QUASI $b$–OPEN AND STRONGLY $b$–OPEN FUNCTIONS

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Abstract. In this paper we introduce $b$–open, $b$–closed, quasi $b$–open, quasi $b$–closed, strongly $b$–open and strongly $b$–closed functions and investigate properties and characterizations of these new types of functions.

1. Introduction

In 1996, Andrijevic [1] introduced the notion of $b$–open sets. This type of sets discussed by El-Atik [3] under the name of $\gamma$–open sets. We continue to explore further properties and characterizations of $b$–open, quasi $b$–open and strongly $b$–open functions. We also introduce and study properties and characterizations of $b$–closed, quasi $b$–closed and strongly $b$–closed functions.

Let $A$ be a subset of a space $(X, \tau)$. The closure (resp. interior) of $A$ will be denoted by $\text{Cl}(A)$ (resp. $\text{Int}(A)$).

A subset $A$ of a space $(X, \tau)$ is called $b$–open [1] if $A \subseteq \text{Cl}(\text{Int}(A)) \cup \text{Int}(\text{Cl}(A))$. The complement of a $b$–open set is called a $b$–closed set. The union of all $b$–open sets contained in $A$ is called the $b$–interior of $A$, denoted by $b\text{Int}(A)$ and the intersection of all $b$–closed sets containing $A$ is called the $b$–closure of $A$, denoted by $b\text{Cl}(A)$. The family of all $b$–open (resp. $b$–closed) sets in $(X, \tau)$ is denoted by $BO(X)$ (resp. $BC(X)$).

A subset $A$ of a space $(X, \tau)$ is called semi–open [4] if $A \subseteq \text{Cl}(\text{Int}(A))$. The complement of a semi–open set is called semi–closed [2]. The family of all semi–open (resp. semi–closed) sets in $(X, \tau)$ is denoted by $SO(X)$ (respectively $SC(X)$).

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2. $b$–Open and $b$–Closed Functions

In this section we define the concept of $b$–open functions as a generalization of open functions and investigate some properties of such functions.

**Definition 2.1.** A function $f : (X, \tau) \to (Y, \rho)$ is called $b$–open if $f(U) \in BO(Y)$ for every open set $U$ in $X$.

The following theorem follows immediately from the above definition.

**Theorem 2.2.** A function $f : (X, \tau) \to (Y, \rho)$ is $b$–open if and only if for each $x \in X$, and each open set $U$ in $X$ with $x \in U$, there exists a set $V \in BO(Y)$ containing $f(x)$ such that $V \subseteq f(U)$.

**Theorem 2.3.** Let $f : (X, \tau) \to (Y, \rho)$ be $b$–open. If $V \subseteq Y$ and $C$ is a closed subset of $X$ containing $f^{-1}(V)$, then there exists a set $F \in BC(Y)$ containing $V$ such that $f^{-1}(F) \subseteq C$.

**Proof.** Let $F = Y - f(X - C)$. Then $F \in BC(Y)$. Since $f^{-1}(V) \subseteq C$, we have $f(X - C) \subseteq (Y - V)$ and so $V \subseteq F$.

Also $f^{-1}(F) = X - f^{-1}[f(X - C)] \subseteq X - (X - C) = C$. $\square$

**Theorem 2.4.** A function $f : (X, \tau) \to (Y, \rho)$ is $b$–open if and only if $f[\text{Int}(A)] \subseteq b\text{Int}[f(A)]$, for every $A \subseteq X$.

**Proof.** $\Rightarrow$). Let $A \subseteq X$ and $x \in \text{Int}(A)$. Then there exists an open set $U_x$ in $X$ such that $x \in U_x \subseteq A$. Now $f(x) \in f(U_x) \subseteq f(A)$. Since $f$ is $b$–open, $f(U_x) \in BO(Y)$. Then $f(x) \in b\text{Int}[f(A)]$. Thus $f[\text{Int}(A)] \subseteq b\text{Int}[f(A)]$.

$\Leftarrow$). Let $U$ be an open set in $X$. Then by assumption, $f[\text{Int}(U)] \subseteq b\text{Int}[f(U)]$. Since $b\text{Int}[f(U)] \subseteq f(U)$, $f(U) = b\text{Int}[f(U)]$. Thus $f(U) \in BO(Y)$.

So $f$ is $b$–open. $\square$

The equality in the last theorem need not be true as shown in the following example.

**Example 2.5.** Let $X = Y = \{a, b\}$. Let $\tau$ be the indiscrete topology on $X$ and $\rho$ be the discrete topology on $Y$. Then $BO(X) = \{\emptyset, X, \{a\}, \{b\}\}$ and $BO(Y) = \rho$. Let $f : (X, \tau) \to (Y, \rho)$ be the identity function and $A = \{a\}$. Then $f[\text{Int}(A)] = \emptyset$ and $b\text{Int}[f(A)] = \{a\}$.

**Theorem 2.6.** A function $f : (X, \tau) \to (Y, \rho)$ is $b$–open if and only if $\text{Int}[f^{-1}(B)] \subseteq f^{-1}[b\text{Int}(B)]$, for every $B \subseteq Y$.

**Proof.** $\Rightarrow$). Let $B \subseteq Y$. Then $f[\text{Int}(f^{-1}(B))] \subseteq f[f^{-1}(B)] \subseteq B$.

But $f[\text{Int}(f^{-1}(B))] \in BO(Y)$ since $\text{Int}[f^{-1}(B)]$ is open in $X$ and $f$ is $b$–open. Hence, $f[\text{Int}(f^{-1}(B))] \subseteq b\text{Int}(B)$. Therefore $\text{Int}[f^{-1}(B)] \subseteq f^{-1}[b\text{Int}(B)]$. 


\( \Leftarrow \). Let \( A \subseteq X \). Then \( f(A) \subseteq Y \). Hence by assumption, we obtain, \( \text{Int}(A) \subseteq \text{Int}[f^{-1}(f(A))] \subseteq f^{-1}[\text{bInt}(f(A))] \). Thus \( f[\text{Int}(A)] \subseteq \text{bInt}[f(A)] \), for every \( A \subseteq X \). Hence, by Theorem 2.4, \( f \) is \( b \)-open. \( \Box \)

**Theorem 2.7.** A function \( f : (X, \tau) \to (Y, \rho) \) is \( b \)-open if and only if 
\( f^{-1}[\text{bCl}(B)] \subseteq \text{Cl}[f^{-1}(B)] \), for every \( B \subseteq Y \).

**Proof.** \( \Rightarrow \). Assume that \( f \) is \( b \)-open and \( B \subseteq Y \). Let \( x \in f^{-1}[\text{bCl}(B)] \). Then \( f(x) \in \text{bCl}(B) \). Let \( U \) be an open set in \( X \) such that \( x \in U \). Since \( f \) is \( b \)-open, then \( f(U) \in \text{BO}(Y) \). Therefore, \( B \cap f(U) \neq \emptyset \). Then \( U \cap f^{-1}(B) \neq \emptyset \). Hence \( x \in \text{Cl}[f^{-1}(B)] \). We conclude that \( f^{-1}[\text{bCl}(B)] \subseteq \text{Cl}[f^{-1}(B)] \).

\( \Leftarrow \). Let \( B \subseteq Y \). Then \( (Y - B) \subseteq Y \). By assumption,
\[
 f^{-1}[\text{bCl}(Y - B)] \subseteq \text{Cl}[f^{-1}(Y - B)].
\]
This implies,
\[
 X - \text{Cl}[f^{-1}(Y - B)] \subseteq X - f^{-1}[\text{bCl}(Y - B)].
\]
Hence
\[
 X - \text{Cl}[X - f^{-1}(B)] \subseteq f^{-1}[Y - \text{bCl}(Y - B)].
\]
Now
\[
 X - \text{Cl}[X - f^{-1}(B)] = \text{Int}[X - (X - f^{-1}(B))] = \text{Int}[f^{-1}(B)]
\]
then we have \( Y - \text{bCl}(Y - B) = \text{bInt}[Y - (Y - B)] = \text{bInt}(B) \).
Then, \( \text{Int}[f^{-1}(B)] \subseteq f^{-1}[\text{bInt}(B)] \). Now from Theorem 2.6, it follows that \( f \) is \( b \)-open. \( \Box \)

Now we introduce \( b \)-closed functions and study certain properties of this type of functions.

**Definition 2.8.** A function \( f : (X, \tau) \to (Y, \rho) \) is called \( b \)-closed if \( f(C) \in \text{BC}(Y) \) for every closed set \( C \) in \( X \).

**Theorem 2.9.** A function \( f : (X, \tau) \to (Y, \rho) \) is \( b \)-closed if and only if 
\( \text{bCl}[f(A)] \subseteq f[\text{Cl}(A)] \), for every \( A \subseteq X \).

**Proof.** \( \Rightarrow \). Let \( f \) be \( b \)-closed and let \( A \subseteq X \). Then \( f[\text{Cl}(A)] \in \text{BC}(Y) \). But \( f(A) \subseteq f[\text{Cl}(A)] \). Then \( \text{bCl}[f(A)] \subseteq f[\text{Cl}(A)] \).

\( \Leftarrow \). Let \( A \subseteq X \) be a closed set. Then by assumption,
\( \text{bCl}[f(A)] \subseteq f[\text{Cl}(A)] = f(A) \). This shows that \( f(A) \in \text{BC}(Y) \). Hence \( f \) is \( b \)-closed. \( \Box \)
Corollary 2.10. Let \( f : (X, \tau) \to (Y, \rho) \) be \( b \)-closed and let \( A \subseteq X \). Then \( \text{bInt}[\text{bCl}(f(A))] \subseteq f[\text{Cl}(A)] \).

Theorem 2.11. Let \( f : (X, \tau) \to (Y, \rho) \) be a surjective function. Then \( f \) is \( b \)-closed if and only if for each subset \( B \) of \( Y \) and each open set \( U \) in \( X \) containing \( f^{-1}(B) \), there exists a set \( V \in \text{BO}(Y) \) containing \( B \) such that \( f^{-1}(V) \subseteq U \).

**Proof.** \( \Rightarrow \). Let \( V = Y - f(X - U) \). Then \( V \in \text{BO}(Y) \). Since \( f^{-1}(B) \subseteq U \), we have \( f(X - U) \subseteq Y - B \) and so \( B \subseteq V \). Also,

\[
\begin{align*}
  f^{-1}(V) &= X - f^{-1}[f(X - U)] \\
  &\subseteq X - (X - U) = U.
\end{align*}
\]

\( \Leftarrow \). Let \( C \) be a closed set in \( X \) and \( y \in Y - f(C) \). Then, \( f^{-1}(y) \subseteq X - f^{-1}(f(C)) \subseteq X - C \) and \( X - C \) is open in \( X \). Hence by assumption, there exists a set \( V_y \in \text{BO}(Y) \) containing \( y \) such that \( f^{-1}(V_y) \subseteq X - C \). This implies that \( y \in V_y \subseteq Y - f(C) \). Thus \( Y - f(C) = \bigcup \{ V_y : y \in Y - f(C) \} \). Hence \( Y - f(C) \in \text{BO}(Y) \). Thus \( f(C) \in \text{BC}(Y) \). \( \square \)

**Definition 2.12.** [3]. A function \( f : (X, \tau) \to (Y, \rho) \) is said to be \( b \)-continuous if \( f^{-1}(V) \in \text{BO}(X) \) for every open set \( V \) in \( Y \).

Theorem 2.13. Let \( f : (X, \tau) \to (Y, \rho) \) be a bijection. Then the following are equivalent:

1) \( f \) is \( b \)-closed
2) \( f \) is \( b \)-open
3) \( f^{-1} \) is \( b \)-continuous

**Proof.** (1) \( \to \) (2). Let \( U \) be an open subset of \( X \). Then \( X - U \) is closed in \( X \). By (1), \( f(X - U) \in \text{BC}(Y) \). But \( f(X - U) = f(X) - f(U) = Y - f(U) \). Thus \( f(U) \in \text{BO}(Y) \).

(2) \( \to \) (3). Let \( U \) be an open subset of \( X \). Since \( f \) is \( b \)-open \( f(U) = (f^{-1})^{-1}(U) \in \text{BO}(Y) \). Hence \( f^{-1} \) is \( b \)-continuous.

(3) \( \to \) (1). Let \( C \) be an arbitrary closed set in \( X \). Then \( X - C \) is open in \( X \). Since \( f^{-1} \) is \( b \)-continuous, \( (f^{-1})^{-1}(X - C) \in \text{BO}(Y) \). But,

\[
(f^{-1})^{-1}(X - C) = f(X - C) = Y - f(C).
\]

Thus, \( f(C) \in \text{BC}(Y) \). \( \square \)

**Definition 2.14.** [3]. A space \( X \) is called:

a) \( b - T_1 \) if for each pair of distinct points \( x \) and \( y \) in \( X \), there exist \( b \)-open sets \( U \) and \( V \) of \( X \) containing \( x \) and \( y \), respectively, such that \( y \notin U \) and \( x \notin V \).

b) \( b - T_2 \) if for each pair of distinct points \( x \) and \( y \) in \( X \), there exist disjoint \( b \)-open sets \( U \) and \( V \) of \( X \) such that \( x \in U \), \( y \in V \) and \( U \cap V = \phi \).
Theorem 2.15. Let \( f : (X, \tau) \rightarrow (Y, \rho) \) be a \( b \)-open bijection. Then the following hold

a) If \( X \) is \( T_1 \) then \( Y \) is \( b-T_1 \).
b) If \( X \) is \( T_2 \) then \( Y \) is \( b-T_2 \).

Proof. (a) Let \( y_1 \) and \( y_2 \) be any distinct points in \( Y \). Then there exist \( x_1 \) and \( x_2 \) in \( X \) such that \( f(x_1) = y_1 \) and \( f(x_2) = y_2 \). Since \( X \) is \( T_1 \) there exist two open sets \( U \) and \( V \) in \( X \) with \( x_1 \in U \), \( x_2 \notin U \) and \( x_2 \in V \), \( x_1 \notin V \). Now \( f(U) \) and \( f(V) \) are \( b \)-open in \( Y \) with \( y_1 \in f(U) \), \( y_2 \notin f(U) \) and \( y_2 \in f(V) \), \( y_1 \notin f(V) \).

(b) Similar to (a). □

Definition 2.16. [3]. A space \( X \) is said to be \( b \)-compact (resp. \( b \)-Lindelöf) if every \( b \)-open cover of \( X \) has a finite (resp. countable) subcover.

Theorem 2.17. Let \( f : (X, \tau) \rightarrow (Y, \rho) \) be a \( b \)-open bijection. Then the following hold

a) If \( Y \) is \( b \)-compact, then \( X \) is compact.
b) If \( Y \) is \( b \)-Lindelöf, then \( X \) is Lindelöf.

Proof. (a) Let \( U = \{U_\alpha : \alpha \in \Delta \} \) be an open cover of \( X \). Then \( O = \{f(U_\alpha) : \alpha \in \Delta \} \) is a cover of \( Y \) by \( b \)-open sets in \( Y \). Since \( Y \) is \( b \)-compact, \( O \) has a finite subcover \( O' = \{f(U_{\alpha_1}), f(U_{\alpha_2}), ..., f(U_{\alpha_n})\} \) for \( Y \). Then \( U' = \{U_{\alpha_1}, U_{\alpha_2}, ..., U_{\alpha_n}\} \) is a finite subcover of \( U \) for \( X \).

(b) Similar to (a). □

Definition 2.18. [3]. A space \( X \) is said to be \( b \)-connected if it cannot be written as a union of two non-empty disjoint \( b \)-open sets.

Theorem 2.19. If \( f : (X, \tau) \rightarrow (Y, \rho) \) is a \( b \)-open surjection and \( Y \) is \( b \)-connected then \( X \) is connected.

Proof. Suppose that \( X \) is not connected. Then there exist two non-empty disjoint open sets \( U \) and \( V \) in \( X \) such that \( X = U \cup V \). Then \( f(U) \) and \( f(V) \) are non-empty disjoint \( b \)-open sets in \( Y \) with \( Y = f(U) \cup f(V) \) which contradicts the fact that \( Y \) is \( b \)-connected. □

3. Quasi \( b \)-Open and Quasi \( b \)-Closed Functions

Definition 3.1. A function \( f : (X, \tau) \rightarrow (Y, \rho) \) is said to be quasi \( b \)-open if \( f(U) \) is open in \( Y \) for every \( U \in BO(X) \).

Clearly, every quasi \( b \)-open function is \( b \)-open.
Definition 3.2. A subset $A$ is called a $b$–neighborhood of a point $x$ in $X$ if there exists a $b$–open set $U$ such that $x \in U \subseteq A$.

Theorem 3.3. Let $f : (X, \tau) \rightarrow (Y, \rho)$ be a function. then the following are equivalent:

1) $f$ is quasi $b$–open.
2) For any subset $A$ of $X$ we have $f[bInt(A)] \subseteq Int[f(A)]$.
3) For any $x \in X$ and any $b$–neighborhood $U$ of $x$, there exists a neighborhood $V$ of $f(x)$ in $Y$ such that $V \subseteq f(U)$.

Proof. (1) $\Rightarrow$ (2). Let $f$ be quasi $b$–open and $A \subseteq X$. Now we have $Int(A) \subseteq A$ and $bInt(A) \subseteq BO(X)$. Hence we obtain that $f[bInt(A)] \subseteq f(A)$. Since $f[bInt(A)]$ is open, $f[bInt(A)] \subseteq Int[f(A)]$.

(2) $\Rightarrow$ (3). Let $x \in X$ and $U$ be a $b$–neighborhood of $x$ in $X$. Then there exists $V \in BO(X)$ such that $x \in V \subseteq U$. Then by (2), we have,

$$f(V) = f[bInt(V)] \subseteq Int[f(V)]$$

and hence $f(V) = Int[f(V)]$. Therefore $f(V)$ is open in $Y$ such that $f(x) \in f(V) \subseteq f(U)$.

(3) $\Rightarrow$ (1). Let $U \in BO(X)$. Then for each $y \in f(U)$, there exists a neighborhood $V_y$ of $y$ in $Y$ such that $V_y \subseteq f(U)$. Since $V_y$ is a neighborhood of $y$, there exists an open set $W_y$ in $Y$ such that $y \in W_y \subseteq V_y$. Thus, $f(U) = \bigcup \{W_y : y \in f(U)\}$ which is an open set in $Y$. This implies that $f$ is quasi $b$–open function. $\square$

Theorem 3.4. A function $f : (X, \tau) \rightarrow (Y, \rho)$ is quasi $b$–open if and only if $bInt[f^{-1}(B)] \subseteq f^{-1}[Int(B)]$ for every subset $B$ of $Y$.

Proof. $\Rightarrow$). Let $B$ be any subset of $Y$. Then, $bInt[f^{-1}(B)] \subseteq BO(X)$ and $f$ is quasi $b$–open, then $f[bInt(f^{-1}(B))] \subseteq Int[f(f^{-1}(B))] \subseteq Int(B)$. Thus , $bInt[f^{-1}(B)] \subseteq f^{-1}[Int(B)]$.

$\Leftarrow$). Let $U \in BO(X)$. Then by assumption $bInt[f^{-1}(f(U))] \subseteq f^{-1}[Int(f(U))]$ then $bInt(U) \subseteq f^{-1}[Int(f(U))]$, but $bInt(U) = U$ so $U \subseteq f^{-1}[Int(f(U))]$ and hence $f(U) \subseteq Int(f(U))$ so $f$ is quasi $b$–open. $\square$

Theorem 3.5. A function $f : (X, \tau) \rightarrow (Y, \rho)$ is quasi $b$–open if and only if for any subset $B$ of $Y$ and for any set $C \in BC(X)$ containing $f^{-1}(B)$, there exists a closed subset $F$ of $Y$ containing $f^{-1}(F) \subseteq C$.

Proof. $\Rightarrow$). Let $f$ be quasi $b$–open and $B \subseteq Y$. Let $C \in BC(X)$ with $f^{-1}(B) \subseteq C$. Now, put $F = Y - f(X - C)$. It is clear that since $f^{-1}(B) \subseteq C$, $B \subseteq F$. Since $f$ is quasi $b$–open, $F$ is a closed subset of $Y$. Also, we have $f^{-1}(F) \subseteq C$.

$\Leftarrow$). Let $U \in BO(X)$ and put $B = Y - f(U)$. Then $X - U \in BC(X)$ with $f^{-1}(B) \subseteq X - U$. By assumption, there exists a closed set $F$ of $Y$ such that $B \subseteq F$.
and $f^{-1}(F) \subseteq X - U$. Hence, we obtain $f(U) \subseteq Y - F$. On the other hand, it follows that $B \subseteq F$, $Y - F \subseteq Y - B = f(U)$. Thus, we have $f(U) = Y - F$ which is open and hence $f$ is a quasi $b$–open function.

**Theorem 3.6.** A function $f : (X, \tau) \rightarrow (Y, \rho)$ is quasi $b$–open if and only if $f^{-1}[Cl(B)] \subseteq bCl[f^{-1}(B)]$ for any subset $B$ of $Y$.

**Proof.** $\Rightarrow$). Suppose that $f$ is quasi $b$–open. For any subset $B$ of $Y$, $f^{-1}(B) \subseteq bCl[f^{-1}(B)]$. Therefore by Theorem 3.5, there exists a closed set $F$ in $Y$ such that $B \subseteq F$ and $f^{-1}(F) \subseteq bCl[f^{-1}(B)]$. Therefore, we obtain,

$$f^{-1}[Cl(B)] \subseteq f^{-1}(F) \subseteq bCl[f^{-1}(B)].$$

$\Leftarrow$). Let $B \subseteq Y$ and $C \in BC(X)$ with $f^{-1}(B) \subseteq C$. Put $F = Cl(B)$, then we have $B \subseteq F$ and $F$ is closed and $f^{-1}(F) \subseteq bCl[f^{-1}(B)] \subseteq C$. Then by Theorem 3.5, the function $f$ is quasi $b$–open.

**Definition 3.7.** A function $f : (X, \tau) \rightarrow (Y, \rho)$ is said to be quasi $b$–closed if $f(C)$ is closed in $Y$ for every $C \in BC(X)$.

Clearly, every quasi $b$–closed function is $b$–closed.

**Theorem 3.8.** If a function $f : (X, \tau) \rightarrow (Y, \rho)$ is quasi $b$–closed then $f^{-1}[Int(B)] \subseteq bInt[f^{-1}(B)]$ for every subset $B$ of $Y$.

**Proof.** Similar to the proof of Theorem 3.4.

**Theorem 3.9.** A function $f : (X, \tau) \rightarrow (Y, \rho)$ is quasi $b$–closed if and only if for any subset $B$ of $Y$ and for any $U \in BO(X)$ containing $f^{-1}(B)$, there exists an open subset $V$ of $Y$ containing $B$ such that $f^{-1}(V) \subseteq U$.

**Proof.** Similar to the proof of Theorem 3.5.

In a similar way used in proving Theorem 2.15, Theorem 2.17 and Theorem 2.19, we can prove the following three theorems

**Theorem 3.10.** Let $f : (X, \tau) \rightarrow (Y, \rho)$ be a quasi $b$–open bijection. Then the following hold

a) If $X$ is $b$–$T_1$ then $Y$ is $T_1$.

b) If $X$ is $b$–$T_2$ then $Y$ is $T_2$.

**Theorem 3.11.** Let $f : (X, \tau) \rightarrow (Y, \rho)$ be a quasi $b$–open bijection. Then the following hold

a) If $Y$ is compact, then $X$ is $b$–compact.

b) If $Y$ is Lindelöf, then $X$ is $b$–Lindelöf.

**Theorem 3.12.** If $f : (X, \tau) \rightarrow (Y, \rho)$ is a quasi $b$–open surjection and $Y$ is connected then $X$ is $b$–connected.
4. **Strongly b–Open and Strongly b–Closed Functions**

**Definition 4.1.** A function \( f : (X, \tau) \to (Y, \rho) \) is said to be strongly b–open if \( f(U) \subseteq BO(Y) \) for every \( U \subseteq BO(X) \).

Clearly, every strongly b–open function is b–open.

**Theorem 4.2.** Let \( f : (X, \tau) \to (Y, \rho) \) and \( g : (Y, \rho) \to (Z, \sigma) \) be two strongly b–open functions. Then the composition function \( g \circ f : (X, \tau) \to (Z, \sigma) \) is strongly b–open.

**Proof.** Let \( U \subseteq BO(X) \). Then \( f(U) \subseteq BO(Y) \) since \( f \) is strongly b–open. But \( g \) is strongly b–open so \( g(f(U)) \subseteq BO(Z) \). Hence \( g \circ f \) is strongly b–open. \( \square \)

**Theorem 4.3.** A function \( f : (X, \tau) \to (Y, \rho) \) is strongly b–open if and only if for each \( x \in X \) and for any \( U \subseteq BO(X) \) with \( x \in U \), there exists \( V \subseteq BO(Y) \) such that \( f(x) \subseteq V \) and \( V \subseteq f(U) \).

**Proof.** It is obvious. \( \square \)

**Theorem 4.4.** A function \( f : (X, \tau) \to (Y, \rho) \) is strongly b–open if and only if for each \( x \in X \) and for any b–neighborhood \( U \) of \( x \) in \( X \), there exists a b–neighborhood \( V \) of \( f(x) \) in \( Y \) such that \( V \subseteq f(U) \).

**Proof.** \( \Rightarrow \). Let \( x \in X \) and let \( U \) be a b–neighborhood of \( x \). Then there exists \( W \subseteq BO(X) \) such that \( x \in W \subseteq U \). Then \( f(x) \subseteq f(W) \subseteq f(U) \). But, \( f(W) \subseteq BO(Y) \) since \( f \) is strongly b–open. Hence \( V = f(W) \) is a b–neighborhood of \( f(x) \) and \( V \subseteq f(U) \).

\( \Leftarrow \). Let \( U \subseteq BO(X) \) and \( x \in U \). Then \( U \) is a b–neighborhood of \( x \). So by assumption, there exists a b–neighborhood \( V_{f(x)} \) of \( f(x) \) such that, \( f(x) \subseteq V_{f(x)} \subseteq f(U) \). It follows that \( f(U) \) is a b–neighborhood of each of its points. Therefore, \( f(U) \subseteq BO(Y) \). Hence \( f \) is strongly b–open. \( \square \)

**Theorem 4.5.** A function \( f : (X, \tau) \to (Y, \rho) \) is strongly b–open if and only if \( f[bInt(A)] \subseteq bInt[f(A)] \), for every \( A \subseteq X \).

**Proof.** \( \Rightarrow \). Let \( A \subseteq X \) and \( x \in bInt(A) \). Then there exists \( U_x \subseteq BO(X) \) such that \( x \in U_x \subseteq A \). So \( f(x) \subseteq f(U_x) \subseteq f(A) \) and by assumption, \( f(U_x) \subseteq BO(Y) \). Hence, \( f(x) \subseteq bInt[f(A)] \). Thus \( f[bInt(A)] \subseteq bInt[f(A)] \).

\( \Leftarrow \). Let \( U \subseteq BO(X) \). Then by assumption, \( f[bInt(U)] \subseteq bInt[f(U)] \). Since \( bInt(U) = U \) and \( bInt[f(U)] \subseteq f(U) \). Hence, \( f(U) = bInt[f(U)] \). Thus, \( f(U) \subseteq BO(Y) \). \( \square \)

**Theorem 4.6.** A function \( f : (X, \tau) \to (Y, \rho) \) is strongly b–open if and only if \( bInt[f^{-1}(B)] \subseteq f^{-1}[bInt(B)] \), for every \( B \subseteq Y \).
Proof. \(\Rightarrow\). Let \(B \subseteq Y\). Since \(b\text{Int}[f^{-1}(B)] \subseteq BO(X)\) and \(f\) is strongly \(b\)–open, \(f[b\text{Int}(f^{-1}(B))] \subseteq BO(Y)\). Also we have \(f[b\text{Int}(f^{-1}(B))] \subseteq f[f^{-1}(B)] \subseteq B\). Hence, \(f[b\text{Int}(f^{-1}(B))] \subseteq b\text{Int}(B)\). Therefore, \(b\text{Int}[f^{-1}(B)] \subseteq f^{-1}[b\text{Int}(B)]\).

\((\Leftarrow)\). Let \(A \subseteq X\). Then \(f(A) \subseteq Y\). Hence by assumption, we obtain,

\[
b\text{Int}(A) \subseteq b\text{Int}[f^{-1}(f(A))] \subseteq f^{-1}[b\text{Int}(f(A))].
\]

This implies that,

\[
f[b\text{Int}(A)] \subseteq f[f^{-1}(b\text{Int}(f(A)))] \subseteq b\text{Int}[f(A)].
\]

Thus, \(f[b\text{Int}(A)] \subseteq b\text{Int}[f(A)]\), for all \(A \subseteq X\). Hence, by Theorem 4.5, \(f\) is strongly \(b\)–open.

\[\square\]

**Theorem 4.7.** A function \(f : (X, \tau) \to (Y, \rho)\) is strongly \(b\)–open if and only if 
\[f^{-1}[b\text{Cl}(B)] \subseteq b\text{Cl}[f^{-1}(B)],\]
for every \(B \subseteq Y\).

**Proof.** \(\Rightarrow\). Let \(B \subseteq Y\) and \(x \in f^{-1}[b\text{Cl}(B)]\). Then \(f(x) \in b\text{Cl}(B)\). Let \(U \in BO(X)\) such that \(x \in U\). By assumption, \(f(U) \in BO(Y)\) and \(f(x) \in f(U)\). Thus \(f(U) \cap B \neq \emptyset\). Hence \(U \cap f^{-1}(B) \neq \emptyset\). Therefore, \(x \in b\text{Cl}[f^{-1}(B)]\). So we obtain \(f^{-1}[b\text{Cl}(B)] \subseteq b\text{Cl}[f^{-1}(B)]\).

\((\Leftarrow)\). Let \(B \subseteq Y\). Then \(Y - B \subseteq Y\). By assumption,

\[
f^{-1}[b\text{Cl}(Y - B)] \subseteq b\text{Cl}[f^{-1}(Y - B)].
\]

This implies that,

\[
X - b\text{Cl}[f^{-1}(Y - B)] \subseteq X - f^{-1}[b\text{Cl}(Y - B)].
\]

Hence,

\[
X - b\text{Cl}[X - f^{-1}(B)] \subseteq f^{-1}[Y - b\text{Cl}(Y - B)].
\]

Then, \(b\text{Int}[f^{-1}(B)] \subseteq f^{-1}[b\text{Int}(B)]\). Now by Theorem 4.6, it follows that \(f\) is strongly \(b\)–open.

\[\square\]

**Definition 4.8.** [3]. A function \(f : (X, \tau) \to (Y, \rho)\) is said to be \(b\)–irresolute if \(f^{-1}(V) \in BO(X)\) for every \(V \in BO(Y)\).

**Theorem 4.9.** Let \(f : (X, \tau) \to (Y, \rho)\) be a function and \(g : (Y, \rho) \to (Z, \sigma)\) be a strongly \(b\)–open injection. If \(gof : (X, \tau) \to (Z, \sigma)\) is \(b\)–irresolute, then \(f\) is \(b\)–irresolute.

**Proof.** Let \(U \in BO(Y)\). Then \(g(U) \in BO(Z)\) since \(g\) is strongly \(b\)–open. Also \(gof\) is \(b\)–irresolute, so we have \((gof)^{-1}[g(U)] \subseteq BO(X)\). Since \(g\) is an injection, we have \((gof)^{-1}[g(U)] = (f^{-1}og^{-1})[g(U)] = f^{-1}[g^{-1}(g(U))] = f^{-1}(U)\). Then, \(f^{-1}(U) \in BO(X)\). So \(f\) is \(b\)–irresolute.

\[\square\]
**Theorem 4.10.** Let \( f : (X, \tau) \to (Y, \rho) \) be strongly \( b \)– open surjection and \( g : (Y, \rho) \to (Z, \sigma) \) be any function. If \( gof : (X, \tau) \to (Z, \sigma) \) is \( b \)– irresolute, then \( g \) is \( b \)– irresolute.

*Proof.* Let \( V \in BO(Z) \). Then \((gof)^{-1}(V) \in BO(X) \) since \( gof \) is \( b \)– irresolute. Also \( f \) is strongly \( b \)– open, so \( f[(gof)^{-1}(V)] \in BO(Y) \). Since \( f \) is surjective, we note that \( f[(gof)^{-1}(V)] = [fo(gof)^{-1}](V) = [fo(f^{-1}og^{-1})](V) = [(fof^{-1})og^{-1}](V) = g^{-1}(V) \). Hence \( g \) is \( b \)– irresolute. \( \Box \)

**Definition 4.11.** A function \( f : (X, \tau) \to (Y, \rho) \) is said to be strongly \( b \)– closed if \( f(C) \in BC(Y) \) for every \( C \in BC(X) \).

The straightforward proof of the following theorem is omitted.

**Theorem 4.12.** If \( f : (X, \tau) \to (Y, \rho) \) and \( g : (Y, \rho) \to (Z, \sigma) \) are two strongly \( b \)– closed functions, then \( gof : (X, \tau) \to (Z, \sigma) \) is a strongly \( b \)– closed function.

**Theorem 4.13.** Let \( f : (X, \tau) \to (Y, \rho) \) and \( g : (Y, \rho) \to (Z, \sigma) \) be two functions such that \( gof : (X, \tau) \to (Z, \sigma) \) is a strongly \( b \)– closed function. Then

1) If \( f \) is \( b \)– irresolute and surjection then \( g \) is strongly \( b \)– closed.

2) If \( g \) is \( b \)– irresolute and injection, then \( f \) is strongly \( b \)– closed.

*Proof.* (1). Let \( F \in BC(Y) \). Since \( f \) is \( b \)– irresolute, \( f^{-1}(F) \in BC(X) \). Now \( gof \) is strongly \( b \)– closed and \( f \) is surjection, then \((gof)(f^{-1}(F)) = g(F) \in BC(Z) \). This implies that \( g \) is strongly \( b \)– closed.

(2). Let \( C \in BC(X) \). Since \( gof \) is strongly \( b \)– closed, \((gof)(C) \in BC(Z) \). Now \( g \) is \( b \)– irresolute and injection, so \( g^{-1}[(gof)(C)] = f(C) \in BC(Y) \). This shows that \( f \) is strongly \( b \)– closed. \( \Box \)

**Theorem 4.14.** A function \( f : (X, \tau) \to (Y, \rho) \) is strongly \( b \)– closed if and only if \( bCl[f(A)] \subseteq f[bCl(A)] \), for every \( A \subseteq X \).

*Proof.* \( \Rightarrow \). Let \( f \) be strongly \( b \)– closed and \( A \subseteq X \). Then \( f[bCl(A)] \in BC(Y) \). Since \( f(A) \subseteq f[bCl(A)] \), we obtain \( bCl[f(A)] \subseteq f[bCl(A)] \).

\( \Leftarrow \). Let \( C \in BC(X) \). By assumption, we obtain, \( f(C) \subseteq bCl[f(C)] \subseteq f[bCl(C)] = f(C) \).

Hence \( f(C) = bCl[f(C)] \). Thus, \( f(C) \in BC(Y) \). It follows that \( f \) is strongly \( b \)– closed. \( \Box \)

**Theorem 4.15.** Let \( f : (X, \tau) \to (Y, \rho) \) be a function such that \( Int[Cl(f(A))] \subseteq f[bCl(A)] \) for every \( A \subseteq X \). Then \( f \) is strongly \( b \)– closed.
Proof. Let $C \in BC(X)$. Then by assumption we have,

$$Int[Cl(f(C))] \subseteq f[bCl(C)] = f(C).$$

Put $F = Cl[f(C)]$. Then $F$ is closed in $Y$. Also it implies that $Int(F) \subseteq f(C) \subseteq F$. Hence, $f(C)$ is semi closed in $Y$. Since $SO(Y) \subseteq BO(Y)$, $f(C) \in BC(Y)$. This implies that $f$ is strongly $b$–closed.

Theorem 4.16. Let $f : (X, \tau) \to (Y, \rho)$ be a strongly $b$–closed function and $B \subseteq Y$. If $U \in BO(X)$ with $f^{-1}(B) \subseteq U$, then there exists $V \in BO(Y)$ with $B \subseteq V$ such that $f^{-1}(B) \subseteq f^{-1}(V) \subseteq U$.

Proof. Let $V = Y - f(X - U)$. Then $Y - V = f(X - U)$. Since $f$ is strongly $b$–closed, $V \in BO(Y)$. Since $f^{-1}(B) \subseteq U$, we have $Y - V = f(X - U) \subseteq f[f^{-1}(Y - B)] \subseteq Y - B$. Hence, $B \subseteq V$. Also $X - U \subseteq f^{-1}[f(X - U)] = f^{-1}(Y - V) = X - f^{-1}(V)$. So $f^{-1}(V) \subseteq U$.

Theorem 4.17. Let $f : (X, \tau) \to (Y, \rho)$ be a surjective strongly $b$–closed function and $B, C \subseteq Y$. If $f^{-1}(B)$ and $f^{-1}(C)$ have disjoint $b$–neighborhoods, then so have $B$ and $C$.

Proof. Let $E$ and $F$ be the disjoint $b$–neighborhood of $f^{-1}(B)$ and $f^{-1}(C)$ respectively. Then by the last theorem There exist two sets $U, V \in BO(Y)$ with $B \subseteq U$ and $C \subseteq V$ such that $f^{-1}(B) \subseteq f^{-1}(U) \subseteq bInt(E)$ and $f^{-1}(C) \subseteq f^{-1}(V) \subseteq bInt(F)$. Since $E$ and $F$ are disjoint, so are $bInt(E)$ and $bInt(F)$, and hence so $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint as well. It follows that $U$ and $V$ are disjoint too since $f$ is a surjective function.

Theorem 4.18. A surjective function $f : (X, \tau) \to (Y, \rho)$ is strongly $b$–closed if and only if for each subset $B$ of $Y$ and each set $U \in BO(X)$ containing $f^{-1}(B)$, there exists a set $V \in BO(Y)$ containing $B$, such that $f^{-1}(V) \subseteq U$.

Proof. $\Rightarrow$. This follows from Theorem 4.16.

$\Leftarrow$. Let $C \in BC(X)$ and $y \in Y - f(C)$. Then $f^{-1}(y) \subseteq X - f^{-1}(f(C)) \subseteq X - C$ and $X - C \in BO(X)$. Hence by assumption, there exists a set $V_y \in BO(Y)$ containing $y$ such that $f^{-1}(V_y) \subseteq X - C$. This implies that $y \in V_y \subseteq Y - f(C)$. Thus, $Y - f(C) = \bigcup\{V_y : y \in Y - f(C)\}$. Hence, $Y - f(C) \in BO(Y)$. Therefore, $f(C) \in BC(Y)$.

Theorem 4.19. Let $f : (X, \tau) \to (Y, \rho)$ be a bijection. Then the following are equivalent:

1) $f$ is strongly $b$–closed.
2) $f$ is strongly $b$–open.
3) $f^{-1}$ is $b$–irresolute.
Proof. (1) $\rightarrow$ (2). Let $U \in BO(X)$. Then $X - U \in BC(X)$. By (1),
$f(X - U) \in BC(Y)$. But $f(X - U) = f(X) - f(U) = Y - f(U)$. Thus $f(U) \in BO(Y)$.

(2) $\rightarrow$ (3). Let $A \subseteq X$. Since $f$ is strongly $b$-open, so by Theorem 4.7,
$f^{-1}[bCl(f(A))] \subseteq bCl[f^{-1}(f(A))]$. It implies that $bCl[f(A)] \subseteq f[bCl(A)]$. Thus
$bCl[(f^{-1})^{-1}(A)] \subseteq (f^{-1})^{-1}[bCl(A)]$, for all $A \subseteq X$. Then, it follows that $f^{-1}$ is $b$-irresolute.

(3) $\rightarrow$ (1). Let $C \in BC(X)$. Then $X - C \in BO(X)$. Since $f^{-1}$ is $b$-irresolute,
$(f^{-1})^{-1}(X - C) \in BO(Y)$. But $(f^{-1})^{-1}(X - C) = f(X - C) = Y - f(C)$. Thus
$f(C) \in BC(Y)$.

□

In a similar way used in proving Theorem 2.15, Theorem 2.17 and Theorem 2.19 we can prove the following three theorems

**Theorem 4.20.** Let $f : (X, \tau) \rightarrow (Y, \rho)$ be a strongly $b$-open bijection. Then the following hold

a) If $X$ is $b-T_1$ then $Y$ is $b-T_1$.
b) If $X$ is $b-T_2$ then $Y$ is $b-T_2$.

**Theorem 4.21.** Let $f : (X, \tau) \rightarrow (Y, \rho)$ be a strongly $b$-open bijection. Then the following hold

a) If $Y$ is $b$-compact, then $X$ is $b$-compact.
b) If $Y$ is $b$-Lindelöf, then $X$ is $b$-Lindelöf.

**Theorem 4.22.** If $f : (X, \tau) \rightarrow (Y, \rho)$ is a strongly $b$-open surjection and $Y$ is $b$-connected then $X$ is $b$-connected.

**References**


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