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ON THE GENERALIZATION OF SOME WELL KNOWN FIXED POINT THEOREMS FOR NONCOMPATIBLE MAPPINGS

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Abstract. Various investigations have dealt with the problem that, is there a contractive definition strong enough to ensure the existence of a fixed point but the function fails to be continuous. An attempt has been made to generalize fixed point theorem of Pant, Bisht, Arora [9]. Further a fixed point theorem of Bisht and Joshi [1] finds a generalization here.

1. Introduction

Mu et al. [6] presented some common fixed point theorems for compatible and weak compatible self maps under generalized contractive conditions in Menger probabilistic G-metric spaces.

Manro et al. [5] obtained common fixed point theorem for weakly compatible mappings making use of the property E.A. in G-metric spaces. Some of the results found an application in deriving the solution of an integral equation and the bounded solution of a functional equation arising in dynamic programming.

Pant et al. [9] introduced the concept of weak reciprocal continuity towards the generalization of reciprocal continuiuty and obtained fixed point theorems on complete metric spaces.

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Bisht and Joshi [1] established common fixed point theorems for a pair of weakly reciprocally continuous selfmaps satisfying generalized contractions or lipschitz type conditions.

Since the paper of Kannan [3, 4] wherein it was shown that there were maps having a discontinuity in their domain but which had fixed points. The problem that there can be a contractive definition strong enough to generate a fixed point but does not allow the map to be continuous had remained unanswered till the paper of Pant [7] in 1998, when he introduced reciprocal continuity and established a situation in which a collection of mapping had a fixed point which was the point of discontinuity for all mappings. In fact this paper was the source of a good deal of researches.

Following Pant [7], we have

Definition 1.1. Two selfmaps f and g of a metric space (X, d) are called reciprocally continuous if $\lim_{n\to\infty} fgx_n = ft$ and $\lim_{n\to\infty} gfx_n = gt$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some t in X.

Remark 1. If f and g are both continuous then they are obviously reciprocally continuous but the converse is not true.

Due to Pant et al. [9], we have following

Definition 1.2. Two selfmaps f and g of a metric space (X, d) are called weakly reciprocally continuous if $\lim_{n\to\infty} fgx_n = ft$ or $\lim_{n\to\infty} gfx_n = gt$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some t in X.

The following definition is due to Jungck [2]

Definition 1.3. Two selfmaps f and g of a metric space (X, d) are called compatible if $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some t in X.Thus the mapping $f$and g will be noncompatible if there exist atleast one sequence $\{x_n\}$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some t in X but $\lim_{n\to\infty} d(fgx_n, gfx_n)$ is either non zero or not exist.

The following finds place in [10]

Definition 1.4. Two selfmaps f and g of a metric space (X, d) are called f-compatible if $\lim_{n\to\infty} d(fgx_n, ffx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some t in X and g-compatible if $\lim_{n\to\infty} d(ggx_n, fgx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some t in X

Remark 2. It is observed that if two selfmaps f and g are either f-compatible or g-compatible then they commutes at coincidence point.

Bisht and Joshi [1] obtained the follwing:

Theorem 1.1. *Let f and g be weakly reciprocally continuous noncompatible selfmappings of a metric space satisfying*

(1) $fX \subseteq gX$
(2) $d(fx, fy) \leq ad(gx, gy) + b[d(fx, gx) + d(fy, gy)] + c[d(fx, gy) + d(fy, gx)];$
$a, b, c \geq 0,$ *and* $b + c < 1$
(3) $d(fx, f^2x) < d(gx, g^2x)$ *whenever* $gx \neq g^2x$

If f and g are either g-compatible or f-compatible then f and g have a common fixed point.

The following definition is due to Pant [8]

Definition 1.5. Two selfmaps f and g of a metric space (X, d) are called R-weakly commuting at a point x in X if $d(fgx, gfx) \leq Rd(fx, gx)$ for some $R > 0$.

Remark 3. It is obvious that pointwise R-weakly commuting maps commute at their coincidence points and pointwise R-weakly commutativity is equivalent to commutativity at coincidence points.

Further, The following couple of definitions are due to Pathak et al. [11]

Definition 1.6. Two selfmaps f and g of a metric space (X, d) are called R-weakly commuting of type (A_g) if there exist some positive real number R such that $d(ffx, gfx) \leq Rd(fx, gx)$ for all x in X.

Definition 1.7. Two selfmaps f and g of a metric space (X, d) are called R-weakly commuting of type (A_f) if there exist some positive real number R such that $d(fgx, ggx) \leq Rd(fx, gx)$ for all x in X.

Remark 4. Both compatible and noncompatible mappings can be R-weakly commuting of type (A_g) or (A_f)

Pant et al. [9] obtained the following by modifying the procedure of Pant.

Theorem 1.2. *Let f and g be weakly reciprocally continuous noncompatible selfmappings of a complte metric space (X, d) satisfying*

(1) $fX \subseteq gX$
(2) $d(fx, fy) \leq d(gx, gy) + bd(fx, gx) + cd(fy, gy), 0 \leq b, c < 1$
(3) $d(fx, f^2x) < d(gx, g^2x)$ *whenever* $gx \neq g^2x$

for all $x, y \in X$. If f and g are R-weakly commuting of type (A_g) or R-weakly commuting of type (A_f) then f and g have a common fixed point.

2. Main Theorem

To start with a variant of the fixed point theorem obtained by Pant et al. [9], is established below

Theorem 2.1. *Let f and g be weakly reciprocally continuous noncompatible selfmappings of a complete metric space (X, d) satisfying*

(1) $fX \subseteq gX$

(2) $d(fx, fy) \leq d(gx, gy) + bd(fx, gx) + cd(fy, gy), 0 \leq b, c < 1$

(3) $d(fx, f^3x) < d(gx, g^3x)$ *whenever* $gx \neq g^3x$

for all $x, y \in X$. *If* f *and* g *are R-weakly commuting of type* (A_g) *or R-weakly commuting of type* (A_f) *then* f^2 *and* g^2 *have a common fixed point.*

Proof. Since f and g are noncompatible maps, there exist a sequence $\{x_n\}$ in X such that $fx_n \to t$ and $gx_n \to t$ for some t in X but either $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$ or the limit does not exist. Since $fX \subseteq gX$, for each $\{x_n\}$ there exists $\{y_n\}$ in X such that $fx_n = gy_n$. Thus $fx_n \to t, gx_n \to t$ and so $gy_n \to t$ as $n \to \infty$. By virtue of this and using (2) one obtains $fy_n \to t$. Therefore, one finds that

$$fx_n \to t, gx_n \to t, gy_n \to t, fy_n \to t \tag{2.1}$$

Suppose that f and g are R-weakly commuting of type (A_g). Then weak reciprocal continuity of f and g implies that $fgx_n \to ft$ or $gfx_n \to gt$. Similarly, $fgy_n \to ft$ or $gfy_n \to gt$. If one assumes that $gfy_n \to gt$, then R-weak commutativity of type (A_g) of f and g yields

$$d(ffy_n, gfy_n) \leq Rd(fy_n, gy_n)$$

On letting $n \to \infty$, one gets $ffy_n \to gt$. Using (ii) one arrives at

$$d(ffy_n, ft) \leq d(gfy_n, gt) + bd(ffy_n, gfy_n) + cd(ft, gt)$$

which leads to

$$d(gt, ft) \leq cd(ft, gt)$$

This implies that $ft = gt$, since $c < 1$.

Again, by virtue of R-weak commutativity of type $(A_g), d(fft, gft) \leq Rd(ft, gt)$. This yields $fft = gft$, which means $fgt = fft = gft = ggt$.

Again by R-weak commutativity of f and g of type (A_g), one finds that

$$d(ffft, gfft) \leq R^2 d(ft, gt)$$

This results in

$$f^3 t = gf^2 t = g^2 ft.$$

Now if $ft \neq f^2(ft)$, then using (2) one gets

$$d(ft, f^3 t) < d(ft, f^3 t)$$

which is a contradiction. Hence,

$$ft = f^2(ft) = g^2(ft).$$

That is, ft is a common fixed point of f^2 and g^2.

Next, suppose that $fgy_n \to ft$. Then $fX \subseteq gX$ implies that $ft = gu$ for some $u \in X$ and by virtue of (4) one gets

$$fgy_n = ffx_n \to ft$$

Thus, $fgy_n \to ft = gu$ and $ffx_n \to gu$.

Hence R-weak commutativity of type (A_g) of f and g yields

$$d(ffx_n, gfx_n) \leq Rd(fx_n, gx_n) \tag{2.2}$$

This gives $gfx_n \to gu$. That is, $ggy_n \to gu$. Also, using (2) one arrives at

$$d(fgy_n, fu) \leq d(ggy_n, gu) + bd(fgy_n, ggy_n) + cd(fu, gu)$$

This leads to

$$d(gu, fu) \leq cd(fu, gu),$$

which implies that $fu = gu$, since $c < 1$. Again, by virtue of R-weak commutativity of type (A_g),

$$d(ffu, gfu) \leq Rd(fu, gu)$$

This yields

$$ffu = gfu and ffu = fgu = gfu = ggu$$

Again by R-weak commutativity of type (A_g) of f and g one concludes that

$$\begin{aligned} d(fffu, gffu) &\leq Rd(ffu, gfu) \\ &\leq R^2 d(fu, gu) \end{aligned}$$

This gives

$$fffu = gffu = ggfu$$

If $fu \neq fffu$, then by using (3), one notes that

$$d(fu, f^2 fu) < d(fu, f^3 u)$$

which is a contradiction. Hence,

$$fu = f^2(fu) = g^2(fu).$$

That is, fu is a common fixed point of f^2 and g^2.

Finally, Suppose that f and g are R-weakly commuting of type (A_f). Now weak reciprocal continuity of f and g implies that $fgx_n \to ft$ or $gfx_n \to gt$.

Similarly, $fgy_n \to ft$ or $gfy_n \to gt$.

Suppose that $gfx_n \to gt$. Then by virtue of (2.1) one finds that

$$ggy_n = gfx_n \to gt \tag{2.3}$$

Hence, R-weak commutativity of type (A_f) yields

$$d(fgy_n, ggy_n) \leq Rd(fy_n, gy_n)$$

which leads to $fgy_n \to gt$. That is, $ffx_n \to gt$. Also, using (ii) one infers that

$$d(ffx_n, ft) \leq d(gfx_n, gt) + bd(ffx_n, gfx_n) + cd(ft, gt)$$

This gives $d(gt, ft) \leq cd(ft, gt)$, which implies that $ft = gt$, since $c < 1$. Hence, R-weak commutativity of type (A_f) implies that

$$d(fgt, ggt) \leq d(ft, gt)$$

This yields

$$fgt = ggt \text{and} fft = fgt = gft = ggt$$

Again, by R-weak commutativity of type (A_f), one obtains that

$$d(ffft, gggt) \leq R^2 d(ft, gt)$$

This yields

$$f^3t = g^3t = g^2ft$$

Now, if $ft \neq f^2(ft)$ then using (2) one obtains that

$$\begin{aligned} d(ft, f^2ft) &= d(ft, f^3t) \\ &< d(gt, g^3t) \\ &= d(ft, f^3t) \end{aligned}$$

which is a contradiction. Hence,

$$ft = f^2(ft) = g^2(ft).$$

That is, ft is a common fixed point of f^2 and g^2.

Next, suppose that $fgx_n \to ft$. Then $fX \subseteq gX$ implies that $ft = gu$ for some $u \in X$.Then R-weak commutativity of type (A_f) of f and g yields

$$d(fgx_n, ggx_n) \leq Rd(fx_n, gx_n)$$

This results in $ggx_n \to gu$. Also, using (2) one arrives at

$$d(fgx_n, fu) \leq d(ggx_n, gu) + bd(fgx_n, ggx_n) + cd(fu, gu)$$

On letting $n \to \infty$ one gets $fu = gu$, since $c < 1$. Again by virtue of R-weak commutativity of type (A_f),

$$fgu = gguandffu = fgu = gfu = ggu$$

Again, by R-weak commutativity of type (A_f) of f and g one obtains

$$d(fffu, gggu) \leq R^2 d(fu, gu)$$

This yields $f^3u = g^3u = g^2fu$. Now, if $fu \neq fffu$, then by using (3), one concludes that

$$d(fu, f^2fu) < d(fu, f^3u)$$

which is a contradiction. Hence,

$$fu = f^2(fu) = g^2(fu).$$

That is, fu is a common fixed point of f^2 and g^2. □

Infact, a generalization of Pant et al. [9] is contained in

Theorem 2.2. *Let f and g be weakly reciprocally continuous noncompatible selfmappings of a complete metric space (X, d) satisfying*

(1) $fX \subseteq gX$

(2) $d(fx, fy) \leq d(gx, gy) + bd(fx, gx) + cd(fy, gy), 0 \leq b, c < 1$

(3) $d(fx, f^{n+1}x) < d(gx, g^{n+1}x)$ *whenever* $gx \neq g^{n+1}x$

for all $x, y \in X$. *If* f *and* g *are R-weakly commuting of type* (A_g) *or R-weakly commuting of type* (A_f) *then* f^n *and* g^n *have a common fixed point.*

Proof. R-commutativity of type (A_g) of f and g implies

$$\begin{aligned} d(fff^{n-1}t, gff^{n-1}t) &\leq Rd(ff^{n-1}t, gf^{n-1}t) \\ &\leq R^n d(ft, gt) \end{aligned}$$

This implies that

$$f^{n+1}t = gf^n t, \forall n \in \mathbb{N} \text{ as } ft = gt.$$

□

Next the following variant of the fixed point theorem of Bisht and Joshi [1], has been presented by suitably modifying the procedure.

Theorem 2.3. *Let* f *and* g *be weakly reciprocally continuous noncompatible selfmappings of a complete metric space satisfying*

(1) $fX \subseteq gX$

(2) $d(fx, fy) \leq ad(gx, gy) + b[d(fx, gx) + d(fy, gy)] + c[d(fx, gy) + d(fy, gx)];$
$a, b, c \geq 0$ *with* $b + c < 1$

(3) $d(fx, f^3x) < d(gx, g^3x)$ *whenever* $gx \neq g^3x$

for all $x, y \in X$. *If* f *and* g *are either* g*-compatible or* f*-compatible then* f^2 *and* g^2 *have a common fixed point.*

Proof. Since f and g are noncompatible maps, there exist a sequence $\{x_n\}$ in X such that $fx_n \to t$ and $gx_n \to t$ for some t in X but either $\lim_{n\to\infty} d(fgx_n, gfx_n) \neq 0$ or the limit does not exist.

Since $fX \subseteq gX$, for each x_n there exist y_n in X such that $fx_n = gy_n$. Thus $fx_n \to t$, $gx_n \to t$ and $gy_n \to t$ as $n \to \infty$. By virtue of this and using (ii) one obtains $fy_n \to t$. Therefore, one finds that

$$fx_n = gy_n \to t, gx_n \to t, fy_n \to t \tag{2.4}$$

Suppose that f and g are g-compatible. Then weak reciprocal continuity of f and g implies that

$$fgx_n \to ft or gfx_n \to gt$$

Similarly,

$$fgy_n \to ft or gfy_n \to gt$$

Let us first assume that $gfy_n \to gt$. Then g-compatibility of f and g yields

$$\lim_{n\to\infty} d(ffy_n, gfy_n) = 0$$

That is, $ffy_n \to gt$.

By virtue of (2) one arrives at

$$d(ffy_n, ft) \le ad(gfy_n, gt) + b[d(ffy_n, gfy_n) + d(ft, gt)] + c[d(ffy_n, gt) + d(ft, gfy_n)]$$

This results in

$$d(gt, ft) \le (b + c)d(ft, gt).$$

which implies that $ft = gt$, since $(b + c) < 1$.

Since g-compatibility implies commutativity at coincidence points (that is, fgt=gft), we have

$$fft = fgt = gft = ggt$$

Again, ft is a coincidence point of f and g. Hence g-compatibility of f and g implies that $fgft = gfft$, and hence

$$ffft = fgft = gfft = gggt$$

Now if $ft \neq f^2ft$, then

$$d(ft, f^3t) < d(ft, f^3t)$$

which is a contradiction. Thus,

$$ft = f^2(ft) = g^2(ft)$$

and ft is a common fixed point of f^2 and g^2.

Next suppose that $fgy_n \to ft$. Then $fX \subseteq gX$ implies that $ft = gu$ for some $u \in X$ and by virtue of (2.4) one concludes that

$$fgy_n = ffx_n \to ft \tag{2.5}$$

Thus,

$$fgy_n \to ft = gu \text{ and } ffx_n \to gu$$

g-compatibility of f and g yields $gfx_n \to gu$. That is, $ggy_n \to gu$.

Also, using (2) one arrives at the

$$d(fgy_n, fu) \leq ad(ggy_n, gu)+b[d(fgy_n, ggy_n)+d(fu, gu)]+c[d(fgy_n, gu)+d(fu, ggy_n)]$$

On letting $n \to \infty$ one notices that

$$d(gu, fu) \leq cd(fu, gu)$$

This implies $fu = gu$, since $(b + c) < 1$.

Since g-compatibility implies commutativity at coincidence points (that is, $fgu = gfu$), one infers that

$$ffu = fgu = gfu = ggu$$

Again, fu is a coincidence point of f and g. Hence g-compatibility of f and g implies

$$fgfu = gffu$$

and hence

$$fffu = fgfu = gffu = gggu$$

Now if $fu \neq f^2fu$, then

$$d(ft, f^3u) < d(ft, f^3u)$$

which is a contradiction. Hence, $fu = f^2(fu) = g^2(fu)$ and fu is a common fixed point of f^2 and g^2.

Suppose that f and g are f-compatible. Now, weak reciprocal continuity of f and g implies that $fgx_n \to ft$ or $gfx_n \to gt$.

Similarly, $fgy_n \to ft$ or $gfy_n \to gt$.

Assume that $gfy_n \to gt$. Now, by virtue of (2.4),

$$ggy_n = gfx_n \to gt$$

f-compatibility of f and g yields

$$\lim_{n\to\infty} d(fgy_n, ggy_n) = 0$$

That is, $fgy_n \to gt$. Also, using (2) one concludes that

$$d(ffx_n, ft) \leq ad(gfx_n, gt)+b[d(ffx_n, gfx_n)+d(ft, gt)]+c[d(ffx_n, gt)+d(ft, gfx_n)]$$

In view of $fgy_n = ffx_n \to gt$, one obtains

$$d(gt, ft) \leq (b + c)d(ft, gt).$$

This implies that $ft = gt$, since $(b + c) < 1$.

Since f-compatibility implies commutativity at coincidence points (that is, fgt=gft), one finds that

$$fft = fgt = gft = ggt$$

Again, ft is a coincidence point of f and g. Hence f-compatibility of f and g implies

$$fgft = gfft$$

and hence

$$ffft = fgft = gfft = gggt$$

Now if $ft \neq f^2ft$, then

$$d(ft, f^3t) < d(ft, f^3t)$$

which is a contradiction. Hence

$$ft = f^2(ft) = g^2(ft)$$

and ft is a common fixed point of f^2 and g^2.

Next suppose that $fgx_n \to ft$. Then $fX \subseteq gX$ implies that $ft = gu$ for some $u \in X$. f-compatibility of f and g yields $ggx_n \to gu$.
Also, using (2) one arrives at

$$d(fgx_n, fu) \leq ad(ggx_n, gu)+b[d(fgx_n, ggx_n)+d(fu, gu)]+c[d(fgx_n, gu)+d(fu, ggx_n)]$$

On letting $n \to \infty$ one gets

$$d(gu, fu) \leq cd(fu, gu)$$

This implies that $fu = gu$, since $(b + c) < 1$.
Since f-compatibility implies commutativity at coincidence points
(that is, $fgu = gfu$), one finds that

$$ffu = fgu = gfu = ggu$$

Again, fu is a coincidence point of f and g. Hence f-compatibility of f and g implies

$$fgfu = gffu$$

and hence

$$fffu = fgfu = gffu = gggu$$

Now if $fu \neq f^2fu$, then

$$d(fu, f^3u) < d(fu, f^3u)$$

which is a contradiction. Hence $fu = f^2(fu) = g^2(fu)$ and fu is a common fixed point of f^2 and g^2. □

Lastly, a generalization of fixed point theorem of Bisht and Joshi [1] appears in the form of

Theorem 2.4. *Let f and g be weakly reciprocally continuous noncompatible selfmappings of a complete metric space satisfying*

(1) $fX \subseteq gX$
(2) $d(fx, fy) \leq ad(gx, gy) + b[d(fx, gx) + d(fy, gy)] + c[d(fx, gy) + d(fy, gx)]$; $a, b, c \geq 0$ *with* $b + c < 1$
(3) $d(fx, f^{n+1}x) < d(gx, g^{n+1}x)$ *whenever* $gx \neq g^{n+1}x$

for all $x, y \in X$. If f and g are either g-compatible or f-compatible then f^n and g^n have a common fixed point.

Proof. The proof follows mutatis mutandis on lines similar to that of the above proof. □

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