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ON CLASSIFICATION OF FACTORABLE SURFACES IN GALILEAN 3-SPACE $\mathbb{G}^3$

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Abstract. In this paper, we study factorable surfaces in Galilean 3-space $\mathbb{G}^3$. Then we describe, up to a congruence, factorable surfaces and the several results in this respect are obtained. In particular, factorable surfaces in terms of an isometric immersion, finite type Gauss map and the pointwise 1-type Gauss map of the surfaces are considered and the characterization results on the factorable surfaces with respect to these conditions are obtained.

1. Introduction

Let $\mathbb{M}$ be a connected n-dimensional submanifold of m-dimensional Euclidean space $\mathbb{E}^m$, equipped with the induced metric. Then, whenever the position vector x of $\mathbb{M}$ in $\mathbb{E}^m$ can be decomposed as a finite sum of $\mathbb{E}^m$- valued eigenfunctions of Δ, we say that $\mathbb{M}$ is of *finite type*, where Δ is the Laplacian of $\mathbb{M}$ with respect to the induced metric. Now, $\mathbb{M}$ is said to be of *k-type* if the position vector x of $\mathbb{M}$ in $\mathbb{E}^m$ can be expressed in the following form [1]:

$$x = x_0 \ + \ x_{i_1} \ + \ ... \ + \ x_{i_k},$$

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where x_0 is a constant vector, and x_{i_j} (j = 1, ..., k) are non-constant $\mathbb{E}^m$-valued functions on $\mathbb{M}$ satisfying

$$\Delta X_{i_j} = \lambda_{i_j} X_{i_j},\ \lambda_{i_j} \in \mathbb{R},\ \lambda_{i_1} < ... < \lambda_{i_k}. \tag{1.1}$$

We can consider a submanifold $\mathbb{M}$ of $\mathbb{E}^m$ whose coordinate functions are eigenfunctions of the Laplacian of $\mathbb{M}$, that is,

$$\Delta X_i = \lambda_i X_i.$$

Such submanifold is said to be a *coordinate finite type submanifold* [11]. Bekkar et al. [9] and Baba-Hamed et al. [4] studied respectively, coordinate finite type surfaces of revolution and translation surfaces in a 3-dimensional Minkowski space $\mathbb{E}^3_1$. In [11], Garay proved that coordinate finite type hypersurfaces are minimal in $\mathbb{E}^m$ and an open pieces of either round hyperspheres or generalized right spherical cylinders. Moreover, he studied the hypersurfaces in an m-dimensional Euclidean space which satisfy the condition

$$\Delta X = AX + B,\ A : \mathbb{R}^m \to \mathbb{R}^m \text{ is an endomorphism, } B \in \mathbb{R}^{m\times 1}.$$

Later, Chen and Piccinni [2] gave the idea of Gauss map $\mathbf{U}$ on submanifolds in Euclidean space. Dillen et al. [6], Baikoussi and Blair [3] respectively studied surfaces of revolution and ruled surfaces in Euclidean 3-space such that their Gauss map $\mathbf{U}$ satisfies

$$\Delta \mathbf{U} = A\mathbf{U}.$$

On the other hand, Aydin et al. found some geometric properties concerning factorable surface in pseudo-galilean space [8] and *Š*ipu*š* et al. studied some constant curvature properties in [14]. Yoon studied classification of translation surfaces in Galilean 3-space [5]. Also in ([7], [13]), authors studied ruled surfaces and factorable surfaces in Minkoswki space. Furthermore, one can note that the Laplacian of the

Gauss map of some known surfaces, a helicoid, a catenoid, a right cone in Euclidean 3-space take the form, namely

$$\Delta \mathbf{U} = \phi(\mathbf{U} + C),$$

for a smooth function ϕ and a constant vector C. These surfaces satisfying above equation are said to have *pointwise 1-type Gauss map*. In particular, a pointwise 1-type Gauss map is said to be of the *first kind* if $C = 0$, otherwise, it is said to be of the *second kind*. On the other hand, if a function ϕ is constant, the Gauss map of surface is called of *1-type*.

In this paper, we obtain characterization theorems on factorable surfaces satisfying Laplacian with respect to the first fundamental form (or with respect to the induced metric) in terms of an isometric immersion, finite type Gauss map and the pointwise 1-type Gauss map of the surface.

2. Preliminaries

The Galilean space $\mathbb{G}^3$ is a Cayley-Klein space defined from a 3-dimensional projective space $P(\mathbb{R}^3)$ with the *absolute figure* that consists of an ordered triple ω, f, I, where ω is the ideal (*absolute*) *plane*, f the line (*absolute line*) in ω and I the fixed elliptic involution of points of f. We discuss homogenous coordinates in $\mathbb{G}^3$ in such a way that the absolute plane ω is given by $x_0 = 0$, the absolute line f by $x_0 = x_1 = 0$ and the elliptic involution by $(x_0 : x_1 : x_2 : x_3)(0 : 0 : x_3 : -x_2)$. In affine coordinates defined by $(x_0 : x_1 : x_2 : x_3) = (1 : x : y : z)$, the distance between the points $P_i = (x_i, y_i, z_i),\ i = 1,\ 2$ is defined by

$$d(P_1, P_2) = \begin{cases} |x_2 - x_1|, & \text{if } x_1 \neq x_2 \\ \sqrt{(y_2 - y_1)^2 + (z_2 - z_1)^2}, & \text{if } x_1 = x_2. \end{cases}$$

The *group of motions* of $\mathbb{G}^3$ is a six-parameter group given (in affine coordinates) by ([10], [15])

$$\bar{x} = a + x$$

$$\bar{y} = b + cx + y\cos\varphi + z\sin\varphi$$
$$\bar{z} = d + ex - y\sin\varphi + z\cos\varphi,$$

where $a, b, c, d, e, \varphi \in \mathbb{R}$.

With respect to the absolute figure, there are two types of lines in the Galilean space, *isotropic lines* which intersect the absolute line f and *non-isotropic lines* which do not. A plane is called *Euclidean* if it contains f, otherwise it is called *isotropic*. In the given affine coordinates, isotropic vectors are of the form $(0,\ y,\ z)$. For further study about this Galilean geometry in detail (see [12]).

A C^r-surface $\mathbb{M}(r \geq 1)$, immersed in the Galilean space $\mathbb{G}^3$ and C^r-mapping $X:\ S \subset \mathbb{R}^2 \to \mathbb{G}^3$, satisfying $\mathbb{M} = X(S)$ which is regularly parameterized by

$$X(u,\ v) = (x(u,v),\ y(u,v),\ z(u,v)),$$

has the following first fundamental form

$$I = (g_1 du + g_2 dv)^2 + \epsilon(h_{11}du^2 + 2h_{12}dudv + h_{22}dv^2), \tag{2.1}$$

where the symbols $g_i = x_i,\ h_{ij} = \tilde{X}_i.\tilde{X}_j$ stands for derivatives of the first coordinate function $x(u,v)$ with respect to $u,\ v$ and for the Euclidean scalar product of the projections $\tilde{X}_k$ of vectors X_k onto the yz-plane, respectively. Furthermore,

$$\epsilon = \begin{cases} 0, & \text{if direction } du : dv \text{ is nonisotropic;} \\ 1, & \text{if direction } du : dv \text{ is isotropic.} \end{cases}$$

A surface is called *admissible* if it has no Euclidean tangent planes. Therefore, for an admissible surface either $g_1 \neq 0$ *or* $g_2 \neq 0$ holds. An admissible surface can always locally be expressed as

$$z = f(x, y).$$

The Gaussian curvature K and the Mean curvature H are defined by [15]

$$K = \frac{LN - M^2}{W^2} \text{ and } H = \frac{g_2^2 L - 2g_1 g_2 M + g_1^2 N}{2W^2},$$

where

$$L_{ij} = \frac{x_1 X_{ij} - x_{ij} X_1}{x_1}.\mathbf{U}, \; x_1 = g_1 \neq 0.$$

We use $L_{ij}(i, \; j = 1, \; 2)$ for $L, \; M, \; N$ ahead and $\mathbf{U}$ define the Gauss map of the surface by

$$\mathbf{U} = \frac{1}{W}(0, \; -x_2 z_1 + z_2 x_1, \; x_2 y_1 - y_2 x_1),$$

where $W^2 = (x_2 X_1 - x_1 X_2)^2$. Let $X = (x, y, z)$ and $Y = (\tilde{x}, \tilde{y}, \tilde{z})$ be vectors in Galilean space $\mathbb{G}^3$. A vector X is called *isotropic* if $x = 0$, otherwise it is called *non-isotropic*. The *Galilean scalar product* of X and Y is defined by [10]

$$\langle X, Y \rangle = \begin{cases} x\tilde{x}, & \text{if } x \neq 0 \text{ or } \tilde{x} \neq 0 \\ y\tilde{y} + z\tilde{z}, & \text{if } x = 0 \text{ and } \tilde{x} = 0. \end{cases}$$

From this, the Galilean norm of a vector X in $\mathbb{G}^3$ is given by $\|X\| = \sqrt{\langle X, \; X \rangle}$ and all unit non-isotropic vectors are the form $(1, \; y, \; z)$.

The *Galilean cross product* of X and Y on $\mathbb{G}^3$ defined by [10]

$$X \times Y = \begin{vmatrix} 0 & e_2 & e_3 \\ x & y & z \\ \tilde{x} & \tilde{y} & \tilde{z} \end{vmatrix},$$

where $e_2 = (0, \; 1, \; 0)$ and $e_3 = (0, \; 0, \; 1)$.

Let $u_1 (= u), \; u_2 (= v)$ be a local coordinate system of $\mathbb{M}$. For the components $g_{ij}(i, j = 1, 2)$ of the first fundamental form I on M, we denote by (g^{ij}) (respectively D) the inverse (respectively the determinant) of the matrix (g_{ij}). The Laplacian operator Δ of the first fundamental form I on $\mathbb{M}$ is defined by

$$\Delta = -\frac{1}{\sqrt{D}} \sum_{i=1}^{2} \frac{\partial}{\partial u_i} \left(\sqrt{D} g^{ij} \frac{\partial}{\partial u_j} \right) .$$

3. Factorable surfaces in $\mathbb{G}^3$

Suppose the factorable surface $\mathbb{M}$ in Galilean space $\mathbb{G}^3$ admits the parametric representation

$$X(u, v) = (u, \; v, \; f(u) g(v)), \tag{3.1}$$

where $f(u)$ and $g(v)$ are smooth functions.

Then, we have the frame X_u, X_v given by

$$X_u = (1,\ 0,\ f'(u)g(v)) \text{ and } X_v = (0,\ 1,\ f(u)g'(v)),$$

where $f' = \frac{df}{du}$ and $g' = \frac{dg}{dv}$.

Next, the Gauss map $\mathbf{U}$ is given by

$$\mathbf{U} = \frac{1}{\sqrt{1+f^2g'^2}}(0, -fg', 1).$$

Accordingly, we also have the coefficients of the first and second fundamental form on $\mathbb{M}$ by

$$E = 1,\ F = 0 \text{ and } G = 1 + f^2g'^2 = W, \tag{3.2}$$

$$L = \frac{f''g}{\sqrt{W}},\ M = \frac{f'g'}{\sqrt{W}} \text{ and } N = \frac{fg''}{\sqrt{W}},$$

where we put $EG - F^2 = 1 + f^2g'^2 = W$.

Then, the Gaussian and the Mean curvature are given by

$$K = \frac{ff''gg'' - (f'^2g'^2)}{W^2} \text{ and } H = \frac{fg''}{2W^{3/2}}$$

respectively.

4. Finite type Factorable surfaces in $\mathbb{G}^3$

In this section, we explore the classification of factorable surfaces in $\mathbb{G}^3$ satisfying

$$\Delta X_i = \lambda_i X_i,\ \lambda_i \in \mathbb{R}. \tag{4.1}$$

Now using (3.2) and the frame X_u, X_v, one can get the Laplacian

$$\Delta X = -\frac{1}{W^2}\begin{pmatrix} ff'g'^2W \\ -f^2g'g'' \\ W(ff'^2gg'^2 + f''g + f^2f''gg'^2) + fg'' \end{pmatrix} \tag{4.2}$$

Now using (4.2), the relation (4.1) is found to be equivalent to the following system of ordinary differential equations,

$$-ff'g'^2 = \lambda_1 uW, \tag{4.3}$$

$$f^2g'g'' = \lambda_2 vW^2, \tag{4.4}$$

$$W(ff'^2gg'^2 \ + \ f''g \ + \ f^2f''gg'^2) \ + \ fg'' = -\lambda_3 fgW^2. \tag{4.5}$$

Now the problem of classifying the factorable surface $\mathbb{M}$ in $\mathbb{G}^3$ which satisfy equation (3.1) is reduced to the integration of the above system of ODE's. Next, we study this system of ODE's by concerning values to (eigenvalues) λ_i $(i = 1,\ 2,\ 3)$.

case 1 : We assume that $\mathbb{M}$ satisfies the condition $\Delta X = 0$. We call such surface, a harmonic surface. In this case, (4.3), (4.4), (4.5) reduces to

$$f'g' = 0\ , \tag{4.6}$$

$$g'g'' = 0 \text{ and} \tag{4.7}$$

$$W(ff'^2gg'^2 + f''g + f^2f''gg'^2) + fg'' = 0. \tag{4.8}$$

subcase (1.1) : Let $f =$ constant and $g =$ constant. Thus, the coefficients of second fundamental form vanish. Hence, the surface is totally geodesic with parabolic points and is given by $X(u, v) = (u,\ v,\ c_1),\ c_1 \in \mathbb{R}$.

subcase (1.2) : Let $f =$ constant. Consequently, g is linear function of v with $g' \neq 0$. Also, the coefficients of second fundamental form vanish. Then the surface is totally geodesic with parabolic points and given by $X(u, v) = (u,\ v,\ c_1v + c_2)$, $c_1, c_2 \in \mathbb{R}$.

subcase (1.3) : Let $g =$ constant. Consequently, f is linear function with $f' \neq 0$. Thus, the coefficients of second fundamental form vanish. Hence, again the surface is totally geodesic with parabolic points and is given by $X(u,v) = (u,\ v,\ c_1u + c_2)$, $c_1,\ c_2 \in \mathbb{R}$.

Thus, we have the following theorem.

Theorem 4.1. *Let* $\mathbb{M}$ *be a harmonic factorable surface in Galilean 3-space given by (3.1). Then* $\mathbb{M}$ *is totally geodesic with parabolic points and is congruent to an open part of plane of one of the following types :*

(1) $X(u,v) = (u,\ v,\ c_1)$

(2) $X(u,v) = (u,\ v,\ c_1v + c_2)$

(3) $X(u,v) = (u,\ v,\ c_1u + c_2)$, *where* $c_i \in \mathbb{R}(i = 1,\ 2)$.

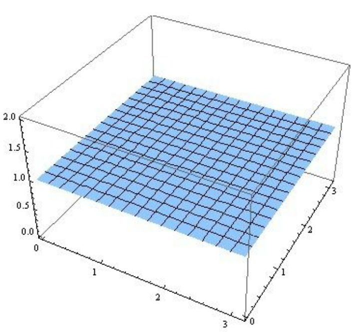

Figure 1

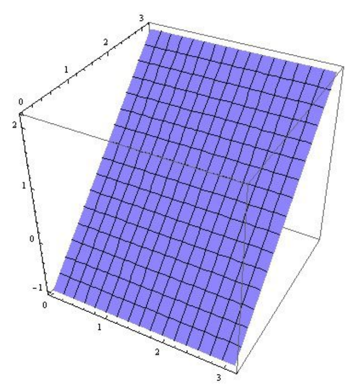

Figure 2

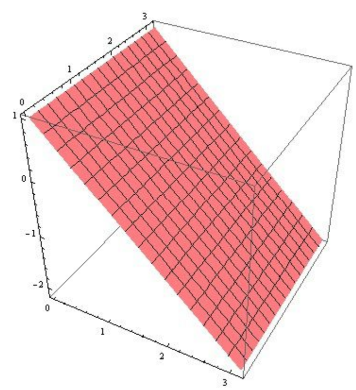

Figure 3

Harmonic Factorable Surfaces with parabolic points

subcase (1.4) : Let f and g are linear functions of u and v resp. with $f' \neq 0,\ g' \neq 0$. Then relation (4.8) yields $f'g' = 0$, which is not possible. This shows that there doesn't exist harmonic factorable surface of type $X(u,v) = (u,\ v,\ (c_1u+c_2)(c_3v+c_4))$, where $c_i \in \mathbb{R}(i = 1, 2, 3, 4)$.

Then, we have

Theorem 4.2. *There is no non-flat minimal harmonic factorable surface without parabolic points of type* $X(u,v) = (u,\ v,\ (c_1u + c_2)(c_3v + c_4)), c_i \in \mathbb{R}\ (i = 1,\ 2,\ 3,\ 4)$ *in Galilean 3- space.*

Next, we distinguish the non-harmonic factorable surface :

case 2 : Let $\lambda_2 = 0,\ \lambda_1 = \lambda_3 = \lambda \neq 0$. Then, we have

$$-ff'g'^2 = \lambda uW, \tag{4.9}$$

$$g'g'' = 0 \text{ and} \tag{4.10}$$

$$W(ff'^2gg'^2 + f''g + f^2f''gg'^2) + fg'' = -\lambda fgW^2. \tag{4.11}$$

Then (4.10) yields either $g' = 0$ *or* $g'' = 0$ but (4.9) with $g' = 0$ gives a contradiction as we assumed that $\lambda \neq 0$. Thus, we obtain $g'' = 0$ which vanishes the Mean curvature i.e. surface is minimal. Now, by the virtue of $g'' = 0$ together with (4.9), (4.11) gives

$$f'' = \lambda(uf' - f) \tag{4.12}$$

Hence, we get that the surface is minimal and is given by $X(u,v) = (u,\ v,\ f(u)(c_1v + c_2))$, where f is solution of $f'' = \lambda(uf' - f)$ and $c_1 \in \mathbb{R} - \{0\},\ c_2 \in \mathbb{R}$.

case 3 : Let $\lambda_1 = 0,\ \lambda_2 = \lambda_3 = \lambda \neq 0$. Then, we get

$$f'g' = 0\ , \tag{4.13}$$

$$f^2g'g'' = \lambda vW^2 \text{ and} \tag{4.14}$$

$$W(ff'^2gg'^2 + f''g + f^2f''gg'^2) + fg'' = -\lambda fgW^2, \tag{4.15}$$

where (4.13) yields either $f' = 0$ *or* $g' = 0$. But $g' = 0$ gives a contradiction to our assumption that $\lambda \neq 0$. Thus, we get $f =$ constant $= a \neq 0$ (say) which vanishes the Gaussian curvature. Now, using $f = a$ and combining with (4.14), (4.15) yields

$$\frac{-gg'}{v} = \frac{1}{a^2}.$$

On solving above relation, we have $g(v) = \sqrt{c_1 - \frac{v^2}{a^2}}$, where $c_1 \in \mathbb{R}$ with $c_1 a^2 > v^2$. Hence surface is flat and is given by $X(u, v) = (u,\ v,\ \sqrt{c_1 a^2 - v^2})$.

case 4 : Let $\lambda_1 = \lambda_2 = 0,\ \lambda_3 \neq 0$. Then, we have the system

$$f'g' = 0\ , \tag{4.16}$$

$$g'g'' = 0 \text{ and} \tag{4.17}$$

$$W(ff'^2 gg'^2 + f''g + f^2 f'' gg'^2) + fg'' = -\lambda_3 fgW^2. \tag{4.18}$$

Now, (4.16) and (4.17) gives three possibilities either $f' = 0$ *and* $g' = 0$, $f' = 0$ *and* $g'' = 0$ or $g' = 0$. But, if we take first of the two possibilities together with (4.18) we get $\lambda_3 = 0$ which contradicts our assumption. So, we have only the case $g' = 0$ (which vanishes the mean curvature) which further on combining with (4.18) yields $f'' + \lambda_3 f = 0$, a second order ordinary differential equation which gives

$$f(u) = \begin{cases} c_1 \cos(\sqrt{\lambda_3} u)\ +\ c_2 \sin(\sqrt{\lambda_3} u), & \lambda_3 > 0; \\ c_1 e^{\sqrt{\lambda_3} ui}\ +\ c_2 e^{-\sqrt{\lambda_3} ui}, & \lambda_3 < 0. \end{cases}, \text{ where } c_i \in \mathbb{R}\ (i = 1,\ 2).$$

Hence the surface is minimal and is given by

$$X(u, v) = \begin{cases} (u,\ v,\ c_1 \cos(\sqrt{\lambda_3} u)\ +\ c_2 \sin(\sqrt{\lambda_3} u)), & \lambda_3 > 0; \\ (u,\ v,\ c_1 e^{\sqrt{\lambda_3} ui}\ +\ c_2 e^{-\sqrt{\lambda_3} ui}), & \lambda_3 < 0. \end{cases},$$

where $c_i \in \mathbb{R}$ $(i = 1, 2)$.

case 5 : Let $\lambda_2 = \lambda_3 = 0,\ \lambda_1 \neq 0$. Then, system is reduced equivalently to

$$-ff'g'^2 = \lambda_1 uW \ , \tag{4.19}$$

$$g'g'' = 0 \text{ and} \tag{4.20}$$

$$W(ff'^2gg'^2 + f''g + f^2f''gg'^2) + fg'' = 0. \tag{4.21}$$

Now (4.20) gives either $g' = 0$ or $g'' = 0$. But, $g' = 0$ together with (4.19) immidietely yields $\lambda_1 = 0$, which is a contradiction. So, we have $g'' = 0$ which further on combining with (4.19) and (4.21) gives $f'' - \lambda_1 uf' = 0$, from which we obtain f by

$$f(u) = c_1 \int e^{\frac{\lambda_1 u^2}{2}} du + c_2, \text{ where } c_i \in \mathbb{R},\ (i = 1,\ 2).$$

Thus, surface is $X(u, v) = (u,\ v,\ (c_1 \int e^{\frac{\lambda_1 u^2}{2}} du + c_2)(c_3 v + c_4))$, where $c_i \in \mathbb{R}, (i = 1,\ 2,\ 3,\ 4)$.

Now, we summarize all of the above discussion in the following theorem

Theorem 4.3. ***(Classification Theorem)*** *Let $\mathbb{M}$ be non-harmonic factorable surface given by (3.1) satisfying $\Delta X_i = \lambda_i X_i$ in $\mathbb{G}^3$. Then $\mathbb{M}$ is an open part of one of the following :*

(1) *$\mathbb{M}$ is minimal and is given by $X(u, v) = (u,\ v,\ f(u)(c_1 v\ +\ c_2))$, where f is solution of $f'' = \lambda(uf' - f),\ , c_1, c_2 \in \mathbb{R}$*

(2) *$\mathbb{M}$ is flat and is given by*
$X(u, v) = (u,\ v,\ \sqrt{c_1 a^2 - v^2}$, where $c_1 \in \mathbb{R}$ with $c_1 a^2 > v^2$

(3) *$\mathbb{M}$ is minimal and is given by*
$$X(u, v) = \begin{cases} (u,\ v,\ c_1 \cos(\sqrt{\lambda_3} u)\ +\ c_2 \sin(\sqrt{\lambda_3} u)), & \lambda_3 > 0; \\ (u,\ v,\ c_1 e^{\sqrt{\lambda_3} ui}\ +\ c_2 e^{-\sqrt{\lambda_3} ui}), & \lambda_3 < 0. \end{cases},$$
where $c_i \in \mathbb{R}(i = 1,\ 2,\ 3,\ 4)$

(4) $\mathbb{M}$ *is minimal and is given by* $X(u,v) = (u,\ v,\ (c_1 \int e^{\frac{\lambda_1 u^2}{2}} du + c_2)(c_3 v + c_4))$, *where* $c_i \in \mathbb{R}$ $(i = 1,\ 2,\ 3,\ 4)$.

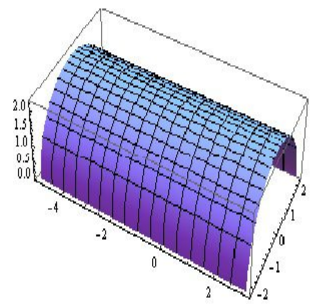

Figure 4. Theorem(4.3)-(2) surface

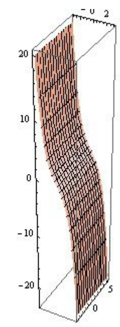

Figure 5. Theorem(4.3-(3)first surface

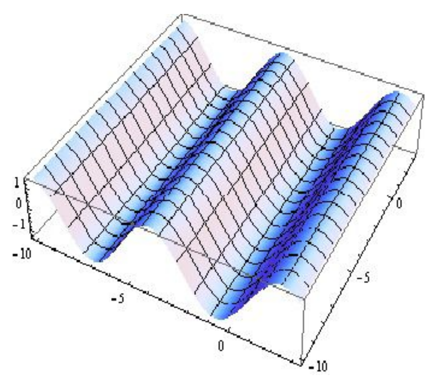

Figure 6. Theorem(4.3)-(3)second surface

Figure 7. Theorem(4.3)-(4) surface

case 6 : Let $\lambda_1 = \lambda_3 = 0,\ \lambda_2 \neq 0$. Then, system reduces to

$$f'g' = 0, \tag{4.22}$$

$$f^2g'g'' = \lambda_2 vW^2 \text{ and} \tag{4.23}$$

$$W(ff'^2gg'^2 + f''g + f^2f''gg'^2) + fg'' = 0. \tag{4.24}$$

Now, from (4.22) we may have $f' = 0$ or $g' = 0$. Now, taking both of the possibilities into account (4.23) and (4.24), we obtain $\lambda_2 = 0$, which again contradict our assumption.

Thus, we conclude

Proposition 4.1. *There is no non-harmonic factorable surfaces satisfying $\Delta X_i = \lambda_i X_i$ in $\mathbb{G}^3$ of the following kinds:*
(1) $X(u, v) = (u,\ v,\ c_1 g(v))$
(2) $X(u, v) = (u,\ v,\ c_1 f(u))$
(3) $X(u, v) = (u,\ v,\ c_1)$, where $c_1 \in \mathbb{R}$.

5. Factorable surfaces in $\mathbb{G}^3$ with finite type Gauss map

This section is devoted to classify the factorable surfaces given by (3.1) in $\mathbb{G}^3$ that satisfy

$$\Delta \mathbf{U} = \lambda_i \mathbf{U},\ \lambda_i \in \mathbb{R},\ (i = 1,\ 2,\ 3). \tag{5.1}$$

Now, by using first fundamental form, one can show that

(5.2)
$$\Delta \mathbf{U} = \frac{-1}{W^{7/2}} \begin{pmatrix} 0 \\ -f''g'W^2 + (2ff'^2g'^3 - fg''')W + 4f^3g'g''^2 \\ (-f'^2g'^2 - ff''g'^2)W^2 + (2f^2f'^2g'^4 - f^2g''^2 - f^2g'g''')W + 4f^4g'^2g''^2 \end{pmatrix}$$

which further on combining with (5.1), give rise the following equations

$$-f''g'W^2 + (2ff'^2g'^3 - fg''')W + 4f^3g'g''^2 = \lambda_2 fg'W^3$$

and

$$(-f'^2g'^2 - ff''g'^2)W^2 + (2f^2f'^2g'^4 - f^2g''^2 - f^2g'g''')W + 4f^4g'^2g''^2 = -\lambda_3 W^3.$$

On account of above equations we are left with this equation

$$f^2g'^2\lambda_2 + \lambda_3 = \frac{f'^2g'^2W + f^2g''^2}{W^2}. \tag{5.3}$$

Now, we first characterize the surface by taking the case of harmonic Gauss map of surface (that is, $\lambda_i = 0, (i = 1,\ 2,\ 3)$).

case 1 : Assume that $\lambda_2 = \lambda_3 = 0$. Then (5.3) reduces to

$$f'^2g'^2 = \tfrac{-f^2g''^2}{W},$$

or, equivalent to

$$f'^2g'^2 = -4H^2W^2.$$

Then, the Mean curvature vanishes (i.e. $g'' = 0$) if and only if either $f' = 0$ or $g' = 0$ or $f' = 0$ and $g' = 0$.

Here, we conclude the following theorem

Theorem 5.1. *Let the factorable surface $\mathbb{M}$ has harmonic Gauss map with parabolic points in $\mathbb{G}^3$ given by (3.1). Then the surface is minimal if and only if it is isometric to one of the following types of surface*

(1) $X(u, v) = (u,\ v,\ c_1v + c_2)$

(2) $X(u, v) = (u,\ v,\ c_1f(u))$

(3) $X(u, v) = (u,\ v,\ c_1)$, where $c_i \in \mathbb{R}$ $(i = 1,\ 2)$.

Next, we consider the cases of non-harmonic Gauss map of the surface :

case 2 : We assume $\lambda_2 = 0,\ \lambda_3 \neq 0$. Then, (5.3) reduces to

$$\lambda_3 = \frac{f'^2g'^2W + f^2g''^2}{W^2}. \tag{5.4}$$

subcase 2.1 : Assume g is linear function of v (that is, $g = c_1v + c_2$) with $g' = c_1 \neq 0$ that is, $\mathbb{M}$ has zero Mean curvature. Then, (5.4) becomes

$$\lambda_3 = \tfrac{f'^2 g'^2}{W},$$

or

$$\tfrac{\lambda_3}{g'^2} = f'^2 - \lambda_3 f^2 = b(say).$$

Thus, we get $f'^2 - \lambda_3 f^2 = b,\ b \in \mathbb{R} - \{0\}$ (we are taking here $b = 0$ otherwise, it gives $\lambda_3 = 0$, a contradiction) and the third eigenvalue is given by $\lambda_3 = bc_1^2 \in \mathbb{R} - \{0\}$.

subcase 2.2 : Assume $f' = 0$ *or* $f = c_1 \in \mathbb{R} - \{0\}$ (that is, $\mathbb{M}$ is flat) then we get $W^2\lambda_3 = c_1^2 g''^2$.

case 3 : If we assume $\lambda_2 \neq 0,\ \lambda_3 = 0$. Then, (5.3) reduces to $f^2 g'^2 \lambda_2 = \frac{f'^2 g'^2 W + f^2 g''^2}{W^2}$.

subcase 3.1 : Assume $f' = 0$ which implies $\mathbb{M}$ is flat then from above equation we get $g''^2 - \lambda_2 g'^2 W^2 = 0$.

subcase 3.2 : Assume g is linear function of v with $g' \neq 0$. Thus, $\mathbb{M}$ is minimal as its Mean curvature vanishes. Now, using our assumption $g'' = 0$ we get $f'^2 - \lambda_2 f^2 W = 0$. Now, we have the theorem

Theorem 5.2. *Let $\mathbb{M}$ be factorable surface in Galilean 3- space given by (3.1) satisfying $\Delta \boldsymbol{U} = \lambda_i \boldsymbol{U}$. Then the following holds true*

(1) $\mathbb{M}$ is minimal iff it is isometric to $X(u,\ v) = (u,\ v,\ f(u)(c_1v + c_2))$, where either f is solution of $f'^2 - \lambda_3 f^2 = b$ with $\lambda_3 = bc_1^2 \in \mathbb{R} - \{0\}$ and $\lambda_2 = 0$ or f is solution of $f'^2 - \lambda_2 f^2 W = 0$ with $\lambda_2 \neq 0$ and $\lambda_3 = 0$.

(2)$\mathbb{M}$ *is flat iff it is congruent to $X(u,\ v) = (u,\ v,\ c_1 f(u))$, where either g is solution of $c_1 g''^2 - \lambda_3 W^2 = 0$ with $\lambda_2 = 0$ and $\lambda_3 \neq 0$ or g is solution of $g''^2 - \lambda_2 g'^2 W^2 = 0$ with $\lambda_2 \neq 0$ and $\lambda_3 = 0$ where $c_1 \in \mathbb{R} - \{0\},\ c_2 \in \mathbb{R}$.*

6. Factorable surfaces in $\mathbb{G}^3$ with pointwise 1-type Gauss map of first kind

Let $\mathbb{M}$ be a factorable surface given by (3.1) in $\mathbb{G}^3$ that satisfy

$$\Delta \mathbf{U} = \phi \mathbf{U}, \tag{6.1}$$

where ϕ denotes the non-zero smooth function.

Remark : For ϕ to be a zero function, we get a condition of harmonic Gauss map which we had already discussed in previous section.

Thus, by direct computation of above equation using (5.2) we get

$$-f''g'W^2 + (2ff'^2g'^3 - fg''')W + 4f^3g'g''^2 = \phi fg'W^3$$

and

$$(-f'^2g'^2 - ff''g'^2)W^2 + (2f^2f'^2g'^4 - f^2g''^2 - f^2g'g''')W + 4f^4g'^2g''^2 = \phi W^3.$$

On combining above two equations, we are left with

$$f'^2g'^2W + f^2g''^2 = \phi W^3,$$

which is equivalent to

$$\phi = \frac{f'^2g'^2}{W^2} + 4H^2 \tag{6.2}$$

and can also be expressed as

$$\phi = 4H^2 + \tfrac{2Hf''g}{\sqrt{W}} - K.$$

Now, we may have the possibilities on functions f and g :

case 1 : If $f' = 0$ *or* $f = c_1 \in \mathbb{R} - \{0\}$, then $\mathbb{M}$ has zero Gaussian curvature (i.e. $\mathbb{M}$ is flat). Using $f' = 0$ in (6.2), we get ϕ is function of v and is given by $\phi = 4H^2$. Now, if we assume $\phi = 4H^2$ then (6.2) arises $f'g' = 0$. So, either $f' = 0$ *or* $g' = 0$. But $g' = 0$ vanishes ϕ, which is a contradiction. Thus, we have $f' = 0$.

case 2 : If $g'' = 0$ with $g' \neq 0$ then $H = 0$ (i.e. $\mathbb{M}$ is minimal). In fact, by the virtue of $g'' = 0$ (6.2) gives ϕ is function of u and is given by $\phi = \frac{f'^2g'^2}{W^2}$. Conversely, if ϕ

$= \frac{f'^2 g'^2}{W^2}$ then (6.2) reduces to $H = 0$ (or $\mathbb{M}$ is minimal) or $g'' = 0$.

Thus, we can state the following proposition and theorem.

Proposition 6.1. *If a factorable surface* $\mathbb{M}$ *in galilean 3-space* G^3 *given by (3.1) satisfies* $\Delta \boldsymbol{U} = \phi\ \boldsymbol{U}$*, where* ϕ *is non-zero smooth function. Then*

$$\phi = 4H^2 + \frac{2Hf''g}{\sqrt{W}} - K,$$

where H *and* K *are the Mean and the Gaussian curvature of* $\mathbb{M}$ *respectively.*

Theorem 6.1. *Let* $\mathbb{M}$ *be a factorable surface in galilean 3-space* $\mathbb{G}^3$ *given by (3.1) satisfying* $\Delta \boldsymbol{U} = \phi \boldsymbol{U}$*, where* ϕ *is non-zero smooth function. Then the following holds*
(1) ϕ *is only dependent on* v *and is given by* $\phi = 4H^2$ *iff* $\mathbb{M}$ *is flat surface parameterized by* $X(u,\ v) = (u,\ v,\ c_1 g(v))$
(2) ϕ *is only dependent on* u *and is given by* $\phi = \frac{c_1^2 f'^2}{(1+c_1^2 f^2)^3}$ *iff* $\mathbb{M}$ *is minimal surface parameterized by* $X(u,\ v) = (u,\ v,\ f(u)(c_1 v + c_2))$*, where* $c_1 \in R - \{0\}$ *and* $c_2 \in R$*.*

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