - 4\ f(\frac{x}{2})\|_2 \leq \phi(\frac{x}{2}, \frac{x}{2})$$

for all $x \in E_1$. Hence

$$\|f(x) - 4\ f(\frac{x}{2})\|_2 \leq \frac{1}{4L}\ \phi(x, x) \leq \frac{1}{L}\ \phi(x, x) \tag{2.12}$$

for all $x \in E_1$. We consider the set $X := \{g|\ g : E_1 \to E_2\}$ and introduce the metric d on X by:

$$d(h, g) := \inf\{C \in \mathbb{R}^+; \|h(x) - g(x)\|_2 \leq\ C\ \phi(x, x)\ for\ all\ x \in E_1\}$$

for all h , $g \in X$. It is easy to show that (X, d) is a complete generalized metric space. We put the partial order $\leq$ on X as follows:

$$h, g \in X \qquad h \leq g\ \Leftrightarrow\ h(x) \leq_2\ g(x)\ for\ all\ x \in E_1.$$

Now, we define the mapping $J : X \to X$ by

$$J(h)(x) := 4\ h(\frac{x}{2})$$

for all $x \in E_1$. For any $g, h \in X$ with $g \leq h$, we have

$$d(g, h) < C \Rightarrow \|g(x) - h(x)\|_2 \leq C\ \phi(x, x)\ for\ all\ x \in E_1$$

$$\Rightarrow \|4g(\frac{x}{2}) - 4h(\frac{x}{2})\|_2 \leq 4C\ \phi(\frac{x}{2}, \frac{x}{2})\ for\ all\ x \in E_1$$

$$\Rightarrow \|J(g)(x) - J(h)(x)\|_2 \leq L^{-1}\ C\ \phi(x, x)\ for\ all\ x \in E_1.$$

It follows that

$$d(J(g), J(h)) \leq L^{-1}\ d(g, h).$$

Applying inequalities (2.8) and (2.12), we can see that $f \leq J(f)$ and $d(J(f), f) \leq L^{-1}$, also, using the condition (i) of E_1 we can show that J is a nondecreasing mapping. By the same method of Theorem 2.1 we can prove that J is a continuous mapping and we get J has a fixed point. Let $H \in X$ is a fixed point of J, then $\lim_{n\to\infty} d(J^n(f), H) = 0$. It follows that

$$H(x) = \lim_{n\to\infty} 4^n\ f(\frac{x}{2^n}) \tag{2.13}$$

for all $x \in E_1$. On the other hand, it follows from (2.8) that for all $x \in E_1$, the sequence $\{4^n\ f(\frac{x}{2^n})\}_{n=0}^{\infty}$ is a nondecreasing sequence in E_2, hence, by using the condition (iii), we find that $f(x) \leq T(x)$, for all $x \in E_1$. This shows that $f \leq T$. Now, we can see that

$$d(J(f), J(T)) \leq L^{-1}\ d(f, T)$$

and hence

$$d(f, T) \leq \frac{1}{L-1}.$$

This implies the inequality (2.11). The inequality (2.10) shows that

$$\phi(\frac{x}{2^n}, \frac{y}{2^n}) \leq 4^{-n}\ L^{-n}\ \phi(x, y) \tag{2.14}$$

for all $x, y \in E_1$ which x is comparable to y and for all $n \in \mathbb{N}$. Let $x, y \in E_1$ are arbitrary elements, then there exists $z \in E_1$ such that z is comparable to x and y. This implies that $\frac{z}{2^n}$ is comparable to $\frac{x}{2^n}$ and $\frac{y}{2^n}$for all $n \in \mathbb{N}$. It follows from (2.9) that

$$\begin{aligned}
&\|f(\frac{x+y}{2^n}) + f(\frac{x-y}{2^n}) - 2f(\frac{x}{2^n}) - 2f(\frac{y}{2^n})\|_2 \\
&\quad = \|f(\frac{x}{2^n} + \frac{y}{2^n} + \frac{z}{2^n} - \frac{z}{2^n})) + f(\frac{x}{2^n} - \frac{y}{2^n} - \frac{z}{2^n} + \frac{z}{2^n}) \\
&\qquad f(\frac{x}{2^n}) - f(\frac{y}{2^n} + \frac{z}{2^n} - \frac{z}{2^n})\|_2 \\
&\quad \leq \phi(\frac{x}{2^n}, \frac{z}{2^n}) + \phi(\frac{y}{2^n}, \frac{z}{2^n})
\end{aligned}$$

for all $n \in \mathbb{N}$. Since $L^{-1} \in (0,1)$ and by using (2.13) and (2.14), we find that H is a Cauchy mapping. To prove the uniqueness property of H, we suppose that T_1 is another quadratic function satisfying (2.11). It is clear that $J(T_1) = T_1$. Fix the arbitrary element $x \in E_1$, then there exists $h(x) \in E_2$ such that $h(x) =$ *upper bound* $\{H(x), T_1(x)\}$. This shows that $h : E_1 \to E_2$ is a function comparable to H and T_1. Hence,

$$\begin{aligned} d(H,T_1) &\leq d(H, J^n(h)) + d(J^n(h), T_1) \\ &= d(J^n(H), J^n(h)) + d(J^n(h), J^n(T_1)) \\ &\leq L^{-n}\ d(H,h) + L^{-n}\ d(h,T_1) \end{aligned}$$

for all $n \in \mathbb{N}$. Therefore, $H = T_1$ and this completes the proof. □

Corollary 2.1. *Let $\epsilon \in (0,\infty)$ and $f : E_1 \to E_2$ be a function with $f(0) = 0$ and satisfies the following*

$$4f(x) \leq_2 f(2x) \qquad ; \quad (x \in E_1)$$

$$\|f(x+y+z-w) + f(x-y-z+w) - 2f(x) - 2f(y+z-w)\|_2 \leq \epsilon$$

for all $x, y, z, w \in E_1$ which x is comparable to z and y is comparable to w. Then there exists a unique quadratic mapping $H : E_1 \to E_2$ such that

$$\|H(x) - f(x)\|_2 \leq \epsilon$$

for all $x \in E_1$.

Proof. Set $\phi(x,y) = \frac{\epsilon}{2}$ for all $x, y \in E_1$ with $x, y \neq 0$ and $\phi(0,0) = 0$ and let $L = \frac{1}{4}$ in Theorem 2.1. Then we get the desired result. □

Corollary 2.2. *Let* $p \in (0,2)$ $(p \in (2,\infty))$ *and* $\epsilon \in (0,\infty)$ *are real numbers. Suppose that* $f : E_1 \to E_2$ *is a mapping satisfies*

$$4f(x) \leq_2 f(2x) \quad (f(x) \leq_2 4\ f(\frac{x}{2})) \qquad ; \quad (x \in E_1)$$

for all $x, y, z, w \in E_1$ *which* x *is comparable to* z *and* y *is comparable to* w*. Then there exists a unique quadratic mapping* $H : E_1 \to E_2$ *such that*

$$\|H(x) - f(x)\|_2 \leq \frac{2^{3-p}}{2^{2-p}\ - 1}\ \epsilon\ \|x\|^p \quad (\|H(x) - f(x)\|_2 \leq \frac{2}{2^{p-2}\ - 1})\ \epsilon\ \|x\|^p)$$

for all $x \in E_1$.

Proof. Set $\phi(x,y) = \epsilon\ (\|x\|^p + \|y\|^p)$ for all $x, y \in E_1$ and $L = 2^{p-2}$ ($L = 2^{p-2}$) in Theorem 2.1 (2.2). Then we get the desired result. □

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