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A ZERO-TRUNCATED POISSON-ARADHANA DISTRIBUTION WITH APPLICATIONS

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ABSTRACT: In this paper, a zero-truncation of Poisson-Aradhana distribution proposed by Shanker (2017) named 'zero-truncated Poisson-Aradhana distribution' has been introduced and investigated. A general expression for the r th factorial moment about origin has been obtained and thus the first four moments about origin and the central moments have been given. Also, the expressions for coefficient of variation, skewness, kurtosis, and the index of dispersion of the distribution have been presented and their natures have been discussed graphically. The method of moments and the method of maximum likelihood estimation have also been discussed for estimating its parameter. Two examples of observed real datasets have been given to test the goodness of fit of of the proposed distribution over zero-truncated Poisson-Sujatha distribution, zero-truncated Poisson-Lindley distribution and zero-truncated Poisson distribution.

1. INTRODUCTION

Suppose X is a discrete random variable having distribution $P_0\left(x;\theta\right)$, where the parameter $\theta>0$. Then the zero-truncated version of $P_0\left(x;\theta\right)$ can be defined as

$$P\left(x;\theta\right)=\frac{P_0\left(x;\theta\right)}{1-P_0\left(0;\theta\right)}\qquad ;x=1,2,3,... \tag{1.1}$$

In probability theory, zero-truncated distributions are certain discrete distributions having support the set of positive integers. Generally, zero-truncated (or size-biased) distributions are suitable models for modeling data when the data to be modeled originate from a mechanism which generates data excluding zero counts.

The discrete random variable X following Poisson-Aradhana distribution (PAD) having probability mass function (pmf)

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(1.2) $$P_0(x;\theta)=\frac{\theta^3}{\theta^2+2\theta+2}\frac{x^2+(2\theta+5)x+(\theta^2+4\theta+5)}{(\theta+1)^{x+3}}\quad ;x=0,1,2,3,...,\ \theta>0$$

was introduced by Shanker (2017) to model count data in different fields of knowledge. The PAD ia a Poisson mixture of Aradhana distribution introduced by Shanker (2016 a) when the parameter λ of the Poisson distribution follows Aradhana distribution with probability density function (pdf)

(1.3) $$f_1(\lambda;\theta)=\frac{\theta^3}{\theta^2+2\theta+2}\left(1+2\lambda+\lambda^2\right)e^{-\theta\lambda}\quad ;\lambda>0\ ,\theta>0$$

Detailed discussion about its various statistical properties, estimation of parameter and applications for modeling lifetime data has been mentioned in Shanker (2016 a) and shown by Shanker (2016 a) that (1.3) is a better model than the exponential and Lindley (1958) distributions for modeling lifetime data from engineering and biomedical sciences.

Since PAD gives better fit than Poisson distribution, Poisson-Lindley distribution (PLD) suggested by Sankaran (1970), and Poisson-Sujatha distribution (PSD) proposed by Shanker (2016 b), it is expected that zero-truncated Poisson-Aradhana distribution (ZTPAD) will provide better fit than zero-truncated Poisson distribution (ZTPD), zero-truncated Poisson-Lindley distribution (ZTPLD) introduced by Ghitany et al (2008) and zero-truncated Poisson-Sujatha distribution (ZTPSD) suggested by Shanker and Hagos (2015 b). Considering these points, in this paper ZTPAD has been suggested by taking the zero-truncated version of PAD. The first four moments about origin and the moments about mean of ZTPAD have been obtained and thus expressions for coefficient of variation, skewness, kurtosis, and index of dispersion have been given. Estimation of its parameter has been discussed using both the method of moments and the maximum likelihood estimation. Finally, the goodness of fit of ZTPAD over ZTPSD, ZTPLD and ZTPD for two observed real data sets have been given.

2. ZERO-TRUNCATED POISSON-ARADHANA DISTRIBUTION

Using (1.1) and (1.2), the pmf of zero-truncated Poisson-Aradhana distribution (ZTPAD) can be obtained as

(2.1) $$P_1(x;\theta)=\frac{\theta^3}{\theta^4+6\theta^3+13\theta^2+8\theta+2}\frac{x^2+(2\theta+5)x+(\theta^2+4\theta+5)}{(\theta+1)^{x}}\quad ;x=1,2,3,...,\ \theta>0$$

Since it is difficult and complicated to obtain the moments of ZTPAD directly, an attempt has been made to derive the pmf of ZTPAD as a size-biased Poisson mixture of an assumed continuous distribution which will be helpful to obtain the moments. The ZTPAD can also be obtained by compounding size-biased Poisson distribution (SBPD) having pmf

(2.2) $$g(x\mid\lambda)=\frac{e^{-\lambda}\lambda^{x-1}}{(x-1)!}\quad ;x=1,2,3,...\ ,\lambda>0$$

when the parameter λ of the SBPD follows a continuous distribution having pdf

(2.3) $$f_2(\lambda;\theta)=\frac{\theta^3}{\theta^4+6\theta^3+13\theta^2+8\theta+2}\left[(\theta+1)^2\lambda^2+2(\theta+1)(\theta+2)\lambda+\left(\theta^2+4\theta+5\right)\right]e^{-\theta\lambda},\lambda>0,\theta>0$$

The pmf of ZTPAD can thus be obtained as

(2.4) $$P_1(x;\theta)=\int_0^\infty g(x\mid\lambda)\cdot f_2(\lambda;\theta)\,d\lambda$$

$$=\int_0^\infty \frac{e^{-\lambda}\lambda^{x-1}}{(x-1)!}\cdot\frac{\theta^3}{\theta^4+6\theta^3+13\theta^2+8\theta+2}\left[(\theta+1)^2\lambda^2+2(\theta+1)(\theta+2)\lambda+\left(\theta^2+4\theta+5\right)\right]e^{-\theta\lambda}d\lambda$$

$$=\frac{\theta^3}{\left(\theta^4+6\theta^3+13\theta^2+8\theta+2\right)(x-1)!}\int_0^\infty e^{-(\theta+1)\lambda}\cdot\left[\begin{array}{l}(\theta+1)^2\lambda^{x+2-1}+2(\theta+1)(\theta+2)\lambda^{x+1-1}\\+\left(\theta^2+4\theta+5\right)\lambda^{x-1}\end{array}\right]d\lambda$$

$$=\frac{\theta^3}{\left(\theta^4+6\theta^3+13\theta^2+8\theta+2\right)}\left[\frac{(x+1)x}{(\theta+1)^x}+\frac{2(\theta+2)x}{(\theta+1)^x}+\frac{\theta^2+4\theta+5}{(\theta+1)^x}\right]$$

$$=\frac{\theta^3}{\theta^4+6\theta^3+13\theta^2+8\theta+2}\frac{x^2+(2\theta+5)x+\left(\theta^2+4\theta+5\right)}{(\theta+1)^x};x=1,2,3,...,\ \theta>0$$

which is the pmf of ZTPAD with parameter θ as obtained earlier in (2.1). Nature of the pmf of ZTPAD for varying values of parameter θ have been drawn and shown in figure 1.

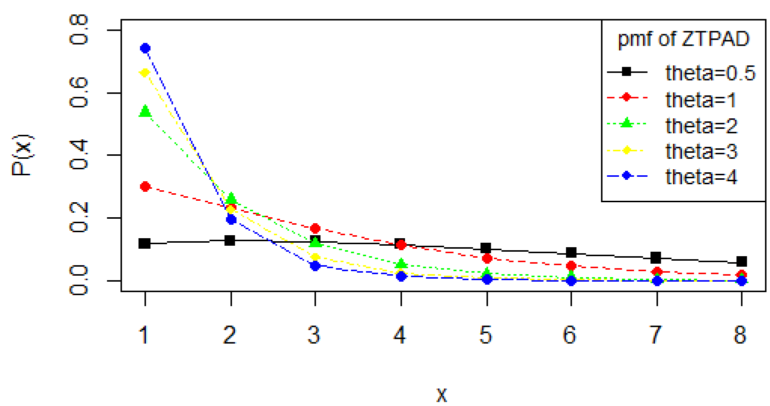

FIGURE 1. Nature of the pmf of ZTPAD for varying values of the parameter θ

Since $\frac{P(x+1;\theta)}{P(x;\theta)}=\left(\frac{1}{\theta+1}\right)\left[1+\frac{2(x+\theta+3)}{x^2+(2\theta+5)x+\left(\theta^2+4\theta+5\right)}\right]$ is a decreasing function of x, $P(x;\theta)$ is log-concave. Therefore, ZTPAD is unimodal, has increasing failure rate (IFR), and hence increasing failure rate average (IFRA). It is new better than used (NBU), new better than used in expectation (NBUE), and has decreasing mean residual life (DMRL). Detailed discussions and interrelationships between these aging concepts are available in Barlow and Proschan (1981).

Recall that the pmf of zero-truncated Poisson- Lindley distribution (ZTPLD) given by

$$P_2(x;\theta)=\frac{\theta^2}{\theta^2+3\theta+1}\frac{x+\theta+2}{(\theta+1)^x} \quad ;x=1,2,3,....,\theta>0 \tag{2.5}$$

has been introduced by Ghitany *et al* (2008). Further, Poisson-Lindley distribution (PLD) has been introduced by Sankaran (1970) by compounding Poisson distribution with Lindley (1958) distribution. Shanker and Hagos (2015 a) have discussed the applications of PLD for modeling data from biological sciences. Shanker *et al.* (2015) have done extensive study on the comparison of ZTPD and ZTPLD with respect to their applications in datasets excluding zero-counts and showed that in demography and biological sciences ZTPLD gives better fit than ZTPD while in social sciences ZTPD gives better fit than ZTPLD.

Note that the pmf of zero-truncated Poisson-Sujatha distribution (ZTPSD) given by

$$P_3(x;\theta)=\frac{\theta^3}{\theta^4+4\theta^3+10\theta^2+7\theta+2}\frac{x^2+(\theta+4)x+(\theta^2+3\theta+4)}{(\theta+1)^x} \quad ;x=1,2,3,..., \theta>0 \tag{2.6}$$

has been introduced by Shanker and Hagos (2015 b) for modeling count data excluding zero counts from different fields of knowledge. Various interesting properties, estimation of parameter and applications of ZTPSD have been mentioned in Shanker and Hagos (2015 b). Shanker (2016 b) has obtained the Poisson-Sujatha distribution (PSD) having pmf

$$P_4(x;\theta)=\frac{\theta^3}{\theta^2+\theta+2}\frac{x^2+(\theta+4)x+(\theta^2+3\theta+4)}{(\theta+1)^{x+3}} \quad ;x=1,2,3,..., \theta>0 \tag{2.7}$$

Shanker (2016 b) obtained the PSD as a Poisson mixture of Sujatha distribution when the parameter λ of the Poisson distribution follows Sujatha distribution, introduced by Shanker (2016 c) having pdf

$$f_3(x,\theta)=\frac{\theta^3}{\theta^2+\theta+2}(1+\lambda+\lambda^2)e^{\theta\lambda} \quad ;\lambda>0,\theta>0 \tag{2.8}$$

Shanker (2016 c) has detailed study about various mathematical and statistical properties, estimation of parameter and applications of Sujatha distribution for modeling lifetime data from biomedical science and engineering and it has been observed that Sujatha distribution is a better model than both exponential and Lindley (1958) distributions. Shanker and Hagos (2016 a) have discussed applications of PSD to model count data in ecology and genetics, and have shown that PSD is a better model than both Poisson and Poisson-Lindley distributions. Shanker and Hagos (2016 b) have comparative study on ZTPD, ZTPLD and ZTPSD for modeling data from biological science, demography and thunderstorms.

3. MOMENTS

The r th factorial moment about origin of ZTPAD (2.1) can be obtained as

$$\mu_{(r)}' = E\left[E\left(X^{(r)} \mid \lambda\right)\right] \text{ ; where } X^{(r)} = X\left(X-1\right)\left(X-2\right)....\left(X-r+1\right).$$

Using (2.4), we have

$$\mu_{(r)}' = \frac{\theta^3}{\theta^4+6\theta^3+13\theta^2+8\theta+2}\int_0^\infty\left[\sum_{x=1}^{\infty} x^{(r)}\frac{e^{-\lambda}\lambda^{x-1}}{(x-1)!}\right]\cdot\begin{bmatrix}(\theta+1)^2\lambda^2+2(\theta+1)(\theta+2)\lambda \\ +\left(\theta^2+4\theta+5\right)\end{bmatrix}e^{-\theta\lambda}d\lambda$$

$$= \frac{\theta^3}{\theta^4+6\theta^3+13\theta^2+8\theta+2}\int_0^\infty\left[\lambda^{r-1}\sum_{x=r}^{\infty} x\frac{e^{-\lambda}\lambda^{x-r}}{(x-r)!}\right]\cdot\begin{bmatrix}(\theta+1)^2\lambda^2+2(\theta+1)(\theta+2)\lambda \\ +\left(\theta^2+4\theta+5\right)\end{bmatrix}e^{-\theta\lambda}d\lambda$$

Taking $y = x - r$, we get

$$\mu_{(r)}' = \frac{\theta^3}{\theta^4+6\theta^3+13\theta^2+8\theta+2}\int_0^\infty\left[\lambda^{r-1}\sum_{y=0}^{\infty}(y+r)\frac{e^{-\lambda}\lambda^{y}}{y!}\right]\cdot\begin{bmatrix}(\theta+1)^2\lambda^2+2(\theta+1)(\theta+2)\lambda \\ +\left(\theta^2+4\theta+5\right)\end{bmatrix}e^{-\theta\lambda}d\lambda$$

$$= \frac{\theta^3}{\theta^4+6\theta^3+13\theta^2+8\theta+2}\int_0^\infty \lambda^{r-1}(\lambda+r)\cdot\begin{bmatrix}(\theta+1)^2\lambda^2+2(\theta+1)(\theta+2)\lambda \\ +\left(\theta^2+4\theta+5\right)\end{bmatrix}e^{-\theta\lambda}d\lambda$$

Using gamma integral and a little algebraic simplification, we get the expression for the r th factorial moment about origin of ZTPAD as

(3.1) $$\mu_{(r)}' = \frac{r!(\theta+1)\left[(r+1)(\theta+1)(r\theta+r+2)+2\theta(\theta+2)(r\theta+r+1)+\theta^2\left(\theta^2+4\theta+5\right)\right]}{\theta^r\left(\theta^4+6\theta^3+13\theta^2+8\theta+2\right)} ; r=1,2,3,...$$

Substituting $r = 1, 2, 3, \text{and } 4$ in equation (3.1), the first four factorial moments about origin can be obtained and using the relationship between moments about origin and factorial moments about origin, the first four moments about origin of ZTPAD can be obtained as follows.

$$\mu_1' = \frac{\theta^5+7\theta^4+21\theta^3+31\theta^2+22\theta+6}{\theta\left(\theta^4+6\theta^3+13\theta^2+8\theta+2\right)}$$

$$\mu_2' = \frac{\theta^6+9\theta^5+39\theta^4+97\theta^3+132\theta^2+90\theta+24}{\theta^2\left(\theta^4+6\theta^3+13\theta^2+8\theta+2\right)}$$

$$\mu_3' = \frac{\theta^7+13\theta^6+81\theta^5+295\theta^4+640\theta^3+768\theta^2+480\theta+120}{\theta^3\left(\theta^4+6\theta^3+13\theta^2+8\theta+2\right)}$$

$$\mu_4' = \frac{\theta^8 + 21\theta^7 + 183\theta^6 + 913\theta^5 + 2796\theta^4 + 5166\theta^3 + 5520\theta^2 + 3120\theta + 720}{\theta^4\left(\theta^4 + 6\theta^3 + 13\theta^2 + 8\theta + 2\right)}$$

Again using the relationship between moments about origin and moments about mean, the moments about mean of ZTPAD are thus obtained using MAPLE software as

$$\mu_2 = \sigma^2 = \frac{\theta^9 + 15\theta^8 + 100\theta^7 + 376\theta^6 + 851\theta^5 + 1165\theta^4 + 948\theta^3 + 440\theta^2 + 108\theta + 12}{\theta^2\left(\theta^4 + 6\theta^3 + 13\theta^2 + 8\theta + 2\right)^2}$$

$$\mu_3 = \frac{\begin{pmatrix}\theta^{14} + 23\theta^{13} + 249\theta^{12} + 1663\theta^{11} + 7523\theta^{10} + 23931\theta^9 + 541758\theta^8 + 87177\theta^7 \\ + 98940\theta^6 + 78538\theta^5 + 43176\theta^4 + 16284\theta^3 + 4128\theta^2 + 648\theta + 48\end{pmatrix}}{\theta^3\left(\theta^4 + 6\theta^3 + 13\theta^2 + 8\theta + 2\right)^3}$$

$$\mu_4 = \frac{\begin{pmatrix}\theta^{19} + 36\theta^{18} + 612\theta^{17} + 6455\theta^{16} + 47052\theta^{15} + 250442\theta^{14} + 1003883\theta^{13} + 3083012\theta^{12} \\ + 7321011\theta^{11} + 13488762\theta^{10} + 19279990\theta^9 + 21325325\theta^8 + 18182016\theta^7 + 11885864\theta^6 \\ + 5907808\theta^5 + 2198840\theta^4 + 595536\theta^3 + 111360\theta^2 + 12960\theta + 720\end{pmatrix}}{\theta^4\left(\theta^4 + 6\theta^3 + 13\theta^2 + 8\theta + 2\right)^4}$$

Finally, the coefficient of variation (C.V), coefficient of Skewness $\left(\sqrt{\beta_1}\right)$, coefficient of Kurtosis $\left(\beta_2\right)$ and index of dispersion $\left(\gamma\right)$ of ZTPAD are obtained as

$$C.V = \frac{\sigma}{\mu_1'} = \frac{\sqrt{\theta^9 + 15\theta^8 + 100\theta^7 + 376\theta^6 + 851\theta^5 + 1165\theta^4 + 948\theta^3 + 440\theta^2 + 108\theta + 12}}{\theta^5 + 7\theta^4 + 21\theta^3 + 31\theta^2 + 22\theta + 6}$$

$$\sqrt{\beta_1} = \frac{\mu_3}{\left(\mu_2\right)^{3/2}} = \frac{\begin{pmatrix}\theta^{14} + 23\theta^{13} + 249\theta^{12} + 1663\theta^{11} + 7523\theta^{10} + 23931\theta^9 + 541758\theta^8 + 87177\theta^7 \\ + 98940\theta^6 + 78538\theta^5 + 43176\theta^4 + 16284\theta^3 + 4128\theta^2 + 648\theta + 48\end{pmatrix}}{\left(\theta^9 + 15\theta^8 + 100\theta^7 + 376\theta^6 + 851\theta^5 + 1165\theta^4 + 948\theta^3 + 440\theta^2 + 108\theta + 12\right)^{3/2}}$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{\begin{pmatrix}\theta^{19} + 36\theta^{18} + 612\theta^{17} + 6455\theta^{16} + 47052\theta^{15} + 250442\theta^{14} + 1003883\theta^{13} + 3083012\theta^{12} \\ + 7321011\theta^{11} + 13488762\theta^{10} + 19279990\theta^9 + 21325325\theta^8 + 18182016\theta^7 + 11885864\theta^6 \\ + 5907808\theta^5 + 2198840\theta^4 + 595536\theta^3 + 111360\theta^2 + 12960\theta + 720\end{pmatrix}}{\left(\theta^9 + 15\theta^8 + 100\theta^7 + 376\theta^6 + 851\theta^5 + 1165\theta^4 + 948\theta^3 + 440\theta^2 + 108\theta + 12\right)^2}$$

$$\gamma = \frac{\sigma^2}{\mu} = \frac{\theta^9 + 15\theta^8 + 100\theta^7 + 376\theta^6 + 851\theta^5 + 1165\theta^4 + 948\theta^3 + 440\theta^2 + 108\theta + 12}{\theta\left(\theta^4 + 6\theta^3 + 13\theta^2 + 8\theta + 2\right)\left(\theta^5 + 7\theta^4 + 21\theta^3 + 31\theta^2 + 22\theta + 6\right)}$$

Behaviors of coefficient variation, coefficient of skewness, coefficient of kurtosis and Index of dispersion of ZTPAD for varying values of parameter θ have been drawn and presented in figure 2. It is obvious that the coefficient of variation and index of dispersion are monotonically decreasing while the coefficient of skewness and coefficient of kurtosis are monotonically increasing for increasing values of the parameter θ. It is obvious from the graphs in figure 2 that coefficient of variation is a decreasing value of parameter $\theta > 1$, whereas the index if dispersion is monotonically decreasing for increasing value of parameter θ. Further, the coefficients of skewness and kurtosis are monotonically increasing for increasing values of the parameter θ.

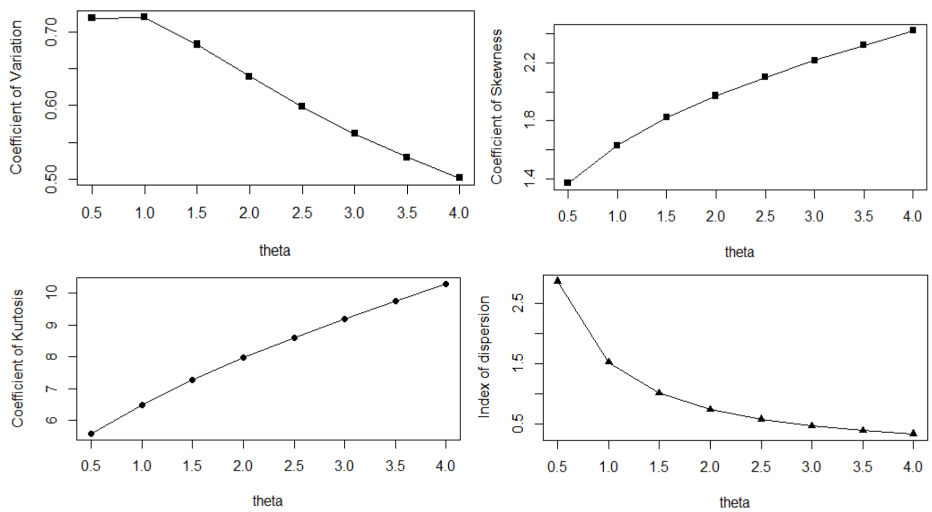

FIGURE 2. Behaviors of coefficient variation, coefficient of skewness, coefficient of kurtosis and Index of dispersion of ZTPAD for varying values of the parameter θ

The over-dispersion $\left(\mu < \sigma^2\right)$, equi-dispersion $\left(\mu = \sigma^2\right)$ and under-dispersion $\left(\mu > \sigma^2\right)$ of ZTPAD, ZTPSD, and ZTPLD for values of parameter are obtained and presented in table 1.

TABLE 1. Over-dispersion $\left(\mu < \sigma^2\right)$, equi-dispersion $\left(\mu = \sigma^2\right)$ and under-dispersion $\left(\mu > \sigma^2\right)$ of ZTPAD, ZTPSD, and ZTPLD

Distributions	Over-dispersion $\left(\mu < \sigma^2\right)$	Equi-dispersion $\left(\mu = \sigma^2\right)$	Under-dispersion $\left(\mu > \sigma^2\right)$
ZTPAD	$\theta < 1.514936$	$\theta = 1.514936$	$\theta > 1.514936$
ZTPSD	$\theta < 1.548329$	$\theta = 1.548329$	$\theta > 1.548329$
ZTPLD	$\theta < 1.258627$	$\theta = 1.258627$	$\theta > 1.258627$

4. PARAMETER ESTIMATION

4.1. Method of Moment Estimate (MOME): Let $x_1, x_2, ..., x_n$ be a random sample of size n from the ZTPAD (2.1). Equating the population mean to the corresponding sample mean, the MOME $\tilde{\theta}$ of the parameter θ is the solution of the following fifth degree polynomial equation

$$(1-\overline{x})\theta^5 + (7-6\overline{x})\theta^4 + (21-13\overline{x})\theta^3 + (31-8\overline{x})\theta^2 + (22-2\overline{x})\theta + 6 = 0,$$

where $\overline{x}$ is the sample mean.

Since it is fifth degree polynomial equation in θ, MOME of the parameter θ can be obtained using Newton-Raphson iterative formula which is always unique to certain decimal places but approximate.

4.2. Maximum Likelihood Estimate (MLE): Let $x_1, x_2, ..., x_n$ be a random sample of size n from the ZTPAD (2.1) and let f_x be the observed frequency in the sample corresponding to $X = x\ (x = 1,2,3,...,k)$ such that $\sum_{x=1}^{k} f_x = n$, where k is the largest observed value having non-zero frequency. The likelihood function L of the ZTPAD is given by

$$L = \left(\frac{\theta^3}{\theta^4 + 6\theta^3 + 13\theta^2 + 8\theta + 2}\right)^n \frac{1}{(\theta+1)^{\sum_{x=1}^{k} x f_x}} \prod_{x=1}^{k} \left[x^2 + (2\theta+5)x + (\theta^2 + 4\theta + 5)\right]^{f_x}$$

The log likelihood function is given by

$$\log L = n \log\left(\frac{\theta^3}{\theta^4 + 6\theta^3 + 13\theta^2 + 8\theta + 2}\right) - \sum_{x=1}^{k} x\, f_x \log(\theta+1)$$
$$+ \sum_{x=1}^{k} f_x \log\left[x^2 + (2\theta+5)x + (\theta^2 + 4\theta + 5)\right]$$

and the log likelihood equation is thus obtained as

$$\frac{d \log L}{d\theta} = \frac{3n}{\theta} - \frac{n(4\theta^3 + 18\theta^2 + 26\theta + 8)}{\theta^4 + 6\theta^3 + 13\theta^2 + 8\theta + 2} - \frac{n\overline{x}}{\theta+1} + \sum_{x=1}^{k} \frac{2(x+\theta+2) f_x}{x^2 + (2\theta+5)x + (\theta^2 + 4\theta + 5)}$$

The maximum likelihood estimate $\hat{\theta}$ of θ is the solution of the equation $\frac{d \log L}{d\theta} = 0$ and is given by the solution of the following non-linear equation

$$\frac{3n}{\theta} - \frac{n(4\theta^3 + 18\theta^2 + 26\theta + 8)}{\theta^4 + 6\theta^3 + 13\theta^2 + 8\theta + 2} - \frac{n\overline{x}}{\theta+1} + \sum_{x=1}^{k} \frac{2(x+\theta+2) f_x}{x^2 + (2\theta+5)x + (\theta^2 + 4\theta + 5)} = 0$$

where $\bar{x}$ is the sample mean. This non-linear equation can be solved by any numerical iteration methods such as Newton- Raphson method, Bisection method, Regula –Falsi method etc. An initial value of the parameter θ can be taken as the value given by MOME of θ. In this paper Newton-Raphson method has been used to solve the above non-linear equation to estimate the parameter.

5. APPLICATIONS

In this section two examples of observed real datasets has been given to test the goodness of fit of ZTPAD over ZTPSD, ZTPLD and ZTPD using maximum likelihood estimate of their parameter. The first dataset in table 2 is the number of household having at least one migrant according to the number of migrants reported by Singh and Yadav (1971). The second dataset in table 3 is the number of counts of sites with particles from immunogold data reported by Mathews and Appleton (1993).

TABLE 2. Number of households having at least one migrant according to the number of migrants, reported by Singh and Yadav (1971)

Number of migrants	Observed frequency	Expected Frequency			
		ZTPD	ZTPLD	ZTPSD	ZTPAD
1	375	354.0	379.0	378.3	377.1
2	143	167.7	137.2	137.8	139.2
3	49	53.0	48.4	48.7	48.9
4	17	12.5	16.8	16.8	16.6
5	2	2.4	5.7	5.6	5.5
6	2	0.4	1.9	1.8	1.8
7	1	0.1	0.6	0.6	0.6
8	1	0.0	0.3	0.4	0.3
Total	590	590.0	590.0	590.0	590.0
ML Estimate		$\hat{\theta}=0.947486$	$\hat{\theta}=2.284782$	$\hat{\theta}=2.722929$	$\hat{\theta}=2.751750$
$-2\log L$		1203.3	1186.1	1186.2	1177.6
AIC		1205.3	1188.1	1188.2	1179.6
χ^2		8.933	1.031	0.912	0.70
d.f.		2	3	3	3
P-value		0.0115	0.7937	0.8225	0.8732

TABLE 3. The number of counts of sites with particles from Immunogold data, reported by Mathews and Appleton (1993)

Number of sites with particles	Observed frequency	Expected Frequency			
		ZTPD	ZTPLD	ZTPSD	ZTPAD
1	122	115.9	124.8	124.4	124.0
2	50	57.4	46.8	47.0	47.5
3	18	18.9	17.1	17.2	17.3
4	4	4.7	6.1	6.1	6.0
5	4	1.1	3.2	3.3	3.2
Total	198	198.0	198.0	198.0	198.0
ML Estimate		$\hat{\theta}=0.990586$	$\hat{\theta}=2.18307$	$\hat{\theta}=2.614691$	$\hat{\theta}=2.638410$
$-2\log L$		411.9	409.2	409.1	408.9
AIC		413.9	411.2	411.1	410.9
χ^2		2.14	0.51	0.46	0.34
d.f.		2	2	2	2
P-value		0.3430	0.7749	0.7945	0.8437

It is obvious from the values of $-2\log L$, and Akaike Information criterion (AIC), chi-square $\left(\chi^2\right)$ and p-values that ZTPAD gives much closer fit than ZTPSD, ZTPLD and ZTPD. Therefore, ZTPAD can be considered as an important tool for modeling count data excluding zero-count over ZTPSD, ZTPLD and ZTPD.

6. CONCLUDING REMARKS

A zero-truncated Poisson- Aradhana distribution (ZTPAD) has been introduced. The general expression for the rth factorial moment about origin has been obtained and thus the first four moments about origin and central moments, coefficient of variation, skewness, kurtosis, and the index of dispersion of ZTPAD have been obtained. The method of maximum likelihood and the method of moments have also been discussed for estimating its parameter. Two examples of observed real datasets have been given to test the goodness of fit of ZTPAD over ZTPSD, ZTPLD and ZTPD and fit by ZTPAD has been quite satisfactory.

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REFERENCES

[1] R. E. Barlow and F. Proschan, *Statistical Theory of Reliability and Life Testing*, Silver Spring, MD, 1981.

[2] M.E. Ghitany, D.K. Al-Mutairi and S. Nadarajah, Zero-truncated Poisson-Lindley distribution and its Applications, *Mathematics and Computers in Simulation*, 79(3) (2008), 279 – 287.

[3] D.V. Lindley, Fiducial distributions and Bayes' theorem, *Journal of the Royal Statistical Society, Series B*, 20 (1958), 102- 107.

[4] J.N.S. Mathews and D.R Appleton, An application of the truncated Poisson distribution to Immunogold assay, *Biometrics*, 49 (1993), 617 – 621.

[5] M. Sankaran, The discrete Poisson-Lindley distribution, *Biometrics*, 26(1970), 145 – 149.

[6] R. Shanker, Aradhana Distribution and Its Applications**,** *International Journal of Statistics and Application*s, 6(1) (2016 a), 23 – 34.

[7] R. Shanker, The discrete Poisson-Sujatha distribution, *International Journal of Probability and Statistics*, 5(1) (2016 b), 1 – 9.

[8] R. Shanker, Sujatha distribution and Its Applications, *Statistics in Transition new Series*, 17(3) (2016 c), 1 – 20.

[9] R. Shanker, The discrete Poisson-Aradhana distribution, *Turkiye Klinikleri Journal of Biostatistics*, 9(1) (2017), 12 – 22.

[10] R. Shanker and F. Hagos, On Poisson-Lindley distribution and Its Applications to biological sciences, *Biometrics & Biostatistics International Journal*, 2(4) (2015 a), 1– 5.

[11] R. Shanker and F. Hagos, Zero-truncated Poisson-Sujatha distribution with Applications**,** *Journal of Ethiopian Statistical Association*, 24 (2015 b), 55 – 63.

[12] R. Shanker and F. Hagos, On Poisson-Sujatha distribution and its Applications to model count data from biological sciences, *Biometrics &Biostatistics International Journal,* 3(4) (2016 a), 1 -7.

[13] R. Shanker and F. Hagos, On Zero-truncation of Poisson, Poisson-Lindley and Poisson-Sujatha distributions and their Applications, *Biometrics & Biostatistics International Journal*, 3(5) (2016 b), 1 – 13.

[14] R. Shanker, F. Hagos, S. Sujatha and Y. Abrehe, On zero-truncation of Poisson and Poisson-Lindley distribution and Their Applications, *Biometrics & Biostatistics International Journal*, 2(6) (2015), 1 – 14.

[15] S.N. Singh and R.C. Yadav, Trends in Rural out-migration at household level, *Rural Demography*, 8(1971), 53 – 61.

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