

ON ALMOST CONTRA $e^*\theta$ -CONTINUOUS FUNCTIONS

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ABSTRACT. The aim of this paper is to introduce and investigate some of fundamental properties of almost contra $e^*\theta$ -continuous functions via $e^*\theta$ -closed sets which are defined by Farhan and Yang [15]. Also, we obtain several characterizations of almost contra $e^*\theta$ -continuous functions. Furthermore, we investigate the relationships between almost contra $e^*\theta$ -continuous functions and separation axioms and $e^*\theta$ -closedness of graphs of functions.

1. INTRODUCTION

In 2006, the concept of almost contra continuity [4], which is stronger than almost contra precontinuity [8] is introduced by Ekici and almost contra β -continuity [4] introduced by Baker, is defined. In 2017, some properties and characterizations of the notion of almost contra $\beta\theta$ -continuous function [5] defined by Caldas via $\beta\theta$ -closed sets are obtained. The notion of almost contra $e^*\theta$ -continuity is stronger than almost contra e^* -continuity which is defined by us in this manuscript. In this paper, we introduce some new forms of contra e^* -continuity [9] defined by Ekici. Also, we obtain some characterizations of almost contra $e^*\theta$ -continuous functions and investigate their some fundamental properties. Moreover, we investigate the relationships between almost contra $e^*\theta$ -continuity and other related generalized forms of contra continuity.

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2. PRELIMINARIES

Throughout this present paper, X and Y represent topological spaces. For a subset A of a space X , $cl(A)$ and $int(A)$ denote the closure of A and the interior of A , respectively. The family of all closed (resp. open) sets of X is denoted by $C(X)$ (resp. $O(X)$). A subset A is said to be regular open [28] (resp. regular closed [28]) if $A = int(cl(A))$ (resp. $A = cl(int(A))$). A point $x \in X$ is said to be δ -cluster point [30] of A if $int(cl(U)) \cap A \neq \emptyset$ for each open neighbourhood U of x . The set of all δ -cluster points of A is called the δ -closure [30] of A and is denoted by $cl_\delta(A)$. If $A = cl_\delta(A)$, then A is called δ -closed [30], and the complement of a δ -closed set is called δ -open [30]. The set $\{x | (\exists U \in \tau)(x \in U)(int(cl(U)) \subseteq A)\}$ is called the δ -interior of A and is denoted by $int_\delta(A)$.

A subset A is called α -open [19] (resp. semiopen [17], δ -semiopen [23], preopen [18], δ -preopen [24], b -open [1], e -open [11], e^* -open [12], a -open [10]) if $A \subseteq int(cl(int(A)))$ (resp. $A \subseteq cl(int(A))$, $A \subseteq cl(int_\delta(A))$, $A \subseteq int(cl(A))$, $A \subseteq int(cl_\delta(A))$, $A \subseteq cl(int(A)) \cup int(cl(A))$, $A \subseteq cl(int_\delta(A)) \cup int(cl_\delta(A))$, $A \subseteq cl(int(cl_\delta(A)))$, $A \subseteq int(cl(int_\delta(A)))$). The complement of an α -open (resp. semiopen, δ -semiopen, preopen, δ -preopen, b -open, e -open, e^* -open, a -open) set is called α -closed [19] (resp. semiclosed [17], δ -semiclosed [23], preclosed [18], δ -preclosed [24], b -closed [1], e -closed [11], e^* -closed [12], a -closed [10]). The intersection of all e^* -closed (resp. a -closed, semiclosed, δ -semiclosed, preclosed, δ -preclosed) sets of X containing A is called the e^* -closure [12] (resp. a -closure [10], semiclosure [17], δ -semiclosure [23], preclosure [18], δ -preclosure [24]) of A and is denoted by $e^*-cl(A)$ (resp. $a-cl(A)$, $scl(A)$, $\delta-scl(A)$, $pcl(A)$, $\delta-pcl(A)$). The union of all e^* -open (resp. a -open, semiopen, δ -semiopen, preopen, δ -preopen) sets of X contained in A is called the e^* -interior [12] (resp. a -interior [10], semiinterior [17], δ -semiinterior [23], preinterior [18], δ -preinterior [24]) of A and is denoted by $e^*-int(A)$ (resp. $a-int(A)$, $sint(A)$, $\delta-sint(A)$, $pint(A)$, $\delta-pint(A)$).

A point x of X is called a θ -cluster [30] point of A if $cl(U) \cap A \neq \emptyset$ for every open set U of X containing x . The set of all θ -cluster points of A is called the θ -closure [30] of A and is denoted by $cl_\theta(A)$. A subset A is said to be θ -closed [30] if $A = cl_\theta(A)$. The complement of a θ -closed set is called a θ -open [30] set. A point x of X said to be a θ -interior [30] point of a subset A , denoted by $int_\theta(A)$, if there exists an open set U of X containing x such that $cl(U) \subseteq A$.

A point $x \in X$ is said to be a θ -semicluster point [16] of a subset S of X if $cl(U) \cap A \neq \emptyset$ for every semiopen U containing x . The set of all θ -semicluster points of A is called the θ -semiclosure of A and is denoted by $\theta-scl(A)$. A subset A is called θ -semiclosed [16] if $A = \theta-scl(A)$. The complement of a θ -semiclosed set is called θ -semiopen.

The union of all e^* -open sets of X contained in A is called the e^* -interior [12] of A and is denoted by $e^*-int(A)$. A subset A is said to be e^* -regular [15] if it is e^* -open and e^* -closed. The family of all e^* -regular subsets of X is denoted by $e^*R(X)$.

A point x of X is called an $e^*\theta$ -cluster point of A if $e^*-cl(U) \cap A \neq \emptyset$ for every e^* -open set U containing x . The set of all $e^*\theta$ -cluster points of A is called the $e^*\theta$ -closure [15] of A and is denoted by $e^*-cl_\theta(A)$. A subset A is said to be $e^*\theta$ -closed if $A = e^*-cl_\theta(A)$. The complement of an $e^*\theta$ -closed set is called an $e^*\theta$ -open [15] set. A point x of X said to be an $e^*\theta$ -interior [15] point of a subset A , denoted by $e^*-int_\theta(A)$, if there exists an e^* -open set U of X containing x such that $e^*-cl(U) \subseteq A$. Also it is noted in [15] that

$$e^*\text{-regular} \Rightarrow e^*\theta\text{-open} \Rightarrow e^*\text{-open}.$$

The family of all $e^*\theta$ -open (resp. $e^*\theta$ -closed, e^* -open, e^* -closed, regular open, regular closed, δ -open, δ -closed, θ -open, θ -closed, θ -semiopen, θ -semiclosed, semiopen, semi-closed, preopen, preclosed, δ -semiopen, δ -semiclosed, δ -preopen, δ -preclosed, a -open, a -closed) subsets of X is denoted by $e^*\theta O(X)$ (resp. $e^*\theta C(X)$, $e^*O(X)$, $e^*C(X)$, $RO(X)$, $RC(X)$, $\delta O(X)$, $\delta C(X)$, $\theta O(X)$, $\theta C(X)$, $\theta SO(X)$, $\theta SC(X)$, $SO(X)$, $SC(X)$),

$PO(X), PC(X), \delta SO(X), \delta SC(X), \delta PO(X), \delta PC(X), aO(X), aC(X)$). The family of all open (resp. closed, e^* - θ -open, e^* - θ -closed, e^* -open, e^* -closed, regular open, regular closed, δ -open, δ -closed, θ -open, θ -closed, θ -semiopen, θ -semiclosed, semiopen, semiclosed, preopen, preclosed, δ -semiopen, δ -semiclosed, δ -preopen, δ -preclosed, a -open, a -closed) sets of X containing a point x of X is denoted by $O(X, x)$ (resp. $C(X, x), e^*\theta O(X, x), e^*\theta C(X, x), e^*O(X, x), e^*C(X, x), RO(X, x), RC(X, x), \delta O(X, x), \delta C(X, x), \theta O(X, x), \theta C(X, x), \theta SO(X, x), \theta SC(X, x), SO(X, x), SC(X, x), PO(X, x), PC(X, x), \delta SO(X, x), \delta SC(X, x), \delta PO(X, x), \delta PC(X, x), aO(X, x), aC(X, x)$).

We shall use the well-known accepted language almost in the whole of the proofs of the theorems in this article.

Lemma 2.1. [12] *Let A be a subset of a space X , then the followings hold:*

- (1) $e^*cl(X \setminus A) = X \setminus e^*int(A)$,
- (2) $x \in e^*cl(A)$ if and only if $A \cap U \neq \emptyset$ for every $U \in e^*O(X, x)$,
- (3) A is $e^*C(X)$ if and only if $A = e^*cl(A)$,
- (4) $e^*cl(A) \in e^*C(X)$,
- (5) $e^*int(A) = A \cap cl(int(cl_\delta(A)))$.

Lemma 2.2. [10, 23, 24] *Let A be a subset of a space X , then the followings hold:*

- (1) $a-cl(A) = A \cup cl(int(cl_\delta(A)))$,
- (2) $\delta-scl(A) = A \cup int(cl_\delta(A))$,
- (3) $\delta-pcl(A) = A \cup cl(int_\delta(A))$.

Lemma 2.3. [15] *The following properties hold for the $e^*\theta$ -closure of a subset A of a topological space X .*

- (1) $A \subseteq e^*cl(A) \subseteq e^*cl_\theta(A)$,
- (2) If $A \in e^*\theta O(X)$, then $e^*cl_\theta(A) = e^*cl(A)$,
- (3) If $A \subseteq B$, then $e^*cl_\theta(A) \subseteq e^*cl_\theta(B)$,

- (4) $e^*cl_\theta(A) \in e^*\theta C(X)$ and $e^*cl_\theta(e^*cl_\theta(A)) = e^*cl_\theta(A)$,
 (5) If $A_\alpha \in e^*\theta C(X)$ for each $\alpha \in \Lambda$, then $\cap\{A_\alpha|\alpha \in \Lambda\} \in e^*\theta C(X)$,
 (6) If $A_\alpha \in e^*\theta O(X)$ for each $\alpha \in \Lambda$, then $\cup\{A_\alpha|\alpha \in \Lambda\} \in e^*\theta O(X)$,
 (7) $e^*cl_\theta(X \setminus A) = X \setminus e^*int_\theta(A)$.

Lemma 2.4. [15] *Let A be a subset of a topological space X , then the followings hold:*

- (1) If $A \in e^*O(X)$, then $e^*cl_\theta(A) \in e^*R(X)$,
 (2) $A \in e^*R(X)$ if and only if $A \in e^*\theta O(X) \cap e^*\theta C(X)$,
 (3) A is $e^*\theta$ -open in X if and only if for each $x \in A$ there exists $U \in e^*R(X, x)$ such that $x \in U \subseteq A$.

Definition 2.1. Let A be a subset of a space X . The intersection of all regular open sets in X containing A is called the r -kernel of A [9] and is denoted by $rker(A)$.

Lemma 2.5. [9] *The following properties hold for subsets A and B of a space X .*

- (1) $x \in rker(A)$ if and only if $A \cap F \neq \emptyset$ for any $F \in RC(X, x)$,
 (2) $A \subseteq rker(A)$,
 (3) If A is regular open in X , then $A = rker(A)$,
 (4) If $A \subseteq B$, then $rker(A) \subseteq rker(B)$.

Lemma 2.6. [11] *The following properties hold for a subset A of a space X .*

- (1) $cl(int_\delta(A)) = cl_\delta(int_\delta(A))$,
 (2) $int(cl_\delta(A)) = int_\delta(cl_\delta(A))$.

Lemma 2.7. *Let A be a subset of a topological space X . If A is an e^* -open set in X , then $int_\delta(X \setminus A) = X \setminus cl_\delta(A) \in RO(X)$.*

Proof. Let $A \in e^*O(X)$.

$$\begin{aligned}
 A \in e^*O(X) &\Rightarrow A \subseteq cl(int(cl_\delta(A))) \\
 &\Rightarrow cl_\delta(A) \subseteq cl_\delta(cl(int(cl_\delta(A)))) \stackrel{\text{Lemma 2.6}}{=} cl_\delta(cl_\delta(int_\delta(cl_\delta(A)))) \\
 &\Rightarrow cl_\delta(A) \subseteq cl_\delta(cl(int(cl_\delta(A)))) = cl_\delta(int_\delta(cl_\delta(A))) \\
 &\Rightarrow cl_\delta(A) \subseteq cl_\delta(cl(int(cl_\delta(A)))) \stackrel{\text{Lemma 2.6}}{=} cl(int(cl_\delta(A))) \\
 &\Rightarrow \setminus cl(int(cl_\delta(A))) = int(cl(\setminus cl_\delta(A))) \subseteq \setminus cl_\delta(A) \dots (*) \\
 int(cl_\delta(A)) \subseteq cl_\delta(A) &\Rightarrow cl(int(cl_\delta(A))) = cl_\delta(int(cl_\delta(A))) \subseteq cl_\delta(cl_\delta(A)) = cl_\delta(A) \\
 &\Rightarrow \setminus cl_\delta(A) \subseteq \setminus cl(int(cl_\delta(A))) = int(cl(\setminus cl_\delta(A))) \dots (**) \\
 (*), (**) &\Rightarrow \setminus cl_\delta(A) = int(cl(\setminus cl_\delta(A))) \Rightarrow \setminus cl_\delta(A) \in RO(X). \quad \square
 \end{aligned}$$

Definition 2.2. A function $f : X \rightarrow Y$ is said to be:

- a) $e^*\theta$ -continuous (briefly $e^*\theta.c.$) if $f^{-1}[V]$ is $e^*\theta$ -closed in X for every $V \in C(Y)$,
- b) almost $e^*\theta$ -continuous (briefly a. $e^*\theta.c.$) if $f^{-1}[V]$ is $e^*\theta$ -closed in X for every regular closed set V in Y ,
- c) contra R -map [9] (resp. contra continuous [7], contra $e^*\theta$ -continuous [3], contra e^* -continuous [13]) if $f^{-1}[V]$ is regular closed (resp. closed, $e^*\theta$ -closed, e^* -closed) in X for every regular open (resp. open, open, open) set V in Y ,
- d) almost contra precontinuous [8] (resp. almost contra continuous [4], almost contra β -continuous [4], almost contra e^* -continuous) if $f^{-1}[V]$ is preclosed (resp. closed, β -closed, e^* -closed) in X for every regular open set V in Y .

Lemma 2.8. [25] For a topological space (X, τ) the followings are equivalent:

- (1) (X, τ) is almost regular;
- (2) For each point $x \in X$ and each neighbourhood M of x , there exists a regular open neighbourhood V of x such that $cl(V) \subseteq int(cl(M))$.

3. ALMOST CONTRA $e^*\theta$ -CONTINUOUS FUNCTIONS

Definition 3.1. A function $f : X \rightarrow Y$ is said to be almost contra $e^*\theta$ -continuous (briefly a.c. $e^*\theta$.c.) if $f^{-1}[V]$ is $e^*\theta$ -closed in X for each regular open set V of Y .

Theorem 3.1. For a function $f : X \rightarrow Y$, the following properties are equivalent:

- (1) f is almost contra $e^*\theta$ -continuous;
- (2) The inverse image of each regular closed set in Y is $e^*\theta$ -open in X ;
- (3) For each point $x \in X$ and each $V \in RC(Y, f(x))$, there exists $U \in e^*\theta O(X, x)$ such that $f[U] \subseteq V$;
- (4) For each point $x \in X$ and each $V \in SO(Y, f(x))$, there exists $U \in e^*\theta O(X, x)$ such that $f[U] \subseteq cl(V)$;
- (5) $f[e^*cl_\theta(A)] \subseteq rker(f[A])$ for every subset A of X ;
- (6) $e^*cl_\theta(f^{-1}[B]) \subseteq f^{-1}[rker(B)]$ for every subset B of Y ;
- (7) $f^{-1}[cl_\delta(V)]$ is $e^*\theta$ -open for every $V \in e^*O(Y)$;
- (8) $f^{-1}[cl_\delta(V)]$ is $e^*\theta$ -open for every $V \in \delta SO(Y)$;
- (9) $f^{-1}[int(cl_\delta(V))]$ is $e^*\theta$ -closed for every $V \in \delta PO(Y)$;
- (10) $f^{-1}[int(cl_\delta(V))]$ is $e^*\theta$ -closed for every $V \in O(Y)$;
- (11) $f^{-1}[cl(int_\delta(V))]$ is $e^*\theta$ -open for every $V \in C(Y)$.

Proof. (1) \Rightarrow (2) : Let $V \in RC(Y)$.

$$V \in RC(Y) \Rightarrow \left. \begin{array}{l} \setminus V \in RO(Y) \\ (1) \end{array} \right\} \Rightarrow \setminus f^{-1}[V] = f^{-1}[\setminus V] \in e^*\theta C(X)$$

$$\Rightarrow f^{-1}[V] \in e^*\theta O(X).$$

(2) \Rightarrow (3) : Let $x \in X$ and $V \in RC(Y, f(x))$.

$$(x \in X)(V \in RC(Y, f(x))) \left. \right\} \Rightarrow (U := f^{-1}[V] \in e^*\theta O(X, x))(f[U] \subseteq V).$$

(2)

(3) \Rightarrow (4) : Let $x \in X$ and $V \in SO(Y, f(x))$.

$$V \in SO(Y, f(x)) \Rightarrow cl(int(V)) \in RC(Y, f(x)) \left. \vphantom{V \in SO(Y, f(x))} \right\} \Rightarrow$$

(3)

$$\Rightarrow (\exists U \in e^*\theta O(X, x))(f[U] \subseteq cl(int(V)) \subseteq cl(V)).$$

$$(4) \Rightarrow (5) : \text{Let } A \subseteq X \text{ and } x \notin f^{-1}[rker(f[A])].$$

$$x \notin f^{-1}[rker(f[A])] \Rightarrow f(x) \notin rker(f[A]) \Rightarrow (\exists F \in RC(Y, f(x)))(F \cap f[A] = \emptyset)$$

$$\Rightarrow (\exists F \in SO(Y, f(x)))(f^{-1}[F] \cap A = \emptyset) \left. \vphantom{(\exists F \in SO(Y, f(x)))(f^{-1}[F] \cap A = \emptyset)} \right\} \Rightarrow$$

(4)

$$\Rightarrow (\exists U \in e^*\theta O(X, x))(f[U] \subseteq cl(F) = F)(f^{-1}[F] \cap A = \emptyset)$$

$$\Rightarrow (\exists U \in e^*\theta O(X, x))(U \subseteq f^{-1}[F])(f^{-1}[F] \cap A = \emptyset)$$

$$\Rightarrow (\exists U \in e^*\theta O(X, x))(U \cap A = \emptyset)$$

$$\Rightarrow x \notin e^*-cl_\theta(A).$$

$$(5) \Rightarrow (6) : \text{Let } B \subseteq Y.$$

$$B \subseteq Y \Rightarrow f^{-1}[B] \subseteq X \left. \vphantom{B \subseteq Y} \right\} \Rightarrow f[e^*-cl_\theta(f^{-1}[B])] \subseteq rker(f[f^{-1}[B]]) \subseteq rker(B)$$

(5)

$$\Rightarrow e^*-cl_\theta(f^{-1}[B]) \subseteq f^{-1}[rker(B)].$$

$$(6) \Rightarrow (7) : \text{Let } V \in e^*O(Y).$$

$$V \in e^*O(Y) \stackrel{\text{Lemma 2.7}}{\Rightarrow} \setminus cl_\delta(V) \in RO(Y) \left. \vphantom{V \in e^*O(Y)} \right\} \Rightarrow$$

(6)

$$\Rightarrow e^*-cl_\theta(f^{-1}[\setminus cl_\delta(V)]) \subseteq f^{-1}[rker(\setminus cl_\delta(V))] = f^{-1}[\setminus cl_\delta(V)]$$

$$\Rightarrow \setminus e^*-int_\theta(f^{-1}[cl_\delta(V)]) \subseteq \setminus f^{-1}[cl_\delta(V)]$$

$$\Rightarrow f^{-1}[cl_\delta(V)] \subseteq e^*-int_\theta(f^{-1}[cl_\delta(V)])$$

$$\Rightarrow f^{-1}[cl_\delta(V)] \in e^*\theta O(X).$$

$$(7) \Rightarrow (8) : \text{This is obvious since every } \delta\text{-semiopen set is } e^*\text{-open.}$$

$$(8) \Rightarrow (9) : \text{Let } V \in \delta PO(Y).$$

$$V \in \delta PO(Y) \Rightarrow int_\delta(\setminus V) \in \delta SO(Y) \left. \vphantom{V \in \delta PO(Y)} \right\} \Rightarrow f^{-1}[cl_\delta(int_\delta(\setminus V))] \in e^*\theta O(X)$$

(8)

$$\Rightarrow \setminus f^{-1} [int_{\delta}(cl_{\delta}(V))] \in e^*\theta O(X)$$

$$\Rightarrow f^{-1} [int(cl_{\delta}(V))] \in e^*\theta C(X).$$

(9) \Rightarrow (10) : This is obvious since every open set is δ -preopen.

(10) \Rightarrow (11) : Clear.

(11) \Rightarrow (1) : Let $V \in RO(Y)$.

$$V \in RO(Y) \Rightarrow (V = int(cl_{\delta}(V))) (\setminus V \in C(Y)) \left. \vphantom{V \in RO(Y)} \right\} \Rightarrow$$

$$(11) \left. \vphantom{V \in RO(Y)} \right\}$$

$$\Rightarrow f^{-1}[\setminus V] = \setminus f^{-1}[V] = \setminus f^{-1} [int(cl_{\delta}(V))] = f^{-1} [cl(int_{\delta}(\setminus V))] \in e^*\theta O(X)$$

$$\Rightarrow f^{-1}[V] \in e^*\theta C(X). \quad \square$$

Lemma 3.1. *For a subset A of a topological space X , the following properties hold:*

(1) *If $A \in e^*O(X)$, then $a-cl(A) = cl_{\delta}(A)$,*

(2) *If $A \in \delta SO(X)$, then $\delta-pcl(A) = cl_{\delta}(A)$,*

(3) *If $A \in \delta PO(X)$, then $\delta-scl(A) = int(cl_{\delta}(A))$,*

(4) *If $A \in PO(X)$, then $scl(A) = int(cl(A))$.*

Proof. (1) Let $A \in e^*O(X)$.

$$A \in e^*O(X) \Rightarrow A \subseteq cl(int(cl_{\delta}(A)))$$

$$\Rightarrow cl_{\delta}(A) \subseteq cl_{\delta}(cl(int(cl_{\delta}(A)))) = cl(int(cl_{\delta}(A)))$$

$$\Rightarrow A \cup cl_{\delta}(A) = cl_{\delta}(A) \subseteq A \cup cl(int(cl_{\delta}(A))) = a-cl(A) \dots (*)$$

$$\delta C(X) \subseteq aC(X) \Rightarrow a-cl(A) \subseteq cl_{\delta}(A) \dots (**)$$

$$(*), (**) \Rightarrow a-cl(A) = cl_{\delta}(A).$$

(2) Let $A \in \delta SO(X)$.

$$A \in \delta SO(X) \Rightarrow A \subseteq cl(int_{\delta}(A)) \stackrel{\text{Lemma 2.6}}{=} cl_{\delta}(int_{\delta}(A))$$

$$\Rightarrow cl_{\delta}(A) \subseteq cl_{\delta}(cl_{\delta}(int_{\delta}(A))) = cl_{\delta}(int_{\delta}(A)) = cl(int_{\delta}(A)) \left. \vphantom{\Rightarrow} \right\} \Rightarrow$$

$$\delta-pcl(A) = A \cup cl(int_{\delta}(A)) \left. \vphantom{\Rightarrow} \right\}$$

$$\Rightarrow \delta-pcl(A) \supseteq A \cup cl_{\delta}(A) = cl_{\delta}(A) \left. \vphantom{\Rightarrow} \right\} \Rightarrow \delta-pcl(A) = cl_{\delta}(A).$$

$$\delta C(X) \subseteq \delta PC(X) \Rightarrow \delta-pcl(A) \subseteq cl_{\delta}(A)$$

(3) Let $A \in \delta PO(X)$.

$$\left. \begin{array}{l} A \in \delta PO(X) \Rightarrow A \subseteq \text{int}(cl_\delta(A)) \\ \delta\text{-scl}(A) = A \cup \text{int}(cl_\delta(A)) \end{array} \right\} \Rightarrow \delta\text{-scl}(A) = \text{int}(cl_\delta(A)).$$

(4) [20]. □

Corollary 3.1. For a function $f : X \rightarrow Y$, the following properties are equivalent:

- (1) f is almost contra $e^*\theta$ -continuous;
- (2) $f^{-1}[a\text{-cl}(A)]$ is e^* - θ -open for every $A \in e^*O(Y)$;
- (3) $f^{-1}[\delta\text{-pcl}(A)]$ is e^* - θ -open for every $A \in \delta SO(Y)$;
- (4) $f^{-1}[\delta\text{-scl}(A)]$ is e^* - θ -closed for every $A \in \delta PO(Y)$.

Proof. It follows from Lemma 3.1. □

Theorem 3.2. For a function $f : X \rightarrow Y$, the following properties are equivalent:

- (1) f is almost contra $e^*\theta$ -continuous;
- (2) $f^{-1}[V]$ is e^* - θ -open in X for each θ -semiopen set of Y ;
- (3) $f^{-1}[V]$ is e^* - θ -closed in X for each θ -semiclosed set of Y ;
- (4) $f^{-1}[V] \subseteq e^*\text{-int}_\theta(f^{-1}[cl(V)])$ for every $V \in SO(Y)$;
- (5) $f[e^*\text{-cl}_\theta(A)] \subseteq \theta\text{-scl}(f[A])$ for every subset A of X ;
- (6) $e^*\text{-cl}_\theta(f^{-1}[B]) \subseteq f^{-1}[\theta\text{-scl}(B)]$ for every subset B of Y ;
- (7) $e^*\text{-cl}_\theta(f^{-1}[V]) \subseteq f^{-1}[\theta\text{-scl}(V)]$ for every open subset V of Y ;
- (8) $e^*\text{-cl}_\theta(f^{-1}[V]) \subseteq f^{-1}[scl(V)]$ for every open subset V of Y ;
- (9) $e^*\text{-cl}_\theta(f^{-1}[V]) \subseteq f^{-1}[\text{int}(cl(V))]$ for every open subset V of Y .

Proof. (1) \Rightarrow (2) : Let $V \in \theta SO(Y)$.

$$V \in \theta SO(Y) \Rightarrow (\exists \mathcal{A} \subseteq RC(Y))(V = \cup \mathcal{A}) \left. \vphantom{V \in \theta SO(Y)} \right\} \Rightarrow$$

$$(1) \left. \vphantom{V \in \theta SO(Y)} \right\} \Rightarrow$$

$$\Rightarrow f^{-1}[V] = \cup \{f^{-1}[A] \mid A \in \mathcal{A}\} \in e^*\theta O(X).$$

(2) \Rightarrow (3) : Obvious.

(3) \Rightarrow (4) : Let $V \in SO(Y)$.

$$V \in SO(Y) \Rightarrow \left. \begin{array}{l} \backslash cl(V) \in \theta SC(Y) \\ (3) \end{array} \right\} \Rightarrow$$

$$\Rightarrow f^{-1}[\backslash cl(V)] \in e^*\theta C(X) \Rightarrow \backslash f^{-1}[cl(V)] \in e^*\theta C(X)$$

$$\Rightarrow f^{-1}[cl(V)] \in e^*\theta O(X) \Rightarrow f^{-1}[V] \subseteq f^{-1}[cl(V)] = e^*-int_\theta(f^{-1}[cl(V)]).$$

$$(4) \Rightarrow (5) : \text{Let } A \subseteq X \text{ and } x \notin f^{-1}[\theta-scl(f[A])].$$

$$x \notin f^{-1}[\theta-scl(f[A])] \Rightarrow f(x) \notin \theta-scl(f[A]) \Rightarrow (\exists U \in SO(Y, f(x)))(cl(U) \cap f[A] = \emptyset)$$

$$\Rightarrow (\exists U \in SO(Y, f(x)))(f^{-1}[cl(U)] \cap A = \emptyset)$$

$$\Rightarrow (\exists U \in SO(Y, f(x)))(e^*-int_\theta(f^{-1}[cl(U)]) \cap A = \emptyset) \left. \vphantom{(\exists U \in SO(Y, f(x)))(e^*-int_\theta(f^{-1}[cl(U)]) \cap A = \emptyset)} \right\} \stackrel{(4)}{\Rightarrow} \\ V := e^*-int_\theta(f^{-1}[cl(U)])$$

$$\Rightarrow (\exists V \in e^*\theta O(X, x))(V \cap A = \emptyset)$$

$$\Rightarrow x \notin e^*-cl_\theta(A).$$

$$(5) \Rightarrow (6) : \text{Let } B \subseteq Y.$$

$$B \subseteq Y \Rightarrow \left. \begin{array}{l} f^{-1}[B] \subseteq X \\ (5) \end{array} \right\} \Rightarrow f[e^*-cl_\theta(f^{-1}[B])] \subseteq \theta-scl(f[f^{-1}[B]]) \subseteq \theta-scl(B)$$

$$\Rightarrow e^*-cl_\theta(f^{-1}[B]) \subseteq f^{-1}[\theta-scl(B)].$$

$$(6) \Rightarrow (7) : \text{Obvious.}$$

$$(7) \Rightarrow (8) : \text{This is obvious since } \theta-scl(V) = scl(V) \text{ for an open set } V.$$

$$(8) \Rightarrow (9) : \text{Obvious from Lemma 3.1(4).}$$

$$(9) \Rightarrow (1) : \text{Let } V \in RO(Y).$$

$$V \in RO(Y) \subseteq O(Y) \left. \vphantom{V \in RO(Y) \subseteq O(Y)} \right\} \Rightarrow e^*-cl_\theta(f^{-1}[V]) \subseteq f^{-1}[in(cl(V))] = f^{-1}[V]$$

$$\Rightarrow f^{-1}[V] \in e^*\theta C(X). \quad \square$$

We recall that a topological space X is said to be extremally disconnected if the closure of every open set of X is open in X .

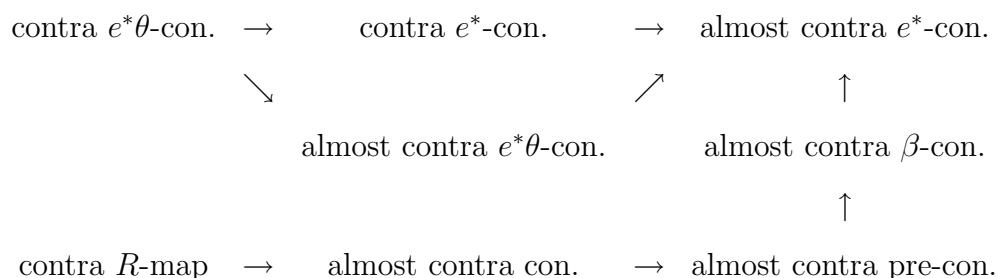
Lemma 3.2. *Let X be a topological space. If X is an extremally disconnected space, then $RO(X) = RC(X)$.*

Theorem 3.3. *Let $f : X \rightarrow Y$ be a function. If Y is extremally disconnected, then the following properties are equivalent:*

- (1) f is almost contra $e^*\theta$ -continuous;
- (2) f is almost $e^*\theta$ -continuous.

Proof. The proof is obvious from Lemma 3.2. □

Remark 1. From Definitions 2.2 and 3.1, we have the following diagram:



Example 3.1. *Let $X := \{a, b, c, d\}$ and $\tau := \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$. It is not difficult to see $e^*\theta O(X) = e^*O(X) = 2^X \setminus \{\{c\}, \{d\}, \{c, d\}\}$. Then the identity function $f : (X, \tau) \rightarrow (X, \tau)$ is almost contra $e^*\theta$ -continuous and so almost contra e^* -continuous but f is neither contra $e^*\theta$ -continuous nor contra e^* -continuous.*

Example 3.2. *Let $X := \{a, b, c, d\}$ and $\tau := \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, b, d\}\}$. It is not difficult to see $e^*\theta O(X) = e^*O(X) = 2^X \setminus \{\{d\}\}$ and $\beta O(X) = 2^X \setminus \{\{c\}, \{d\}, \{b, c\}, \{c, d\}, \{b, c, d\}\}$. Define the function $f : (X, \tau) \rightarrow (X, \tau)$ by $f = \{(a, b), (b, a), (c, c), (d, d)\}$. Then f is almost contra $e^*\theta$ -continuous but it is not almost contra β -continuous.*

Theorem 3.4. *If $f : X \rightarrow Y$ is an almost contra $e^*\theta$ -continuous function which satisfies the property $e^*\text{-int}_\theta(f^{-1}[cl_\delta(V)]) \subseteq f^{-1}[V]$ for each open set V of Y , then f is $e^*\theta$ -continuous.*

Proof. Let $V \in O(Y)$.

$$\left. \begin{array}{l} V \in O(Y) \\ f \text{ is a.c.}e^*\theta.c. \end{array} \right\} \xRightarrow{\text{Theorem 3.1(7)}}$$

$$\Rightarrow f^{-1}[V] \subseteq f^{-1}[cl_\delta(V)] = e^*int_\theta(e^*int_\theta(f^{-1}[cl(V)])) \subseteq e^*int_\theta(f^{-1}[V]) \subseteq f^{-1}[V]$$

$$\Rightarrow f^{-1}[V] = e^*int_\theta(f^{-1}[V])$$

$$\Rightarrow f^{-1}[V] \in e^*\theta O(X). \quad \square$$

We recall that a topological space is said to be P_Σ [29] if for any open set V of X and each $x \in V$, there exists a regular closed set F of X containing x such that $x \in F \subseteq V$.

Theorem 3.5. *If $f : X \rightarrow Y$ is an almost contra $e^*\theta$ -continuous function and Y is P_Σ , then f is $e^*\theta$ -continuous.*

Proof. Let $V \in O(Y)$.

$$\left. \begin{array}{l} y \in V \in O(Y) \xrightarrow{Y \text{ is } P_\Sigma} (\exists F \in RC(Y, y))(F \subseteq V) \\ \mathcal{A} := \{F | y \in V \Rightarrow (\exists F \in RC(Y, y))(F \subseteq V)\} \end{array} \right\} \Rightarrow \left. \begin{array}{l} \cup \mathcal{A} = V \\ f \text{ is a.c.}e^*\theta.c. \end{array} \right\} \Rightarrow$$

$$\Rightarrow f^{-1}[V] = \bigcup_{F \in \mathcal{A}} f^{-1}[F] \in e^*\theta O(X). \quad \square$$

Definition 3.2. A function $f : X \rightarrow Y$ is said to be:

- a) R -map [6] if $f^{-1}[A]$ is regular closed in X for every regular closed A of Y ,
- b) weakly e^* -irresolute [22] if $f^{-1}[A]$ is $e^*\theta$ -open in X for every $e^*\theta$ -open set A of Y ,
- c) pre- $e^*\theta$ -closed if $f[A]$ is $e^*\theta$ -closed in Y for every $e^*\theta$ -closed A of X .

Theorem 3.6. *Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be two functions. Then the following properties hold:*

- (1) *If f is almost contra $e^*\theta$ -continuous and g is an R -map, then $g \circ f : X \rightarrow Z$ is almost contra $e^*\theta$ -continuous,*
- (2) *If f is almost $e^*\theta$ -continuous and g is a contra R -map, then $g \circ f : X \rightarrow Z$ is*

almost contra $e^*\theta$ -continuous,

(3) If f is weakly e^* -irresolute and g is almost contra $e^*\theta$ -continuous, then $g \circ f : X \rightarrow Z$ is almost contra $e^*\theta$ -continuous.

Proof. Routine. □

Theorem 3.7. If $f : X \rightarrow Y$ is a pre- $e^*\theta$ -closed surjection and $g : Y \rightarrow Z$ is a function such that $g \circ f : X \rightarrow Z$ is almost contra $e^*\theta$ -continuous, then g is almost contra $e^*\theta$ -continuous.

Proof. Let $V \in RO(Z)$.

$$\left. \begin{array}{l} V \in RO(Z) \\ g \circ f \text{ is a.c.}e^*\theta\text{.c.} \end{array} \right\} \Rightarrow \left. \begin{array}{l} (g \circ f)^{-1}[V] = f^{-1}[g^{-1}[V]] \in e^*\theta C(X) \\ f \text{ is pre-}e^*\theta\text{-closed surjection} \end{array} \right\} \Rightarrow \\ \Rightarrow f[f^{-1}[g^{-1}[V]]] = g^{-1}[V] \in e^*\theta C(Y). \quad \square$$

Theorem 3.8. Let $\{X_\alpha | \alpha \in \Lambda\}$ be any family of topological spaces. If $f : X \rightarrow \prod X_\alpha$ is an almost contra $e^*\theta$ -continuous function, then $Pr_\alpha \circ f : X \rightarrow X_\alpha$ is almost contra $e^*\theta$ -continuous for each $\alpha \in \Lambda$ where Pr_α is the projection of $\prod X_\alpha$ onto X_α .

Proof. Let $\alpha \in \Lambda$ and $U_\alpha \in RO(X_\alpha)$.

$$\left. \begin{array}{l} \alpha \in \Lambda \Rightarrow Pr_\alpha \text{ is open and continuous} \Rightarrow Pr_\alpha \text{ is } R\text{-map} \\ U_\alpha \in RO(X_\alpha) \end{array} \right\} \Rightarrow \\ \left. \begin{array}{l} \Rightarrow Pr_\alpha^{-1}[U_\alpha] \in RO(\prod X_\alpha) \\ f \text{ is a.c.}e^*\theta\text{.c.} \end{array} \right\} \Rightarrow (Pr_\alpha \circ f)^{-1}[U_\alpha] = f^{-1}[Pr_\alpha^{-1}[U_\alpha]] \in e^*\theta C(X). \quad \square$$

Definition 3.3. A function $f : X \rightarrow Y$ is called weakly $e^*\theta$ -continuous (briefly $w.e^*\theta.c.$) if for each $x \in X$ and each open set V of Y containing $f(x)$, there exists a $U \in e^*\theta O(X, x)$ such that $f[U] \subseteq cl(V)$.

Theorem 3.9. Let $f : X \rightarrow Y$ be a function. Then the following properties hold:

(1) If f is almost contra $e^*\theta$ -continuous, then it is weakly $e^*\theta$ -continuous,

(2) If f is weakly $e^*\theta$ -continuous and Y is extremally disconnected, then f is almost contra $e^*\theta$ -continuous.

Proof. (1) Let $x \in X$ and $V \in O(Y, f(x))$.

$$\left. \begin{aligned} (x \in X)(V \in O(Y, f(x))) \Rightarrow cl(V) \in RC(Y, f(x)) \\ f \text{ is a.c.}e^*\theta.c. \end{aligned} \right\} \Rightarrow$$

$$\left. \begin{aligned} \Rightarrow f^{-1}[cl(V)] \in e^*\theta O(X, x) \\ U := f^{-1}[cl(V)] \end{aligned} \right\} \Rightarrow (U \in e^*\theta O(X, x))(f[U] \subseteq cl(V)).$$

(2) Let $V \in RC(Y)$ and $x \in f^{-1}[V]$.

$$\left. \begin{aligned} (V \in RC(Y))(x \in f^{-1}[V]) \Rightarrow (V \in RC(Y, f(x)))(cl(V) = V) \\ Y \text{ is extremally disconnected} \end{aligned} \right\} \Rightarrow$$

$$\left. \begin{aligned} \Rightarrow cl(V) \in RO(Y, f(x)) \\ f \text{ is w.}e^*\theta.c. \end{aligned} \right\} \Rightarrow (\exists U \in e^*\theta O(X, x))(f[U] \subseteq cl(V) = V)$$

$$\Rightarrow (\exists U \in e^*\theta O(X, x))(U \subseteq f^{-1}[V])$$

$$\Rightarrow f^{-1}[V] \in e^*\theta O(X). \quad \square$$

4. SOME FUNDAMENTAL PROPERTIES

Definition 4.1. A topological space X is said to be:

- a) $e^*\theta$ - T_0 if for any distinct pair of points x and y in X , there is an $e^*\theta$ -open set U in X containing x but not y or an $e^*\theta$ -open set V in X containing y but not x ,
- b) $e^*\theta$ - T_1 if for any distinct pair of points x and y in X , there is an $e^*\theta$ -open set U in X containing x but not y and an $e^*\theta$ -open set V in X containing y but not x ,
- c) $e^*\theta$ - T_2 (resp. e^* - T_2 [13, 14]) if for every pair of distinct points x and y , there exist two $e^*\theta$ -open (resp. e^* -open) sets U and V such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

Theorem 4.1. For a topological space X , the following properties are equivalent:

- (1) (X, τ) is $e^*\theta$ - T_0 ;
- (2) (X, τ) is $e^*\theta$ - T_1 ;
- (3) (X, τ) is $e^*\theta$ - T_2 ;

(4) (X, τ) is e^*-T_2 ;

(5) For every pair of distinct points $x, y \in X$, there exist $U \in e^*O(X, x)$ and $V \in e^*O(X, y)$ such that $e^*-cl(U) \cap e^*-cl(V) = \emptyset$;

(6) For every pair of distinct points $x, y \in X$, there exist $U \in e^*R(X, x)$ and $V \in e^*R(X, y)$ such that $U \cap V = \emptyset$;

(7) For every pair of distinct points $x, y \in X$, there exist $U \in e^*\theta O(X, x)$ and $V \in e^*\theta O(X, y)$ such that $e^*-cl_\theta(U) \cap e^*-cl_\theta(V) = \emptyset$.

Proof. (3) \Rightarrow (2) : Obvious.

(2) \Rightarrow (1) : Obvious.

(1) \Rightarrow (3) : Let $x, y \in X$ and $x \neq y$.

$$(x, y \in X)(x \neq y) \left. \vphantom{(x, y \in X)(x \neq y)} \right\} \Rightarrow (\exists W \in e^*\theta O(X, x))(y \notin W)$$

(1)

$$\stackrel{\text{Lemma 2.4}}{\Rightarrow} (\exists U \in e^*R(X, x))(U = e^*-cl_\theta(U) \subseteq W) \left. \vphantom{(\exists U \in e^*R(X, x))(U = e^*-cl_\theta(U) \subseteq W)} \right\} \Rightarrow$$

$$V := \setminus U = \setminus e^*cl_\theta(U)$$

$$\Rightarrow (U \in e^*\theta O(X, x))(V \in e^*\theta O(X, y))(U \cap V = \emptyset).$$

(3) \Rightarrow (4) : The proof is obvious since $e^*\theta O(X) \subseteq e^*O(X)$.

(4) \Rightarrow (5) : Let $x, y \in X$ and $x \neq y$.

$$(x, y \in X)(x \neq y) \left. \vphantom{(x, y \in X)(x \neq y)} \right\} \Rightarrow (\exists U \in e^*O(X, x))(\exists V \in e^*O(X, y))(U \cap V = \emptyset)$$

X is e^*-T_2

$$\Rightarrow (\exists U \in e^*O(X, x))(\exists V \in e^*O(X, y))(U \subseteq \setminus V)$$

$$\Rightarrow (\exists U \in e^*O(X, x))(\exists V \in e^*O(X, y))(e^*-cl(U) \subseteq \setminus V)$$

$$\Rightarrow (\exists U \in e^*O(X, x))(\exists V \in e^*O(X, y))(e^*-int(e^*-cl(U)) = e^*-cl(U) \subseteq e^*-int(\setminus V))$$

$$\Rightarrow (\exists U \in e^*O(X, x))(\exists V \in e^*O(X, y))(e^*-cl(U) \subseteq e^*-int(\setminus V) = \setminus e^*-cl(V))$$

$$\Rightarrow (\exists U \in e^*O(X, x))(\exists V \in e^*O(X, y))(e^*-int(U) \cap e^*-cl(V) = \emptyset).$$

(5) \Rightarrow (6) : Let $x, y \in X$ and $x \neq y$.

$$\begin{aligned}
 & \left. \begin{array}{l} (x, y \in X)(x \neq y) \\ (5) \end{array} \right\} \Rightarrow \\
 & \Rightarrow \left. \begin{array}{l} (\exists U_1 \in e^*O(X, x))(\exists V_1 \in e^*O(X, y))(e^*cl(U_1) \cap e^*cl(V_1) = \emptyset) \\ (U_2 := e^*cl(U_1))(V_2 := e^*cl(V_1)) \end{array} \right\} \Rightarrow \\
 & \Rightarrow (\exists U_2 \in e^*R(X, x))(\exists V_2 \in e^*R(X, y))(U_2 \cap V_2 = \emptyset).
 \end{aligned}$$

(6) \Rightarrow (7) : Let $x, y \in X$ and $x \neq y$.

$$\begin{aligned}
 & \left. \begin{array}{l} (x, y \in X)(x \neq y) \\ (6) \end{array} \right\} \Rightarrow (\exists U \in e^*R(X, x))(\exists V \in e^*R(X, y))(U \cap V = \emptyset) \\
 & \Rightarrow (\exists U \in e^*\theta O(X, x))(\exists V \in e^*\theta O(X, y))(e^*cl_\theta(U) \cap e^*cl_\theta(V) = \emptyset).
 \end{aligned}$$

(7) \Rightarrow (3) : Obvious. □

Definition 4.2. A topological space X is said to be:

- a) weakly Hausdorff [27] (briefly weakly- T_2) if every point of X is an intersection of regular closed sets of X ,
- b) s -Urysohn [2] if for each pair of distinct points x and y in X , there exist $U \in SO(X, x)$ and $V \in SO(X, y)$ such that $cl(U) \cap cl(V) = \emptyset$.

Theorem 4.2. For a function $f : X \rightarrow Y$, the following properties hold:

- (1) If f is an almost contra $e^*\theta$ -continuous injection of a topological space X into a s -Urysohn space Y , then X is $e^*\theta$ - T_2 ,
- (2) If f is an almost contra $e^*\theta$ -continuous injection of a topological space X into a weakly Hausdorff space Y , then X is $e^*\theta$ - T_1 .

Proof. (1) Let $x, y \in X$ and $x \neq y$.

$$\begin{aligned}
 & \left. \begin{array}{l} (x, y \in X)(x \neq y) \\ f \text{ is injective} \end{array} \right\} \Rightarrow \left. \begin{array}{l} f(x) \neq f(y) \\ Y \text{ is } s\text{-Urysohn} \end{array} \right\} \Rightarrow \\
 & \Rightarrow \left. \begin{array}{l} (\exists V_1 \in SO(Y, f(x)))(\exists V_2 \in SO(Y, f(y)))(cl(V_1) \cap cl(V_2) = \emptyset) \\ f \text{ is a.c.}e^*\theta.c. \end{array} \right\} \xrightarrow{\text{Theorem 3.1(4)}}
 \end{aligned}$$

$$\begin{aligned}
&\Rightarrow (\exists U_1 \in e^*\theta O(X, x))(\exists U_2 \in e^*\theta O(X, y))(f[U_1] \cap f[U_2] \subseteq cl(V_1) \cap cl(V_2) = \emptyset) \\
&\Rightarrow (\exists U_1 \in e^*\theta O(X, x))(\exists U_2 \in e^*\theta O(X, y))(f[U_1 \cap U_2] = f[U_1] \cap f[U_2] = \emptyset) \\
&\Rightarrow (\exists U_1 \in e^*\theta O(X, x))(\exists U_2 \in e^*\theta O(X, y))(U_1 \cap U_2 = \emptyset). \\
(2) \text{ Let } x, y \in X \text{ and } x \neq y.
\end{aligned}$$

$$\begin{aligned}
&\left. \begin{array}{l} (x, y \in X)(x \neq y) \\ f \text{ is injective} \end{array} \right\} \Rightarrow \left. \begin{array}{l} f(x) \neq f(y) \\ Y \text{ is weakly-}T_2 \end{array} \right\} \Rightarrow \\
&\Rightarrow (\exists V_1 \in RC(Y, f(x)))(\exists V_2 \in RC(Y, f(y)))(f(x) \notin V_2)(f(y) \notin V_1) \left. \vphantom{\begin{array}{l} \exists V_1 \in RC(Y, f(x)) \\ \exists V_2 \in RC(Y, f(y)) \end{array}} \right\} \begin{array}{l} \text{Theorem 3.1(3)} \\ \Rightarrow \\ f \text{ is a.c.}e^*\theta\text{.c.} \end{array} \\
&\Rightarrow (\exists U_1 \in e^*\theta O(X, x))(\exists U_2 \in e^*\theta O(X, y))(f[U_1] \subseteq V_1)(f[U_2] \subseteq V_2)(f(x) \notin V_2)(f(y) \notin V_1) \\
&\Rightarrow (\exists U_1 \in e^*\theta O(X, x))(\exists U_2 \in e^*\theta O(X, y))(x \notin U_2)(y \notin U_1). \quad \square
\end{aligned}$$

Remark 2. [15] The intersection of two $e^*\theta$ -open sets is not necessarily $e^*\theta$ -open as shown in the following example.

Example 4.1. [15] Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$. Although the subsets $\{b, c, d\}$ and $\{a, c, d\}$ are $e^*\theta$ -open in X , the set $\{c, d\}$ which is the intersection of these sets is not $e^*\theta$ -open in X .

Definition 4.3. A topological space X is called an $e^*\theta c$ -space if the intersection of any two $e^*\theta$ -open sets is an $e^*\theta$ -open set.

Theorem 4.3. If $f, g : X \rightarrow Y$ are almost contra $e^*\theta$ -continuous functions, X is an $e^*\theta c$ -space and Y is s -Urysohn, then $E = \{x \in X \mid f(x) = g(x)\}$ is $e^*\theta$ -closed in X .

Proof. Let $x \notin E$.

$$\begin{aligned}
&\left. \begin{array}{l} x \notin E \Rightarrow f(x) \neq g(x) \\ Y \text{ is } s\text{-Urysohn} \end{array} \right\} \Rightarrow \\
&\Rightarrow (\exists V_1 \in SO(Y, f(x)))(\exists V_2 \in SO(Y, g(x)))(cl(V_1) \cap cl(V_2) = \emptyset) \left. \vphantom{\begin{array}{l} \exists V_1 \in SO(Y, f(x)) \\ \exists V_2 \in SO(Y, g(x)) \end{array}} \right\} \Rightarrow \\
&\quad f \text{ and } g \text{ are a.c.}e^*\theta\text{.c.}
\end{aligned}$$

$$\begin{aligned} & \Rightarrow (\exists U_1 \in e^*\theta O(X, x))(\exists U_2 \in e^*\theta O(X, x))(f[U_1] \cap g[U_2] \subseteq cl(V_1) \cap cl(V_2) = \emptyset) \\ & \left. \begin{array}{l} \\ \\ \\ \end{array} \right\} \Rightarrow \\ & \qquad \qquad \qquad X \text{ is } e^*\theta c\text{-space} \\ & \Rightarrow (\exists U := U_1 \cap U_2 \in e^*\theta O(X, x))(f[U] \cap g[U] \subseteq f[U_1] \cap g[U_2] = \emptyset) \\ & \Rightarrow (\exists U \in e^*\theta O(X, x))(U \cap E = \emptyset) \\ & \Rightarrow x \notin e^*cl_\theta(E). \qquad \qquad \qquad \square \end{aligned}$$

We say that the product space $X = X_1 \times \dots \times X_n$ has Property $P_{e^*\theta}$ if A_i is an $e^*\theta$ -open set in a topological space X_i for $i = 1, 2, \dots, n$, then $A_1 \times \dots \times A_n$ is also $e^*\theta$ -open in the product space $X = X_1 \times \dots \times X_n$.

Theorem 4.4. *Let $f : X_1 \rightarrow Y$ and $g : X_2 \rightarrow Y$ be two functions, where*

- (i) $X = X_1 \times X_2$ has the Property $P_{e^*\theta}$,
- (ii) Y is a Urysohn space,
- (iii) f and g are almost contra $e^*\theta$ -continuous,

then $A = \{(x_1, x_2) | f(x_1) = g(x_2)\}$ is $e^*\theta$ -closed in the product space $X = X_1 \times X_2$.

Proof. Let $(x_1, x_2) \notin A$.

$$\begin{aligned} & (x_1, x_2) \notin A \Rightarrow f(x_1) \neq g(x_2) \left. \begin{array}{l} \\ \\ \end{array} \right\} \Rightarrow \\ & \qquad \qquad \qquad Y \text{ is Urysohn} \\ & \Rightarrow (\exists V_1 \in O(Y, f(x_1)))(\exists V_2 \in O(Y, g(x_2)))(cl(V_1) \cap cl(V_2) = \emptyset)(cl(V_1), cl(V_2) \in RC(Y)) \left. \begin{array}{l} \\ \\ \end{array} \right\} \Rightarrow \\ & \qquad \qquad \qquad f \text{ and } g \text{ are a.c. } e^*\theta\text{.c.} \\ & \Rightarrow (f^{-1}[cl(V_1)] \in e^*\theta O(X_1, x_1))(g^{-1}[cl(V_2)] \in e^*\theta O(X_2, x_2)) \left. \begin{array}{l} \\ \\ \end{array} \right\} \Rightarrow \\ & \qquad \qquad \qquad X = X_1 \times X_2 \text{ has the Property } P_{e^*\theta} \\ & \Rightarrow ((x_1, x_2) \in f^{-1}[cl(V_1)] \times g^{-1}[cl(V_2)] \in e^*\theta O(X))(f^{-1}[cl(V_1)] \times g^{-1}[cl(V_2)] \subseteq \setminus A) \\ & \Rightarrow \setminus A \in e^*\theta O(X_1 \times X_2) \\ & \Rightarrow A \in e^*\theta C(X_1 \times X_2). \qquad \qquad \qquad \square \end{aligned}$$

Theorem 4.5. *Let $f : X \rightarrow Y$ be a function and $g : X \rightarrow X \times Y$ the graph function, given by $g(x) = (x, f(x))$ for every $x \in X$. If g is almost contra $e^*\theta$ -continuous, then f is almost contra $e^*\theta$ -continuous.*

Proof. Let $V \in RO(Y)$.

$$\left. \begin{array}{l} V \in RO(Y) \Rightarrow X \times V \in RO(X \times Y) \\ g \text{ is a.c.}e^*\theta\text{.c.} \end{array} \right\} \Rightarrow f^{-1}[V] = g^{-1}[X \times V] \in e^*\theta C(X). \quad \square$$

We recall that for a function $f : X \rightarrow Y$, the subset $\{(x, f(x)) | x \in X\}$ of $X \times Y$ is called the graph of f and is denoted by $G(f)$.

Definition 4.4. A function $f : X \rightarrow Y$ has an $e^*\theta$ -closed graph if for each $(x, y) \notin G(f)$, there exist $U \in e^*\theta O(X, x)$ and $V \in O(Y, y)$ such that $(U \times V) \cap G(f) = \emptyset$.

Lemma 4.1. *The graph $G(f)$ of a function $f : X \rightarrow Y$ is $e^*\theta$ -closed if and only if for each $(x, y) \notin G(f)$, there exist $U \in e^*\theta O(X, x)$ and $V \in O(Y, y)$ such that $f[U] \cap V = \emptyset$.*

Proof. Straightforward. □

Theorem 4.6. *Let X and Y be two topological spaces. If $f : X \rightarrow Y$ is a function with an $e^*\theta$ -closed graph, then $\{f(x)\} = \cap\{cl(f[U]) | U \in e^*\theta O(X, x)\}$ for each x in X .*

Proof. Let $G(f)$ be $e^*\theta$ -closed. Suppose that there exists a point of x in X such that $\{f(x)\} \neq \cap\{cl(f[U]) | U \in e^*\theta O(X, x)\}$.

$$\begin{aligned} \{f(x)\} \neq \cap\{cl(f[U]) | U \in e^*\theta O(X, x)\} &\Rightarrow (\exists y \in \cap\{cl(f[U]) | U \in e^*\theta O(X, x)\})(y \neq f(x)) \\ &\Rightarrow (\forall U \in e^*\theta O(X, x))(y \in cl(f[U]))((x, y) \notin G(f)) \left. \vphantom{\Rightarrow} \right\} \Rightarrow \\ &\quad G(f) \text{ is } e^*\theta\text{-closed} \\ &\Rightarrow (\exists V \in O(Y, y))(y \in cl(f[U]))(\emptyset = f[U] \cap V = cl(f[U]) \cap V \neq \emptyset) \end{aligned}$$

This is a contradiction. □

Theorem 4.7. *If $f : X \rightarrow Y$ is almost contra $e^*\theta$ -continuous and Y is Hausdorff, then $G(f)$ is $e^*\theta$ -closed.*

Proof. Let $(x, y) \notin G(f)$.

$$\left. \begin{array}{l} (x, y) \notin G(f) \Rightarrow y \neq f(x) \\ Y \text{ is Hausdorff} \end{array} \right\} \Rightarrow (\exists U \in O(Y, y))(\exists V \in O(Y, f(x)))(U \cap V = \emptyset)$$

$$\Rightarrow (f(x) \notin Y \setminus cl(V))(U \subseteq Y \setminus cl(V) \in RO(Y)) \Rightarrow f(x) \notin rker(U)$$

$$\Rightarrow x \notin f^{-1}[rker(U)] \stackrel{f \text{ is a.c.}, e^*\theta.c.}{\Rightarrow} x \notin e^*cl_\theta(f^{-1}[U]) \left. \vphantom{\Rightarrow} \right\} \Rightarrow$$

$$V := \setminus e^*cl_\theta(f^{-1}[U])$$

$$\Rightarrow (V \in e^*\theta O(X, x))(U \in O(Y, y))(V \times U \subseteq \setminus G(f))$$

$$\Rightarrow (V \in e^*\theta O(X, x))(U \in O(Y, y))((V \times U) \cap G(f) = \emptyset). \quad \square$$

Theorem 4.8. *If $f : X \rightarrow Y$ have an $e^*\theta$ -closed graph and injective, then X is $e^*\theta$ - T_1 .*

Proof. Let $x_1, x_2 \in X$ and $x_1 \neq x_2$.

$$\left. \begin{array}{l} (x_1, x_2 \in X)(x_1 \neq x_2) \\ f \text{ is injective} \end{array} \right\} \Rightarrow f(x_1) \neq f(x_2) \Rightarrow (x_1, f(x_2)) \in (X \times Y) \setminus G(f) \left. \vphantom{\Rightarrow} \right\} \Rightarrow$$

$$G(f) \text{ is } e^*\theta\text{-closed}$$

$$\Rightarrow (\exists U \in e^*\theta O(X, x_1))(\exists V \in O(Y, f(x_2)))(f[U] \cap V = \emptyset)$$

$$\Rightarrow (\exists U \in e^*\theta O(X, x_1))(\exists V \in O(Y, f(x_2)))(U \cap f^{-1}[V] = \emptyset)$$

$$\Rightarrow (\exists U \in e^*\theta O(X, x_1))(x_2 \notin U)$$

Then X is $e^*\theta$ - T_0 . On the other hand, the notions of $e^*\theta$ - T_0 and $e^*\theta$ - T_1 are equivalent from Theorem 4.1. Thus X is $e^*\theta$ - T_1 . □

Theorem 4.9. *If $f : X \rightarrow Y$ has an $e^*\theta$ -closed graph and X is an $e^*\theta c$ -space, then $f^{-1}[K]$ is $e^*\theta$ -closed for every compact subset K of Y .*

Proof. Let K be a compact subset of Y and let $x \notin f^{-1}[K]$.

$$\begin{aligned}
& \left. \begin{aligned} x \notin f^{-1}[K] &\Rightarrow f(x) \notin K \Rightarrow (\forall y \in K)(y \neq f(x)) \Rightarrow (x, y) \in (X \times Y) \setminus G(f) \\ &G(f) \text{ is } e^*\theta\text{-closed} \end{aligned} \right\} \Rightarrow \\
& \left. \begin{aligned} \Rightarrow (\exists U_y \in e^*\theta O(X, x))(\exists V_y \in O(Y, y))(f[U_y] \cap V_y = \emptyset) \\ \mathcal{A} := \{V_y | y \in K\} \end{aligned} \right\} \Rightarrow \\
& \left. \begin{aligned} \Rightarrow (\mathcal{A} \subseteq O(Y))(K \subseteq \cup \mathcal{A}) \\ K \text{ is compact} \end{aligned} \right\} \Rightarrow \left. \begin{aligned} (\exists \mathcal{A}^* \subseteq \mathcal{A})(|\mathcal{A}^*| < \aleph_0)(K \subseteq \cup \mathcal{A}^*) \\ U := \cap \{U_{y_i} | i = 1, 2, \dots, n\} \end{aligned} \right\} \begin{array}{l} X \text{ is } e^*\theta\text{c-space} \\ \xrightarrow{\Rightarrow} \end{array} \\
& \Rightarrow (U \in e^*\theta O(X, x))(f[U] \cap K = \emptyset) \\
& \Rightarrow (U \in e^*\theta O(X, x))(U \cap f^{-1}[K] = \emptyset) \\
& \Rightarrow (U \in e^*\theta O(X, x))(U \subseteq \setminus f^{-1}[K]) \\
& \Rightarrow x \in e^*\text{-int}_\theta(X \setminus f^{-1}[K]) \\
& \stackrel{\text{Lemma 2.3(7)}}{\Rightarrow} x \in X \setminus e^*\text{-cl}_\theta(f^{-1}[K]) \\
& \Rightarrow x \notin e^*\text{-cl}_\theta(f^{-1}[K]). \quad \square
\end{aligned}$$

Definition 4.5. A topological space X is said to be:

- a) strongly $e^*\theta C$ -compact if every $e^*\theta$ -closed cover of X has a finite subcover (resp. $A \subseteq X$ is strongly $e^*\theta C$ -compact if the subspace A is strongly $e^*\theta C$ -compact),
- b) nearly compact [26] if every regular open cover of X has a finite subcover.

Theorem 4.10. *If $f : X \rightarrow Y$ is an almost contra $e^*\theta$ -continuous surjection and X is strongly $e^*\theta C$ -compact, then Y is nearly compact.*

Proof. Let $\mathcal{B} \subseteq RO(Y)$ and $Y = \cup \mathcal{B}$.

$$\left. \begin{aligned} (\mathcal{B} \subseteq RO(Y))(Y = \cup \mathcal{B}) \\ f \text{ is a.c. } e^*\theta\text{.c.} \end{aligned} \right\} \Rightarrow \left. \begin{aligned} (\mathcal{A} := \{f^{-1}[B] | B \in \mathcal{B}\} \subseteq e^*\theta C(X))(X = \cup \mathcal{A}) \\ X \text{ is strongly } e^*\theta C\text{-compact} \end{aligned} \right\} \Rightarrow$$

$$\begin{aligned} & \Rightarrow (\exists \mathcal{A}^* \subseteq \mathcal{A})(|\mathcal{A}^*| < \aleph_0)(X = \cup \mathcal{A}^*) \left. \vphantom{\exists \mathcal{A}^* \subseteq \mathcal{A}} \right\} \Rightarrow \\ & \qquad \qquad \qquad f \text{ is surjective} \\ & \Rightarrow (\mathcal{B}^* := \{f[A] \mid A \in \mathcal{A}^*\} \subseteq \mathcal{B})(|\mathcal{B}^*| < \aleph_0)(Y = \cup \mathcal{B}^*). \quad \square \end{aligned}$$

We recall that a topological space X is said to be almost regular [25] if for each regular closed set F of X and each point $x \in X \setminus F$, there exist disjoint open sets U and V such that $F \subseteq V$ and $x \in U$.

Theorem 4.11. *If a function $f : X \rightarrow Y$ is almost contra $e^*\theta$ -continuous and Y is almost regular, then f is almost $e^*\theta$ -continuous.*

Proof. Let $x \in X$ and $V \in O(Y, f(x))$.

$$\begin{aligned} & (x \in X)(V \in O(Y, f(x))) \left. \vphantom{x \in X} \right\} \xrightarrow{\text{Lemma 2.8}} \\ & \qquad \qquad \qquad Y \text{ is almost regular} \\ & \Rightarrow (\exists W \in RO(Y, f(x)))(cl(W) \subseteq int(cl(V))) \left. \vphantom{\exists W \in RO(Y, f(x))} \right\} \xrightarrow{\text{Theorem 3.1(3)}} \\ & \qquad \qquad \qquad f \text{ is a.c.}e^*\theta.c. \\ & \Rightarrow (\exists U \in e^*\theta O(X, x))(f[U] \subseteq cl(W) \subseteq int(cl(V))). \quad \square \end{aligned}$$

Definition 4.6. The $e^*\theta$ -frontier of a subset A , denoted by $Fr_{e^*\theta}(A)$, is defined as $Fr_{e^*\theta}(A) = e^*-cl_\theta(A) \setminus e^*-int_\theta(A)$, equivalently $Fr_{e^*\theta}(A) = e^*-cl_\theta(A) \cap e^*-cl_\theta(X \setminus A)$.

Theorem 4.12. *The set of points $x \in X$ on which $f : X \rightarrow Y$ is not almost contra $e^*\theta$ -continuous is identical with the union of the $e^*\theta$ -frontiers of the inverse images of regular closed sets of Y containing $f(x)$.*

Proof. Let $A := \{x \mid f \text{ is not a.c.}e^*\theta.c. \text{ at } x \in X\}$.

$$\begin{aligned} x \in A & \Rightarrow f \text{ is not a.c.}e^*\theta.c. \text{ at } x \\ & \Rightarrow (\exists V \in RC(Y, f(x)))(\forall U \in e^*\theta O(X, x))(f[U] \not\subseteq V) \\ & \Rightarrow (\exists V \in RC(Y, f(x)))(\forall U \in e^*\theta O(X, x))(U \cap (X \setminus f^{-1}[V]) \neq \emptyset) \\ & \Rightarrow (x \in f^{-1}[V])(x \in e^*-cl_\theta(X \setminus f^{-1}[V]) = X \setminus e^*-int_\theta(f^{-1}[V])) \\ & \Rightarrow x \in Fr_{e^*\theta}(f^{-1}[V]) \end{aligned}$$

Then we have $A \subseteq \cup \{Fr_{e^*\theta}(f^{-1}[V]) | V \in RC(Y, f(x))\} \dots (*)$

$$\left. \begin{array}{l} x \notin A \Rightarrow f \text{ is a.c.}e^*\theta.c. \text{ at } x \\ V \in RC(Y, f(x)) \end{array} \right\} \Rightarrow (\exists U \in e^*\theta O(X, x))(U \subseteq f^{-1}[V])$$

$\Rightarrow x \in e^*\text{-int}_\theta(f^{-1}[V])$

$\Rightarrow x \notin Fr_{e^*\theta}(f^{-1}[V])$

$\Rightarrow x \notin \cup \{Fr_{e^*\theta}(f^{-1}[V]) | V \in RC(Y, f(x))\}$

Then we have $\cup \{Fr_{e^*\theta}(f^{-1}[V]) | V \in RC(Y, f(x))\} \subseteq A \dots (**)$

$(*), (**) \Rightarrow A = \cup \{Fr_{e^*\theta}(f^{-1}[V]) | V \in RC(Y, f(x))\}.$ □

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