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RESTRICTED HOM-LIE SUPERALGEBRAS

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Abstract. The aim of this paper is to introduce the notion of restricted Hom-Lie superalgebras. This class of algebras is a generalization of both restricted Hom-Lie algebras and restricted Lie superalgebras. In this paper, we present a way to obtain restricted Hom-Lie superalgebras from the classical restricted Lie superalgebras along with algebra endomorphisms. Homomorphisms relations between restricted Hom-Lie superalgebras are defined and studied. Also, we obtain some properties of p-maps and restrictable Hom-Lie superalgebras.

1. Introduction

Hom structures including Hom-algebras, Hom-Lie algebras, Hom-Lie superalgebras, Hom-Lie color algebras, Hom-coalgebras, Hom-modules, and Hom-Hopf modules have been widely investigated during the last years. The motivations to study Hom-Lie structures are related to physics and to deformations of Lie algebras, especially Lie algebras of vector fields. The Hom-Lie algebras were firstly studied by Hartwig, Larsson, and Silvestrov in [5]. Later, Hom-Lie superalgebras are introduced by Ammar and Makhlouf in [1]. Also, Hom-Lie color algebras, which are the natural generalizations of Hom-Lie algebras and Hom-Lie superalgebras, are studied by Yuan ([14]). However, the notion of restricted Hom-Lie algebras was introduced by Guan

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and Chen ([4]).
The Lie superalgebras were originally introduced by Kac ([6]). Later, when char$F = p > 0$, Lie superalgebras were studied, and also the notion of restricted Lie superalgebras was introduced ([7, 15]). In the present work, we introduce the notion of restricted Hom-Lie superalgebras.
The article is organized as follows. In Section 2, we review some definitions, notations, and results in [1, 3]. In Section 3, we introduce the definition of restricted Hom-Lie superalgebra. We provide some properties and their relationships with restricted Hom-Lie superalgebras. As a result, we show that it is enough to know how a p-map behaves for inputs from any basis of the domain. In Section 4, We study the direct sum of (restricted) Hom-Lie superalgebras. In Section 5, we study the homomorphisms between restricted Hom-Lie superalgebras. In particular, we show how arbitrary restricted Lie superalgebras deform into restricted Hom-Lie superalgebras via algebra endomorphisms. Also we discuss the images as well as preimages of restricted Hom-Lie subsuperalgebras under homomorphisms. The restrictable Hom-Lie superalgebras are defined and studied in Section 6.

2. Preliminaries

Let F be the ground field of characteristic $\neq 2, 3$. A linear superspace V over F is merely a $\mathbb{Z}_2$-graded linear space with a direct sum $V = V_0 \oplus V_1$. The elements of V_j, $j \in \{0, 1\}$, are said to be homogeneous of parity j. The parity of a homogeneous element x is denoted by $|x|$. Suppose that $V = V_0 \oplus V_1$ and $V' = V'_0 \oplus V'_1$ are two linear superspaces. A linear map $\alpha : V \to V'$ is an even if $\alpha(V_j) \subseteq V'_j$ for $j \in \{0, 1\}$.

Definition 2.1 ([1]). A Hom-associative superalgebra is a triple (A, μ, α) where A is a linear superspace, $\mu : A \times A \to A$ is an even bilinear map, and $\alpha : A \to A$ is an even linear map such that

$$\mu(\alpha(x), \mu(y,z)) = \mu(\mu(x,y), \alpha(z)).$$

A Hom-associative superalgebra (A, μ, α) is called multiplicative if $\alpha(\mu(x,y)) = \mu(\alpha(x), \alpha(y))$ for all $x, y \in A$.

Definition 2.2 ([1]). A Hom-Lie superalgebra is a triple $(L, [\ ,\ ], \alpha)$ where L is a linear superspace, $[\ ,\ ] : L \times L \to L$ is an even bilinear map, and $\alpha : L \to L$ is an even linear map such that the following identities satisfied for any homogeneous $x, y, z \in L$.

(i) Super skew-symmetry:

$$[x,y] = -(-1)^{|x||y|}[y,x].$$

(ii) Hom-superJacobi identity:

$$(-1)^{|x||z|}[\alpha(x), [y,z]] + (-1)^{|z||y|}[\alpha(z), [x,y]] + (-1)^{|y||x|}[\alpha(y), [z,x]] = 0.$$

Example 2.1 ([1]). *Any Hom-associative superalgebra A, with the bracket*

$$[x,y] = \mu(x,y) - (-1)^{|x||y|}\mu(y,x)$$

for any nonzero homogeneous $x, y \in A$, is a Hom-Lie superalgebra, which will be denoted by $A^{(-)}$.

It is clear that Lie superalgebras are examples of Hom-Lie superalgebras by setting $\alpha = \mathrm{id}_L$. A Hom-Lie superalgebra is called a multiplicative Hom-Lie superalgebra if α is an even homomorphism (that is, $\alpha([x,y]) = [\alpha(x), \alpha(y)]$ for all $x, y \in L$). A subspace $H \subseteq L$ is called a Hom-Lie subsuperalgebra if $\alpha(H) \subseteq H$ and H is closed under the bracket operation (that is, $[x,y] \in L$ for all $x, y \in L$).
Let $L = L_0 \oplus L_1$ be a Lie superalgebra. For a homogenous element $a \in L$, we consider a mapping $\mathrm{ad}a : L \to L;\ b \mapsto [a,b]$.

Definition 2.3 ([3])**.** A Lie superalgebra $L = L_0 \oplus L_1$ is called *restricted* (or *p-superalgebra*) if there is a map

$$^{[p]} : L_0 \to L;\ x \mapsto x^{[p]},$$

satisfying

(i) $(\text{ad}x)^p = \text{ad}\left(x^{[p]}\right)$ for all $x \in L_0$,

(ii) $(kx)^{[p]} = k^p x^{[p]}$ for all $k \in F, x \in L_0$,

(iii) $(x+y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{i=1}^{p-1} s_i(x,y)$ for all $x, y \in L_0$ where $is_i(x,y)$ is the coefficient of λ^{i-1} in $\text{ad}\,(\lambda x + y)^{p-1}(x)$.

3. Restricted Hom-Lie superalgebra

Let $(L, [\ ,\ ], \alpha)$ be a multiplicative Hom-Lie superalgebra. For a homogeneous element $a \in L_0$ with $\alpha(a) = a$, we consider a map

$$\text{ad}_\alpha a : L \to L;\ b \mapsto [a, \alpha(b)].$$

Put $L^0 = \{x \in L_0 \mid \alpha(x) = x\}$. We can easily prove that L^0 is a Hom-Lie subsuperalgebra of L.

Definition 3.1. A multiplicative Hom-Lie superalgebra $(L, [\ ,\ ], \alpha)$ is called restricted if there is a map (called a p-map)

$$[p]\ : L^0 \to L^0;\ x \mapsto x^{[p]},$$

satisfying

(i) $(\text{ad}_\alpha x)^p = \text{ad}_\alpha\left(x^{[p]}\right)$ for all $x \in L^0$,

(ii) $(kx)^{[p]} = k^p x^{[p]}$ for all $k \in F, x \in L^0$,

(iii) $(x+y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{i=1}^{p-1} s_i(x,y)$ for all $x, y \in L^0$ where $is_i(x,y)$ is the coefficient of λ^{i-1} in $\text{ad}_\alpha\,(\lambda x + y)^{p-1}(x)$.

Let $(L, [\ ,\ ], \alpha, [p])$ be a restricted Hom Lie-superalgebra over a field F. A Hom-Lie subsuperalgebra $H = H_0 \oplus H_1$ of L is called a p-subalgebra if $x^{[p]} \in H^0\ \forall x \in H^0$.

Definition 3.2. Let $(L, [\ ,\ ], \alpha)$ be a Hom Lie-superalgebra, and let S be a subset of L.

(i) A map $f : L \to L$ is called a p-semilinear map if $f(kx + y) = k^p f(x) + f(y)$ $\forall x, y \in L$ and $\forall k \in F$.

(ii) The α-centralizer of S in L, denoted by $C_L(S)$, is defined to be

$$C_L(S) = \{x \in L \mid [x, \alpha(y)] = 0\ \forall y \in S\}.$$

In particular, if $S = L$, it is called the α-center of L, and it is denoted by $C(L)$.

The following theorem was proved by Baoling Guan and Liangyun Chen in [4] in the setting of Hom-Lie algebras.

Theorem 3.1. *Let H be a Hom-Lie subsuperalgebra of a multiplicative restricted Hom-Lie superalgebra $(L, [\ ,\ ], \alpha, [p])$ and $[p]_1 : H^0 \to H^0$ is a map. Then the following statements are equivalent:*

(i) *$[p]_1$ is a p-map on H^0,*

(ii) *there is a p-semilinear map $f : H^0 \to C_L(H)$ such that $[p]_1 = [p] + f$.*

Proof. Suppose that $[p]_1$ is a p-map on H^0. Consider

$$f : H^0 \to L;\ x \mapsto x^{[p]_1} - x^{[p]}.$$

Since $\mathrm{ad}_\alpha f(x)(y) = [f(x), \alpha(y)] = [x^{[p]_1}, \alpha(y)] - [x^{[p]}, \alpha(y)] = (\mathrm{ad}_\alpha x)^p - (\mathrm{ad}_\alpha x)^p = 0$ for all $x \in H^0, y \in L$, f actually maps H^0 into $C_L(H)$. Now, for $x, y \in H^0$ and $k \in F$, we obtain

$$\begin{aligned} f(kx+y) &= (kx+y)^{[p]_1} - (kx+y)^{[p]} \\ &= k^p x^{[p]_1} + y^{[p]_1} + \sum_{i=1}^{p-1} s_i(kx,y) - k^p x^{[p]} - y^{[p]} - \sum_{i=1}^{p-1} s_i(kx,y) \\ &= k^p f(x) + f(y). \end{aligned}$$

This shows that f is a p-semilinear map. Conversely, assume there exists a p-semilinear map $f : H^0 \to C_L(H)$ with $[p]_1 = [p] + f$. We check the three conditions given in Definition 3.1. For $x \in H^0$, $y \in H$, we have

$$\begin{aligned} \mathrm{ad}_\alpha x^{[p]_1}(y) &= \mathrm{ad}_\alpha(x^{[p]} + f(x))(y) \\ &= \mathrm{ad}_\alpha x^{[p]}(y) + \mathrm{ad}_\alpha f(x)(y) \\ &= \mathrm{ad}_\alpha x^{[p]}(y) \text{ (since } f(x) \in C_L(H)) \\ &= (\mathrm{ad}_\alpha x)^p(y). \end{aligned}$$

For $x, y \in H^0$, we have

$$\begin{aligned} (x+y)^{[p]_1} &= (x+y)^{[p]} + f(x+y) \\ &= x^{[p]} + y^{[p]} + \sum_{i=1}^{p-1} s_i(x,y) + f(x) + f(y) \\ &= x^{[p]_1} + y^{[p]_1} + \sum_{i=1}^{p-1} s_i(x,y). \end{aligned}$$

and, for $k \in F$, we get

$$\begin{aligned} (kx)^{[p]_1} &= (kx)^{[p]} + f(kx) \\ &= k^p x^{[p]} + k^p f(x) \\ &= k^p (x^{[p]} + f(x)) \\ &= k^p x^{[p]_1}. \end{aligned}$$

□

Corollary 3.1. *Let $(L, [\ ,\ ], \alpha)$ be a multiplicative Hom-Lie superalgebra.*

(i) *If $C(L) = 0$, then L admits at most one p-map.*

(ii) *If two p-maps coincide on a basis of L^0, then they are equal.*

(iii) *If $[p] : L^0 \to L^0$ is a p-map, then there exists a p-mapping $[p]'$ of L such that $x^{[p]'} = 0\ \forall x \in C(L^0)$.*

Proof.

(i) Suppose $[p]_1$ and $[p]_2$ are p-maps of L. By Theorem 3.1, there exists a p-semilinear map f from L^0 into $C(L)$ such that $[p]_2 = [p]_1 + f$, and since $C(L) = 0$, we have $[p]_1 = [p]_2$.

(ii) Let $[p]_1$ and $[p]_2$ be two p-maps coincide on a basis B of L. According to Theorem 3.1, there exists a p-semilinear map $f : L^0 \to C(L)$ such that $f = [p]_2 - [p]_1$, and so $f(b) = 0\ \forall b \in B$. As f is a p-semilinear, we have $f(x) = 0$ $\forall x \in L^0$. Thus, $[p]_1 = [p]_2$.

(iii) $[p]|_{C(L^0)}$ is obviously a p-map. For $x, y \in C(L^0)$ and $k \in F$, we have

$$\begin{aligned}(kx + y)^{[p]} &= (kx)^{[p]} + y^{[p]} + \sum_{i=1}^{p-1} s_i(kx, y)\\ &= k^p x^{[p]} + y^{[p]} \text{ (since C(L}^0\text{) is abelian).}\end{aligned}$$

Thus it is a p-semilinear map. Extend it to a p-semilinear map $f : L^0 \to C(L^0)$. Then $[p]' := [p] - f$ is a p-map with $x^{[p]'} = 0\ \forall x \in C(L^0)$.

□

4. Direct sum of Hom-Lie superalgebras

Let V and V' be two superspace. Then the homogeneous elements of the superspace $V \oplus V'$ have the form (x, y), where x and y are homogeneous elements of the same degrees in V and V', respectively. In this case, $|(x,y)| = |x| = |y|$. We have the following result.

Theorem 4.1. *Given two Hom-Lie superalgebras* $(L, [\ ,\ ]_L, \alpha)$ *and* $(\Gamma, [\ ,\ ]_\Gamma, \beta)$*, then* $(L \oplus \Gamma, [\ ,\ ]_{L\oplus\Gamma}, \gamma)$ *has a multiplicative Hom-Lie superalgebra structure, where the bilinear map* $[\ ,\ ]_{L\oplus\Gamma}$ *is given by (*(x_1, y_1) *and* (x_2, y_2) *are homogeneous elements):* $[(x_1, y_1), (x_2, y_2)]_{L\oplus\Gamma} = ([x_1, x_2]_L, [y_1, y_2]_\Gamma)$*, and the linear map* $\gamma : L \oplus \Gamma \to L \oplus \Gamma$ *is given by:* $\gamma(x_1, y_1) = (\alpha(x_1), \beta(y_1))$*. Moreover,* $(L \oplus \Gamma)^0 = L^0 \oplus \Gamma^0$.

Proof. Suppose that (x_1, y_1) and (x_2, y_2) are two homogeneous elements in $L \oplus \Gamma$. Then

$$\begin{aligned}
[(x_1,y_1),(x_2,y_2)]_{L\oplus\Gamma} &= ([x_1,x_2]_L,[y_1,y_2]_\Gamma)\\
&= (-(-1)^{|x_1||x_2|}[x_2,x_1]_L, -(-1)^{|y_1||y_2|}[y_2,y_1]_\Gamma)\\
&= -(-1)^{|x_1||x_2|}([x_2,x_1]_L,[y_2,y_1]_\Gamma)\ (|x_1| = |y_1| \text{ and } |x_2| = |y_2|)\\
&= -(-1)^{|(x_1,y_1)||(x_2,y_2)|}[(x_2,y_2),(x_1,y_1)]_{L\oplus\Gamma}.
\end{aligned}$$

For all homogeneous elements $(x_1, y_1), (x_2, y_2)$, and (x_3, y_3), we have,

$$\begin{aligned}
&(-1)^{|(x_1,y_1)||(x_3,y_3)|}[\gamma(x_1,y_1),[(x_2,y_2),(x_3,y_3)]_{L\oplus\Gamma}]_{L\oplus\Gamma}\\
+\ &(-1)^{|(x_3,y_3)||(x_2,y_2)|}[\gamma(x_3,y_3),[(x_1,y_1),(x_2,y_2)]_{L\oplus\Gamma}]_{L\oplus\Gamma}\\
+\ &(-1)^{|(x_2,y_2)||(x_1,y_1)|}[\gamma(x_2,y_2),[(x_3,y_3),(x_1,y_1)]_{L\oplus\Gamma}]_{L\oplus\Gamma}\\
=\ &(-1)^{|x_1||x_3|}[(\alpha(x_1),\beta(y_1)),([x_2,x_3]_L,[y_2,y_3]_\Gamma)]_{L\oplus\Gamma}\\
+\ &(-1)^{|x_3||x_2|}[(\alpha(x_3),\beta(y_3)),([x_1,x_2]_L,[y_1,y_2]_\Gamma)]_{L\oplus\Gamma}\\
+\ &(-1)^{|x_2||x_1|}[(\alpha(x_2),\beta(y_2)),([x_3,x_1]_L,[y_3,y_1]_\Gamma)]_{L\oplus\Gamma}\\
=\ &((-1)^{|x_1||x_3|}[\alpha(x_1),[x_2,x_3]_L]_L,(-1)^{|y_1||y_3|}[\beta(y_1),[y_2,y_3]_\Gamma]_\Gamma)\\
+\ &((-1)^{|x_3||x_2|}[\alpha(x_3),[x_1,x_2]_L]_L,(-1)^{|y_3||y_2|}[\beta(y_3),[y_1,y_2]_\Gamma]_\Gamma)
\end{aligned}$$

$$+ \quad ((-1)^{|x_2||x_1|}[\alpha(x_2),[x_3,x_1]_L]_L, (-1)^{|y_2||y_1|}[\beta(y_2),[y_3,y_1]_\Gamma]_\Gamma)$$
$$= \quad (0,0).$$

Also, $(L\oplus\Gamma, [\ ,\ ]_{L\oplus\Gamma}, \gamma)$ is multiplicative, since for all homogeneous elements $(x_1,y_1),(x_2,y_2) \in L \oplus \Gamma$, we have

$$\begin{aligned}
\gamma([(x_1,y_1),(x_2,y_2)]_{L\oplus\Gamma}) &= \gamma([x_1,x_2]_L,[y_1,y_2]_\Gamma) \\
&= ([\alpha(x_1),\alpha(x_2)]_L,[\beta(y_1),\beta(y_2)]_\Gamma) \\
&= ([(\alpha(x_1),\beta(y_1)),(\alpha(x_2),\beta(y_2))]_{L\oplus\Gamma} \\
&= [\gamma(x_1,y_1),\gamma(x_2,y_2)]_{L\oplus\Gamma}.
\end{aligned}$$

Finally,

$$\begin{aligned}
(L\oplus\Gamma)^0 &= \{(x,y)\in(L\oplus\Gamma)_0 \mid \gamma(x,y)=(x,y)\} \\
&= \{(x,y)\in(L\oplus\Gamma)_0 \mid (\alpha(x),\beta(y))=(x,y)\} \\
&= \{(x,y)\in(L\oplus\Gamma)_0 \mid \alpha(x)=x \text{ and } \beta(y)=y\} \\
&= L^0\oplus\Gamma^0.
\end{aligned}$$

□

Theorem 4.2. *Given restricted Hom-Lie superalgebras* $(L,[\ ,\ ]_L,\alpha,[p]_1)$ *and* $(\Gamma,[\ ,\ ]_\Gamma,\beta,[p]_2)$, *then* $(L\oplus\Gamma,[\ ,\ ]_{L\oplus\Gamma},\gamma,[p])$ *has a restricted Hom-Lie superalgebras, where* $[\ ,\ ]_{L\oplus\Gamma}$ *and* γ *are defined as in Theorem 4.1, and*

$$[p] : (L\oplus\Gamma)^0 \to (L\oplus\Gamma)^0;\ \ (u,v)\mapsto (u^{[p]_1},v^{[p]_2}).$$

Proof. According to Theorem 4.1, it is enough to check the three conditions given in Definition 3.1. Let $(x_1,y_1)\in L^0\oplus\Gamma^0$ and $(x_2,y_2)\in L\oplus\Gamma$. We have

$$(\mathrm{ad}_\gamma(x_1,y_1))^p(x_2,y_2) \quad = \quad (\mathrm{ad}_\gamma(x_1,y_1))^{p-1}(\mathrm{ad}_\gamma(x_1,y_1))(x_2,y_2)$$

$$
\begin{aligned}
&= (\mathrm{ad}_\gamma(x_1,y_1))^{p-1}[(x_1,y_1),(\alpha(x_2),\beta(y_2))]_{L\oplus\Gamma}\\
&= (\mathrm{ad}_\gamma(x_1,y_1))^{p-1}([x_1,\alpha(x_2)]_L,[y_1,\beta(y_2)]_\Gamma)\\
&= (\mathrm{ad}_\gamma(x_1,y_1))^{p-2}\left([x_1,[x_1,\alpha^2(x_2)]_L]_L,[y_1,[y_1,\beta^2(y_2)]_\Gamma]_\Gamma\right)\\
&\vdots\\
&= ([x_1,[x_1,\cdots,[x_1,\alpha^p(x_2)]_L\cdots]_L]_L,[y_1,[y_1,\cdots,[y_1,\beta^p(y_2)]_\Gamma\cdots]_\Gamma]_\Gamma)\\
&= ((\mathrm{ad}_\alpha x_1)^p(x_2),(\mathrm{ad}_\beta y_1)^p(y_2))\\
&= \left(\mathrm{ad}_\alpha x_1^{[p]_1}(x_2),\mathrm{ad}_\beta y_1^{[p]_2}(y_2)\right)\\
&= \left([x_1^{[p]_1},\alpha(x_2)]_L,[y_1^{[p]_2},\beta(y_2)]_\Gamma\right)\\
&= \left[\left(x_1^{[p]_1},y_1^{[p]_2}\right),(\alpha(x_2),\beta(y_2))\right]_{L\oplus\Gamma}\\
&= \mathrm{ad}_\gamma(x_1,y_1)^{[p]}(x_2,y_2).
\end{aligned}
$$

This proves that $(\mathrm{ad}_\gamma(x_1,y_1))^{[p]} = \mathrm{ad}_\gamma(x_1,y_1)^{[p]}$. For $k \in F, (x,y) \in L^0 \oplus \Gamma^0$, we obtain

$$(k(x,y))^{[p]} = \left((kx)^{[p]_1},(ky)^{[p]_2}\right) = \left(k^p x^{[p]_1}, ky^{[p]_2}\right) = k^p\left(x^{[p]_1},y^{[p]_2}\right) = k^p(x,y)^{[p]}.$$

Finally, for $(x_1,y_1),(x_2,y_2) \in L^0 \oplus \Gamma^0$, one gets

$$
\begin{aligned}
((x_1,y_1)+(x_2,y_2))^{[p]} &= (x_1+x_2,y_1+y_2)^{[p]}\\
&= \left((x_1+x_2)^{[p]_1},(y_1+y_2)^{[p]_2}\right)\\
&= \left(x_1^{[p]_1}+x_2^{[p]_1}+\sum_{i=1}^{p-1}s_i(x_1,x_2), y_1^{[p]_2}+y_2^{[p]_2}+\sum_{i=1}^{p-1}s_i(y_1,y_2)\right)\\
&= \left(x_1^{[p]_1},y_1^{[p]_2}\right)+\left(x_2^{[p]_1},y_2^{[p]_2}\right)+\left(\sum_{i=1}^{p-1}s_i(x_1,x_2),\sum_{i=1}^{p-1}s_i(y_1,y_2)\right)\\
&= (x_1,y_1)^{[p]}+(x_2,y_2)^{[p]}+\sum_{i=1}^{p-1}s_i\left((x_1,y_1),(x_2,y_2)\right).
\end{aligned}
$$

□

Corollary 4.1. *Suppose that* $(L_1, [\ ,\ ]_{L_1}, \alpha_1.[p]_1), \dots, (L_n, [\ ,\ ]_{L_n}, \alpha_n, [p]_n)$ *are restricted Hom-Lie superalgebras. Then* $(L_1 \oplus \cdots \oplus L_n, [\ ,\ ]_{L_1 \oplus \cdots \oplus L_n}, \gamma, [p])$ *is a restricted Hom-Lie superalgebra, where the bilinear map* $[\ ,\ ]_{L_1 \oplus \cdots \oplus L_n}$ *is defined on homogenous elements by*

$$[(x_1, \dots, x_n), (y_1, \dots, y_n)]_{L_1 \oplus \cdots \oplus L_n} = ([x_1, y_1]_{L_1}, \dots, [x_n, y_n]_{L_n}),$$

and the linear map $\gamma : L_1 \oplus \cdots \oplus L_n \to L_1 \oplus \cdots \oplus L_n$ *is defined as*

$$\gamma(x_1, \dots, x_n) = (\alpha_1(x_1), \dots, \alpha_n(x_n)).$$

Also, the p-map $[p] : (L_1 \oplus \cdots \oplus L_n)^0 \to (L_1 \oplus \cdots \oplus L_n)^0$ *is given by*

$$(x_1, \dots, x_n)^{[p]} = \left(x_1^{[p]_1}, \dots, x_n^{[p]_n}\right).$$

5. On morphisms of Hom-Lie superalgebras

Definition 5.1 ([1])**.** Let $(L, [\ ,\ ]_L, \alpha)$ and $(\Gamma, [\ ,\ ]_\Gamma, \beta)$ be two Hom-Lie superalgebras. An even superspace $\varphi : L \to \Gamma$ is said to be a morphism of Hom-Lie superalgebras if

$$[\varphi(u), \varphi(v)]_\Gamma = \varphi([u, v]_L)\ \forall u, v \in L \text{ and } \varphi \circ \alpha = \beta \circ \varphi.$$

If $\varphi : L \to \Gamma$ is an even superspace, then the graph of φ is the set

$$G_\varphi = \{(x, \varphi(x) \mid x \in L\} \subseteq L \oplus \Gamma.$$

Definition 5.2. A morphism of Hom-Lie superalgebra

$$\varphi : (L, [\ ,\ ]_L, \alpha, [p]_1) \to (\Gamma, [\ ,\ ]_\Gamma, \beta, [p]_2)$$

is said to be restricted if $\varphi\left(x^{[p]_1}\right) = (\varphi(x))^{[p]_2}\ \forall x \in L^0$.

Theorem 5.1. *A linear map* $\varphi : (L, [\ ,\ ]_L, \alpha, [p]_1) \to (\Gamma, [\ ,\ ]_\Gamma, \beta, [p]_2)$ *is a restricted morphism of Hom-Lie superalgebras if and only if the graph* G_φ *is a restricted Hom-Lie subsuperalgebra of* $(L \oplus \Gamma, [\ ,\ ]_{L\oplus\Gamma}, \gamma, [p])$ *where* $[\ ,\ ]_{L\oplus\Gamma}, \gamma$*, and* $[p]$ *are defined as in Theorem 4.2.*

Proof. Suppose $\varphi : (L, [\ ,\ ]_L, \alpha, [p]_1) \to (\Gamma, [\ ,\ ]_\Gamma, \beta, [p]_2)$ is a restricted morphism of Hom-Lie superalgebras. Let $(x_1, \varphi(x_1)), (x_2, \varphi(x_2)) \in G_\varphi$. Then

$$\begin{aligned}[(x_1, \varphi(x_1)), (x_2, \varphi(x_2))]_{L\oplus\Gamma} &= ([x_1, x_2]_L, [\varphi(x_1), \varphi(x_2)]_\Gamma) \\ &= ([x_1, x_2]_L, \varphi([x_1, x_2])) \in G_\varphi.\end{aligned}$$

This shows G_φ is closed under the bracket operation. For $(x_1, \varphi(x_1)) \in G_\varphi$, we have

$$\gamma(x_1, \varphi(x_1)) = (\alpha(x_1), \beta(\varphi(x_1))) = (\alpha(x_1), \varphi(\alpha(x_1))) \in G_\varphi.$$

So far we have shown that G_φ is a Hom-Lie subsuperalgebra. For $(x_1, \varphi(x_1)) \in G_\varphi^0$, we have

$$\begin{aligned}(x_1, \varphi(x_1))^{[p]} &= \left(x_1^{[p]_1}, (\varphi(x_1))^{[p]_2}\right) \\ &= \left(x_1^{[p]_1}, \varphi\left(x_1^{[p]_1}\right)\right) \in G_\varphi^0.\end{aligned}$$

Conversely, suppose G_φ is a restricted Hom-Lie subsuperalgebra of $(L\oplus\Gamma, [\ ,\ ]_{L\oplus\Gamma}, \gamma, [p])$. Let $x_1, x_2 \in L$. Then $[(x_1, \varphi(x_1)), (x_2, \varphi(x_2))]_{L\oplus\Gamma} = ([x_1, x_2]_L, [\varphi(x_1), \varphi(x_2)]_\Gamma) \in G_\varphi$, and so $\varphi([x_1, x_2]_L) = [\varphi(x_1), \varphi(x_2)]_\Gamma$. For $x \in L$, we have $\gamma(x, \varphi(x)) = (\alpha(x), \beta(\varphi(x))) \in G_\varphi$, so that $\varphi(\alpha(x)) = \beta(\varphi(x))$. Finally, if $x \in L^0$, then we have $(x, \varphi(x))^{[p]} = (x^{[p]_1}, (\varphi(x))^{[p]_2}) \in G_\varphi$, and so $\varphi(x^{[p]_1}) = (\varphi(x))^{[p]_2}$. □

We extend, in the following theorem, the result in [1] to restricted Lie superalgebras case. It gives a way to construct restricted Hom-Lie superalgebras starting from a restricted Lie superalgebra and an even superalgebra endomorphism.

Theorem 5.2. *Let $(L, [\ ,\ ], [p])$ be a restricted Lie superalgebra and $\alpha : L \to L$ be an even Lie superalgebra endomorphism. Then $(L, [\ ,\ ]_\alpha, \alpha, [p]|_{L^0})$, where $[x, y]_\alpha = \alpha([x, y])$ is a restricted Hom-Lie superalgebra.*

Proof. The bracket is obviously super skew-symmetric. Let $x, y, z \in L$ be homogenous elements. With a direct computation, we have

$$\begin{aligned}
& (-1)^{|x||z|}[\alpha(x),[y,z]_\alpha]_\alpha + (-1)^{|z||y|}[\alpha(z),[x,y]_\alpha]_\alpha + (-1)^{|y||x|}[\alpha(y),[z,x]_\alpha]_\alpha \\
= \ & (-1)^{|x||z|}\alpha^2([x,[y,z]]) + (-1)^{|z||y|}\alpha^2([z,[x,y]]) + (-1)^{|y||x|}\alpha^2([y,[z,x]]) \\
= \ & \alpha^2\left((-1)^{|x||z|}[x,[y,z]] + (-1)^{|z||y|}[z,[x,y]] + (-1)^{|y||x|}[y,[z,x]]\right) = 0.
\end{aligned}$$

Now, the Hom-Lie superalgebra $(L, [\ ,\ ]_\alpha, \alpha)$ is multiplicative. Indeed, for homogeneous elements $x, y \in L$, we have $\alpha([x,y]_\alpha) = \alpha^2([x,y]) = [\alpha(x), \alpha(y)]_\alpha$.
Next, we check the three conditions given in Definition 3.1. For $x \in L^0$ and $z \in L$, we have

$$\begin{aligned}
(\mathrm{ad}_\alpha x)^p(z) &= (\mathrm{ad}_\alpha x)^{p-1}(\mathrm{ad}_\alpha x)(z) \\
&= (\mathrm{ad}_\alpha x)^{p-1}([x,\alpha(z)]_\alpha) \\
&= (\mathrm{ad}_\alpha x)^{p-1}([x,\alpha^2(z)]) \\
&= (\mathrm{ad}_\alpha x)^{p-2}\left([x,[x,\alpha^2(z)]]_\alpha\right) \\
&= (\mathrm{ad}_\alpha x)^{p-2}([x,[x,\alpha^3(z)]]) \\
&\ \ \vdots \\
&= [x,[x,\cdots,[x,\alpha^{p+1}(z)]\cdots]] \\
&= (\mathrm{ad} x)^p(\alpha(z)) \\
&= \mathrm{ad}\left(x^{[p]}\right)(\alpha(z)) \\
&= \left[x^{[p]}, \alpha(z)\right] \\
&= \mathrm{ad}_\alpha x^{[p]|_{L^0}}(z).
\end{aligned}$$

The second and the third properties are clear. □

Theorem 5.3. *Let* $(L, [\ ,\ ]_L, [p]_1)$ *and* $(\Gamma, [\ ,\ ]_\Gamma, [p]_2)$ *be restricted Lie superalgebras,* $\alpha : L \to L$ *and* $\beta : \Gamma \to \Gamma$ *be an even Lie superalgebra endomorphisms, and* $f : L \to \Gamma$ *be a morphism of restricted Lie superalgebras with* $f \circ \alpha = \beta \circ f$*. Then*

$$f : (L, [\ ,\ ]_\alpha, \alpha, [p]_1) \to (\Gamma, [\ ,\ ]_\beta, \beta, [p]_2),$$

where $[\ ,\ ]_\alpha$ *and* $[\ ,\ ]_\beta$ *are defined as in Theorem 5.2, is a morphism of restricted Hom-Lie superalgebras.*

Proof. If $u, v \in L$, we have

$$f([u,v]_\alpha) = f(\alpha[u,v]) = \beta \circ f([u,v]) = \beta([f(u), f(v)]) = [f(u), f(v)]_\beta.$$

Also. for $u \in L^0$ and using $L^0 \subseteq L_0$, we obtain $f\left(u^{[p]_1}\right) = (f(u))^{[p]_2}$. □

Theorem 5.4. *Let* $(L, [\ ,\ ]_L, \alpha)$ *and* $(\Gamma, [\ ,\ ]_\Gamma, \beta)$ *be multiplicative Hom-Lie superalgebras, and* φ *be a one to one morphism of Hom-Lie superalgebras. If* C *is a Hom-Lie subsuperalgebra of* Γ *and* $[p] : C^0 \to C^0$ *is a p-map, then* $\left(\varphi^{-1}(C), [\ ,\ ]_L, \alpha|_{\varphi^{-1}(C)}, [p]'\right)$*, where*

$$[p]' : \left(\varphi^{-1}(C)\right)^0 \to \left(\varphi^{-1}(C)\right)^0;\ x \mapsto \varphi^{-1}\left((\varphi(x))^{[p]}\right),$$

is a restricted Hom-Lie superalgebra.

Proof. First, we show $\varphi^{-1}(C)$ is a Hom-Lie subsuperalgebra of L. For $x_1, x_2 \in \varphi^{-1}(C)$, there exist $y_1, y_2 \in C$ with $\varphi(x_1) = y_1$ and $\varphi(x_2) = y_2$. Now, $\varphi([x_1, x_2]) = [y_1, y_2]$, implies $[x_1, x_2] \in \varphi^{-1}(C)$. Also, $\alpha(x_1) \in \varphi^{-1}(C)$ follows from $\beta(\varphi(x_1)) = \varphi(\alpha(x_1))$. Next, if $x \in \left(\varphi^{-1}(C)\right)^0$, then it is clear that $\varphi(x) \in C^0$. Also, for $k \in F$, we have

$$\begin{aligned}(kx)^{[p]'} &= \varphi^{-1}\left((\varphi(kx))^{[p]}\right)\\ &= \varphi^{-1}\left((k\varphi(x))^{[p]}\right)\\ &= \varphi^{-1}\left(k(\varphi(x))^{[p]}\right)\\ &= k\left(\varphi(x)\right)^{[p]} = kx^{[p]'}.\end{aligned}$$

Also, for $x_1, x_2 \in \left(\varphi^{-1}(C)\right)^0$ we have

$$\begin{aligned}(x_1+x_2)^{[p]'} &= \varphi^{-1}\left(\varphi(x_1)+\varphi(x_2)\right)^{[p]}\\ &= \varphi^{-1}\left(\varphi(x_1)+\varphi(x_2)\right)^{[p]}\\ &= \varphi^{-1}\left((\varphi(x_1))^{[p]}+(\varphi(x_2))^{[p]}+\sum_{i=1}^{p-1}s_i(\varphi(x_1),\varphi(x_2))\right)\\ &= \varphi^{-1}\left((\varphi(x_1))^{[p]}\right)+\varphi^{-1}\left((\varphi(x_2))^{[p]}\right)+\varphi^{-1}\left(\sum_{i=1}^{p-1}s_i(\varphi(x_1),\varphi(x_2))\right)\\ &= x_1^{[p]'}+x_2^{[p]'}+\sum_{i=1}^{p-1}s_i(x_1,x_2).\end{aligned}$$

Finally, let $x_1 \in \left(\varphi^{-1}(C)\right)^0$ and $x_2 \in \phi^{-1}(C)$. There exist $y_1 \in C^0$ and $y_2 \in C$ with $\varphi(x_1) = y_1$ and $\varphi(x_2) = y_2$. With a direct computation, we have

$$\begin{aligned}\mathrm{ad}_\alpha x^{[p]'}(x_2) &= \left[x_1^{[p]'}, \alpha(x_2)\right]\\ &= \left[\varphi^{-1}\left((\varphi(x_1))^{[p]}\right), \alpha(x_2)\right]\\ &= \varphi^{-1}\left[(\varphi(x_1))^{[p]}, \varphi\circ\alpha(x_2)\right]\\ &= \varphi^{-1}\left[y_1^{[p]}, \beta(y_2)\right]\\ &= \varphi^{-1}\left(\mathrm{ad}_\beta y_1^{[p]}\right)(y_2)\\ &= \varphi^{-1}\left(\mathrm{ad}_\beta y_1\right)^p(y_2)\\ &= \varphi^{-1}\left(\mathrm{ad}_\beta y_1\right)^{p-1}\left[\varphi(x_1), \beta(\varphi(x_2))\right]\end{aligned}$$

$$\begin{aligned}
&= \varphi^{-1} \left(\mathrm{ad}_\beta y_1\right)^{p-1} \left[\varphi(x_1), \varphi(\alpha(x_2))\right] \\
&\vdots \\
&= \varphi^{-1}[\varphi(x_1), [\varphi(x_1), [\cdots, [\varphi(x_1), \varphi(\alpha^p(x_2))] \cdots]]] \\
&= [x_1, [x_1, \cdots, [x_1, \alpha^p(x_2)] \cdots]] \\
&= \left(\mathrm{ad}_\alpha x_1\right)^p (x_2).
\end{aligned}$$

□

Theorem 5.5. *Suppose that $(L, [\ ,\ ]_L, \alpha, [p])$ is a restricted Hom-Lie superalgebra, $(\Gamma, [\ ,\ ]_\Gamma, \beta)$ is a multiplicative Hom-Lie superalgebra, and $\varphi : L \to \Gamma$ is a one to one morphism of Hom-Lie superalgebras. Then $(\varphi(L), [\ ,\ ]_\Gamma, \beta|_{\varphi(L)}, [p]')$ is a restricted Hom-Lie superalgebra, where*

$$[p]' : (\varphi(L))^0 \to (\varphi(L))^0;\ \varphi(x) \mapsto \varphi\left(x^{[p]}\right).$$

Proof. For $\varphi(x_1), \varphi(x_2) \in \varphi(L)$, we have $[\varphi(x_1), \varphi(x_2)] = \varphi([x_1, x_2]) \in \varphi(L)$ and $\beta(\varphi(x_1)) = \varphi(\alpha(x_1)) \in \varphi(L)$. For $\varphi(x) \in (\varphi(L))^0$, we have $\beta(\varphi(x)) = \varphi(\alpha(x)) = \varphi(x)$. Since φ is one to one we have $\alpha(x) = x$. Thus, $x \in L^0$. Next, we check the three conditions given in Definition 3.1. For $\varphi(x) \in (\varphi(L))^0$ and $k \in F$, we obtain

$$(k\varphi(x))^{[p]'} = (\varphi(kx))^{[p]'} = \varphi\left((kx)^{[p]}\right) = \varphi\left(kx^{[p]}\right) = k\varphi\left(x^{[p]}\right) = k(\varphi(x))^{[p]'}.$$

If $\varphi(x_1), \varphi(x_2) \in (\varphi(L))^0$, then we have

$$\begin{aligned}
(\varphi(x_1) + \varphi(x_2))^{[p]'} &= (\varphi(x_1 + x_2))^{[p]'} \\
&= \varphi\left((x_1 + x_2)^{[p]}\right) \\
&= \varphi\left(x_1^{[p]} + x_2^{[p]} + \sum_{i=1}^{p-1} s_i(x_1, x_2)\right) \\
&= \varphi\left(x_1^{[p]}\right) + \varphi\left(x_2^{[p]}\right) + \varphi\left(\sum_{i=1}^{p-1} s_i(x_1, x_2)\right)
\end{aligned}$$

$$= (\varphi(x_1))^{[p]'} + (\varphi(x_2))^{[p]'} + \sum_{i=1}^{p-1} s_i(\varphi(x_1), \varphi(x_2)).$$

In addition, for $\varphi(x_1) \in (\varphi(L))^0$ and $\varphi(x_2) \in \varphi(L)$, we obtain

$$\begin{aligned}
\mathrm{ad}_\beta(\varphi(x_1))^{[p]'}(\varphi(x_2)) &= \left[(\varphi(x_1))^{[p]'}, \beta(\varphi(x_2))\right] \\
&= \left[(\varphi(x_1))^{[p]'}, \varphi(\alpha(x_2))\right] \\
&= \varphi\left(\left[x_1^{[p]}, \alpha(x_2)\right]\right) \\
&= \varphi\left(\mathrm{ad}_\alpha x_1^{[p]}\right)(x_2) \\
&= \varphi\left((\mathrm{ad}_\alpha x_1)^p\,(x_2)\right) \\
&= \varphi\left([x_1, [x_1, \cdots, [x_1, \alpha^p(x_2)] \cdots]]\right) \\
&= [\varphi(x_1), [\varphi(x_1), \cdots, [\varphi(x_1), \varphi(\alpha^p(x_2))] \cdots]] \\
&= [\varphi(x_1), [\varphi(x_1), \cdots, [\varphi(x_1), \beta^p(\varphi(x_2))] \cdots]] \\
&= (\mathrm{ad}_\beta \varphi(x_1))^p\,(\varphi(x_2)).
\end{aligned}$$

□

6. Restrictable Hom-Lie superalgebra

Definition 6.1. A multiplicative Hom-Lie superalgebra $(L, [\ ,\ ], \alpha)$ is called restrictable if $(\mathrm{ad}_\alpha x)^p \in \mathrm{ad}_\alpha L^0$ for all $x \in L^0$, where $\mathrm{ad}_\alpha L^0 = \{\mathrm{ad}_\alpha x \mid x \in L^0\}$.

The following theorem was obtained by Guan and Chen in [4] in the setting of Lie algebras. We extend it to Lie superalgebra case.

Theorem 6.1. *A multiplicative Hom-Lie superalgebra is restrictable if and only if there is a p-map $[p] : L^0 \to L^0$ which makes L a restricted Hom-Lie superalgebra.*

Proof. Suppose $(L, [\ ,\ ], \alpha, [p])$ is a restricted Hom-Lie superalgebra. Let $x \in L^0$. Then $(\mathrm{ad}_\alpha x)^p = \mathrm{ad}_\alpha x^{[p]} \in \mathrm{ad}_\alpha L^0$. Conversely, suppose that $(L, [\ ,\ ], \alpha)$ is restrictable.

Let $\{e_j \mid j \in J\}$ be a basis of L^0. Then for each $j \in J$, there exists $y_j \in L^0$ such that $(\mathrm{ad}_\alpha e_j)^p = \mathrm{ad}_\alpha y_j$. Now, $e_j^p - y_j \in C(L^0)$, where

$$e_j^p = [e_j, [e_j, \cdots, [e_j, e_j] \cdots]],$$

for if $z \in L^0$, then $(\mathrm{ad}_\alpha e_j)^p(z) - \mathrm{ad}_\alpha y_j(z) = [e_j^p - y_j, \alpha(z)] = 0$. Define a function

$$f : L^0 \to C(L^0);\ \sum c_i e_i \mapsto \sum c_i^p (y_i - e_i^p).$$

Clearly f is a p-semilinear map. Set

$$W = \{x \in L^0 \mid x^p + f(x) \in L^0\}.$$

Then W is a subspace of L^0. Indeed, if $x, y \in W$ and $a, b \in F$, then

$$(ax + by)^p + f(ax + by) = a^p x^p + b^p y^p + \sum_{i=1}^{p-1} s_i(ax, by) + k^p f(x) + f(by) \in L^0.$$

Since $e_j^p + f(e_j) \in L^0$, it follows $x^p + f(x) \in L^0$ for all $x \in L^0$. By Theorem 3.1, we have

$$[p] : L^0 \to L^0;\ x \mapsto x^p + f(x)$$

is a p-map with $e_j^{[p]} = y_j$. □

Let $(L, [\ ,\ ], \alpha)$ be a Hom-Lie superalgebra, and let U and W be subspaces of L. Then L is a direct sum of U and W, and we write $L = U \oplus W$, if $L = U + W$ and $U \cap W = \{0\}$. The subspace U is a Hom-Lie ideal of L if $\alpha(U) \subseteq U$ and $[x, y] \in U$ for all $x \in U$ and $y \in L$. We have the following result.

Theorem 6.2. *Let $(L, [\ ,\ ], \alpha)$ be a Hom-Lie superalgebra and let U and W be Hom-Lie ideals of L with $L = U \oplus W$. Then, L is restrictable if and only if U and W are restrictable.*

Proof. Suppose L is restrictable. Then by Theorem 6.1, we have L is restricted. Now, the result follows from the trivial fact that a subalgebra of a restricted Hom-Lie superalgebra is restricted and Theorem 6.1. Conversely, suppose that U and V are restrictable. Let $x = x_1 + x_2$, where $x_1 \in U$ and $x_2 \in W$, be an element of L^0. Then $\alpha(x) = \alpha(x_1) + \alpha(x_2) = x_1 + x_2$, and so $x_1 \in U^0$ and $x_2 \in W^0$. As U and W are restrictable, there exist $y_1 \in Y^0$ and $y_2 \in W^0$ such that $(\mathrm{ad}_\alpha x_1)^p = \mathrm{ad}_\alpha y_1$ and $(\mathrm{ad}_\alpha x_2)^p = y_2$. Now,

$$\begin{aligned}(\mathrm{ad}_\alpha(x_1 + x_2))^p &= (\mathrm{ad}_\alpha x_1 + \mathrm{ad}_\alpha x_2)^p \\ &= (\mathrm{ad}_\alpha x_1)^p + (\mathrm{ad}_\alpha x_2)^p \\ &= \mathrm{ad}_\alpha y_1 + \mathrm{ad}_\alpha y_2 \\ &= \mathrm{ad}_\alpha(y_1 + y_2).\end{aligned}$$

Therefore, L is restrictable. □

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