Jordan Journal of Mathematics and Statistics (JJMS) 13(1), 2020, pp 61 - 88

.

$\mathbb{T}-$RELATIVE FUZZY MAPS AND SOME FIXED POINT RESULTS

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Abstract. The concept of $\mathbb{T}-$Relative fuzzy sets was recently introduced by Osawaru, Olaleru and Olaoluwa [2]. It is a fuzzy set in which the membership grade of an element is dynamic and can change on a time scale. In this paper, we introduce and develop the theory of $\mathbb{T}-$Relative fuzzy maps and prove some fixed point results on these maps. Previous related results in literature are shown as special cases with examples.

1. Introduction and Preliminaries

Research efforts of fixed points of set-valued maps has been carried out by several authors. In [3], Kakutani extended the Brouwer's fixed point theorems for $n-$cell to upper semi-continuous compact, nonempty, convex set-valued mappings of the $n-$cell. The authors in [4] generalized [3] to acyclic absolute neighbourhood retracts and upper semi-continuous mappings such that the function values are nonempty, compact and acyclic. Among other research efforts on set-valued mappings are those of W. L. Strother [5], R. L. Plunkett [6] and L. E. Ward [7]. One characteristic of these theorems is that several conditions are imposed in order to prove the existence of fixed points.

1991 *Mathematics Subject Classification.* 47H10, 54H25.

Key words and phrases. $\mathbb{T}-$Relative fuzzy set, $\mathbb{T}-$Relative fuzzy map, $\mathbb{T}-$Relative fuzzy fixed point.

Received: Sept. 26, 2018 Accepted: Jan. 6, 2019 .

In 1969, S. B. Nadler [8] combined the ideas of set-valued map and Lipschitz map and proved some fixed point results which generalizes those of single-valued maps. His theorem do not place several restrictions on the images of points but only requires that the metric space be complete. The study of set-valued maps and fixed point theorems of set-valued maps have increased thereby. Following [8] are several generalizations of single-valued maps by many authors. See [17-19] for studies related to set-valued maps in literature.

On the advent of fuzzy set theory by L. A. Zadeh [16], several authors have extended and applied fuzzy set concepts in diverse areas. Of importance in this study is the concept of fuzzy mapping introduced by S. Heilpern [1]. In [1], Heilpern defined fuzzy mapping and proved some fixed point results using the ideas of set-valued maps in [8]. See [10-15, 20, 22-24] for some extensions and generalizations of [1].

Recently, the concept of $\mathbb{T}-$Relative fuzzy set, a fundamental generalization of the fuzzy set was introduced by Osawaru et al. [2]. The authors gave a characterization of $\mathbb{T}-$Relative fuzzy set which extends the concept of fuzzy set by accommodating the dynamics of membership grades of elements of a set. Also insights into applications of the $\mathbb{T}-$Relative fuzzy sets were also shown by the authors. The $\mathbb{T}-$Relative fuzzy set has further revolutionized, improved, enriched and extended the application of fuzzy sets and fuzzy set concepts.

In this present paper, we introduce the concept of the $\mathbb{T}-$Relative fuzzy map in the sense of [1] and prove the $\mathbb{T}-$Relative fuzzy set version of some fixed point results in literature. Our work applies the concepts of $\mathbb{T}-$Relative fuzzy set of [2], defines some fuzzy map concepts of [1] in the context of $\mathbb{T}-$Relative fuzzy set of Osawaru et al. [2] and prove fuzzy fixed point of the map which extends fuzzy fixed point results of (see [9-15]) and others in literature.

We restate some definitions of concepts relating to relative fuzzy sets.

Definition 1.1. [2](**Measure Chains**). A (*strong*) measure chain $(\mathbb{M}, \preceq, \mu)$ is any non-empty set $\mathbb{M}$ equipped with a

MC1. relation "$\preceq$" which is reflexive, transitive, antisymmetric and total such that

MC2. the chain $(\mathbb{M}, \preceq)$ is conditionally complete i.e every non-empty bounded subset has a least upper bound and a greatest lower bound and

MC3. the mapping $\mu : \mathbb{M} \times \mathbb{M} \to \mathbb{R}$ has the following properties (for all $r, s, t \in \mathbb{M}$)

  (i): $\mu(r, s) + \mu(s, t) = \mu(r, t)$

  (ii): if $r \succ s$, then $\mu(r, s) > 0$

  (iii): μ is continuous with respect to the product order topology (the order topology is generated by the open intervals of $\mathbb{M}$. A subset S of $\mathbb{M}$ is defined to be open, if for any $t \in S$ there are $r, s \in \mathbb{M}$ such that $t \in]r, s[\subset S$)

Remark 1.1. [2]

1.1a: Time scales $\mathbb{T}$ are specific forms of measure chains. It is defined as a nonempty closed subset of $\mathbb{R}$ ordered by the relation "$\leq$". Examples of time scales are $\mathbb{R}, \mathbb{Z}, h\mathbb{Z}, [0, 1], [0, 1] \cup [2, 3], [0, 1] \cup \mathbb{N}$, the Cantor set, e.t.c.

1.1b: By $MC1$, we can define

  (i): an order topology

  (ii): the forward and backward jump operators i.e the maps $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ such that

$$\sigma(t) = \inf\{s \in \mathbb{T} : t \leq s\}$$

and

$$\rho(t) = \sup\{s \in \mathbb{T} : s \leq t\}$$

respectively. This equips measure chains with the property of connectedness.

1.1c: Property $MC2$ ensures the transfer of important features of the real line to other time scales.

1.1d: Property $MC3$ ensures the measure of distances between elements of the measure chains.

Definition 1.2. [2]. Let $(\mathbb{T}, \leq, \mu)$ be a time scale and $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ be the forward and backward jump operators respectively. Then

1.2a: a nonmaximal (nonminimal) element $t \in \mathbb{T}$ is said to be right (left)-scattered if $\sigma(t) > t(\rho(t) < t)$

1.2b: a nonmaximal (nonminimal) element $t \in \mathbb{T}$ is said to be right (left)-dense if $\sigma(t) = t(\rho(t) = t)$

1.2c: $t \in \mathbb{T}$ is said to be isolated (dense) if it is left-scattered and right-scattered (left-dense and right-dense).

1.2d: the *graininess* function $\mu^* : \mathbb{T} \to [0, \infty)$ is defined by

$$\mu^*(t) = \sigma(t) - t$$

Definition 1.3. [2]($\mathbb{T}-$**Relative Fuzzy Set**). Let X be the universal set of discourse, $\mathbb{T}$ any time scale, R a subset of X and $\sigma : \mathbb{T} \to \mathbb{T}$ a forward difference operator. Then a $\mathbb{T}-$Relative fuzzy set R of X with respect to $\mathbb{T}$ is a set equipped with the membership growth function $\mu_R : X \times \mathbb{T} \to [0, 1]$ such that

$$\mu_R(x, \sigma(t)) = \begin{cases} 0 & \text{if} \quad x \notin R \\ k \in (0,1] & \text{if} \quad x \in R \text{ and } t \in \mathbb{T} \end{cases}$$

for all $x \in X$ and all $t \in \mathbb{T}$.

Remark 1.2. [2]

(i) The nearer the value of μ_R for any $x \in R$ and $t \in \mathbb{T}$ to unity the higher the membership grade.

(ii) If t is nonmaximal and dense (isolated) in $\mathbb{T}$ then $\mu_R(x, \sigma(t)) = \mu_R(x,t)$ $(\mu_R(x, \sigma(t)) \neq \mu_R(x,t))$. Similarly, if t is nonminimal and dense (isolated) in $\mathbb{T}$ then $\mu_R(x, \rho(t)) = \mu_R(x,t)$ $(\mu_R(x, \rho(t)) \neq \mu_R(x,t))$. Thus, if $\mathbb{T} = \mathbb{R}$ $(\mathbb{Z})$ then, $\mu_R(x, \sigma(t)) = \mu_R(x,t) = \mu_R(x, \rho(t))$ $(\mu_R(x, \sigma(t)) = \mu_R(x, t+1)$ and $\mu_R(x, \rho(t)) = \mu_R(x, t-1))$ for all $t \in \mathbb{T}$.

(iii) If for each $x \in R$, we have that $\mu_R(x, \sigma(t)) = c_x \in (0, 1]$ for all $t \in \mathbb{T}$, then R is a fuzzy set. Thus R is a set if $\mu_R(x, \sigma(t)) = 1$ for all $x \in R$ and $t \in \mathbb{T}$.

(iv) Unlike the time-dependent fuzzy set, the $\mathbb{T}-$Relative fuzzy subset of a set is not determined by time but it a fuzzy subset of the Cartesian product of the set X and $\mathbb{T}$. Also while the time-dependent fuzzy set defines a fuzzy set for each subset of X determined by time, a $\mathbb{T}-$Relative fuzzy is characterized by a single membership function defining the dynamics of the membership value(s) of element(s) with respect to an element of a time scale (not necessarily time).

(v) Unlike the non stationary fuzzy set, the $\mathbb{T}-$Relative fuzzy set is not a time-varying fuzzy sets but evolutionary with respect to an element of a time scale.

Example 1.1. [2] The set X of $\mathbb{R}$ equipped with a membership function of real numbers *close to* 1 *relative to the interval* $\mathbb{T} = [1, 4]$, defined as $\mu_X(x, \sigma(t)) = exp(-\beta(x-1)^{2\sigma(t)})$ where β is a positive real number, is a $\mathbb{T}-$Relative fuzzy set.

Remark 1.3. (i) $\sigma(t) = t$ for $\sigma : [1, 4] \to [1, 4]$ as each t is nonmaximal and dense in $[1, 4]$ so that $\mu_X(x, \sigma(t)) = exp(-\beta(x-1)^{2t})$

(ii) If $\mathbb{T} = 2$ say, instead of $[1, 4]$, then R is a $\mathbb{T}-$Relative fuzzy set for $t = 2$ with $\mu_X(x, 2) = exp(-\beta(x-1)^4)$

Operationally, we write t to mean $\sigma(t)$ for any $\sigma : \mathbb{T} \to \mathbb{T}$ in the sequel. Some operations on $\mathbb{T}-$Relative fuzzy sets were also defined in [2] as follows:

Definition 1.4. [2] Let X be a universal set and $\{R_i\}_{i=1}^n$ be $\mathbb{T}-$Relative fuzzy sets of X with $\mathbb{T}-$Relative fuzzy membership functions $\{\mu_{R_i}\}_{i=1}^n$ respectively, where $\mathbb{T}$ is any time scales set. Then

(i) the union of $\{R_i\}_{i=1}^n$ at a membership growth over $\mathbb{T}$ (respectively for any $t \in \mathbb{T}$) is the $\mathbb{T}-$Relative fuzzy set with $\mathbb{T}-$Relative fuzzy membership function defined as

$$\begin{aligned} \mu_{(\cup_{i=1}^n R_i)}(x,t)_{\mathbb{T}} &= \max_{x\in X, t\in\mathbb{T}}\{\mu_{R_i}(x,t)\} \\ \text{(respectively } \mu_{(\cup_{i=1}^n R_i)}(x,t)_t &= \max_{x\in X}\{\mu_{R_i}(x,t)\} \text{ for any } t \in \mathbb{T}) \end{aligned}$$

(ii) the intersection of $\{R_i\}_{i=1}^n$ at a membership growth over $\mathbb{T}$ (respectively for any $t \in \mathbb{T}$) is the $\mathbb{T}-$Relative fuzzy set with $\mathbb{T}-$Relative fuzzy membership function defined as

$$\begin{aligned} \mu_{(\cap_{i=1}^n R_i)}(x,t)_{\mathbb{T}} &= \min_{x\in X, t\in\mathbb{T}}\{\mu_{R_i}(x,t)\} \\ \text{(respectively } \mu_{(\cap_{i=1}^n R_i)}(x,t)_t &= \min_{x\in X}\{\mu_{R_i}(x,t)\}\} \text{ for any } t \in \mathbb{T}) \end{aligned}$$

(iii) R_i is a subset of R_j over $\mathbb{T}$ (respectively for any $t \in \mathbb{T}$) if

$$\mu_{R_i}(x,t)_{\mathbb{T}} \leq \mu_{R_j}(x,t)_{\mathbb{T}} \ \forall x \in U \text{ and } \forall t \in \mathbb{T}$$

(respectively $\mu_{R_i}(x,t)_t \leq \mu_{R_j}(x,t)_t \ \forall x \in U$ and for any $t \in \mathbb{T}$) for any i, j with $i \neq j$.

Definition 1.5. (**Level Sets of $\mathbb{T}-$Relative Fuzzy Sets**). Let R be a $\mathbb{T}-$Relative fuzzy set of X with respect to $\mathbb{T}$ and ϕ, α', α where $\phi : X \times \mathbb{T} \to (0,1]$, $\alpha' : X \to (0,1]$ and $\alpha \in (0,1]$. Then

(i) the $\phi_{\mathbb{T}}$-level sets for R with respect to $\mathbb{T}$ (respectively over any $t \in \mathbb{T}$) is defined as

$$R(\phi)_{\mathbb{T}} = \{x \in X : \mu_R(x,t) \geq \phi(x,t), \text{ for all } t \in \mathbb{T}\}$$

(respectively $R(\phi)_t = \{x \in X : \mu_R(x,t) \geq \phi(x,t), \text{ for any } t \in \mathbb{T}\}$).

(ii) the $\alpha'_{\mathbb{T}}$-level sets for R with respect to $\mathbb{T}$ (respectively over any $t \in \mathbb{T}$) is defined as

$$R(\alpha')_{\mathbb{T}} = \left\{x \in X : \mu_R(x,t) \geq \alpha'(x), \text{ for all } t \in \mathbb{T}\right\}.$$

(respectively $R(\alpha')_t = \left\{x \in X : \mu_R(x,t) \geq \alpha'(x), \text{ for any } t \in \mathbb{T}\right\}$).

(iii) the $\alpha_{\mathbb{T}}$-level sets for R with respect to $\mathbb{T}$ (respectively over any $t \in \mathbb{T}$) is defined as

$$R(\alpha)_{\mathbb{T}} = \{x \in X : \mu_R(x,t) \geq \alpha, \text{ for all } t \in \mathbb{T}\}$$

(respectively $R(\alpha)_t = \{x \in X : \mu_R(x,t) \geq \alpha, \text{ for any } t \in \mathbb{T}\}$).

Remark 1.4. (i) *If for all $x \in X$, we have that $\phi(x,t) = \alpha$ for all $t \in \mathbb{T}$ then $R(\phi)_{\mathbb{T}} = R(\alpha)_{\mathbb{T}}$.*

(ii) *If $\alpha'(x) = \alpha$ for all $x \in X$ then $R(\alpha')_{\mathbb{T}} = R(\alpha)_{\mathbb{T}}$.*

(iii) *If for all $x \in X$, $\phi'(x,t) \leq \phi(x,t)$ for all $t \in \mathbb{T}$, where $\phi', \phi : X \times \mathbb{T} \to (0,1]$ then $R(\phi)_{\mathbb{T}} \subseteq R(\phi')_{\mathbb{T}}$.*

(iv) *If $\beta \leq \alpha$ with $\alpha, \beta \in (0,1]$, then $R(\alpha)_{\mathbb{T}} \subseteq R(\beta)_{\mathbb{T}}$ for all $t \in \mathbb{T}$ and $R(\alpha)_t \subseteq R(\beta)_t$ for any $t \in \mathbb{T}$.*

(v) *$R(\phi)_{\mathbb{T}} \subset R(\phi)_t$, $R(\alpha')_{\mathbb{T}} \subset R(\alpha')_t$ and $R(\alpha)_{\mathbb{T}} \subset R(\alpha)_t$ for any $t \in \mathbb{T}$*

In fixed point theory on the other hand, Banach [21] published a very remarkable result. His result has contributed to the study of many nonlinear systems. The conditions required for the existence and uniqueness of solutions which can be reformulated as a fixed point problem was proved. The theorem is as given below:

Theorem 1.1. [21]. Let (X, d) be a complete metric space and $T : X \to X$ such that

$$d(Tx, Ty) \leq cd(x, y)$$

for each $x, y \in X$, where $c \in (0, 1]$. Then there exists a unique point $x^* \in X$ such that $x^* = Tx^*$.

In order to generalize the result of Banach and other fixed point results of single-valued maps, Nadler among others introduced set-valued maps and proved condition for existence of fixed points for such maps. The Banach-type fixed point results due to Nadler is as stated below:

Theorem 1.2. [8]. Let (X, d) be a complete metric space and $T : X \to 2^X$ such that

$$H(Tx, Ty) \leq cd(x, y)$$

for each $x, y \in X$, where $c \in (0, 1]$ and $H(A, B)$ is the Hausdorf distance between $A, B \in 2^X$. Then there exists a fixed point $x^* \in X$ such that $x^* \in Tx^*$.

Using the idea of Nadler, Heilpern introduced fuzzy mapping and investigated fixed point results for fuzzy maps that are useful for solving problems that can be reformulated as fuzzy fixed map problem. The following definitions of distance functions for fuzzy sets were introduced in [1] by Heilpern:

Definition 1.6. [1]. Let (X, d) be a metric space and $W(X)$ a collection of fuzzy sets of X. A subcollection of $W(X)$ denoted $Q(X)$ is called a subfamily of approximate quantities if for any $A \in Q(X)$, we have that $A(\alpha)$ is nonempty and compact for all $\alpha \in (0, 1]$.

Definition 1.7. [1]. Let (X,d) be a metric space and $Q(X)$ the approximate quantities of X. Then for all $A, B \in Q(X)$ and any $\alpha \in (01]$

$$\begin{aligned}
P_\alpha(a,B) &= \inf_{b\in B} d(a,b),\\
P_\alpha(A,B) &= \inf_{a\in A, b\in B} d(a,b),\\
D_\alpha(A,B) &= \max\{\sup_{a\in A} P_\alpha(a,B), \sup_{b\in B} P_\alpha(A,b)\},\\
D(A,B) &= \sup_{\alpha} D_\alpha(A,B)
\end{aligned}$$

The fuzzy version of the Nadler fixed point result is as stated below:

Theorem 1.3. [1]. Let (X,d) be a complete metric space and $T : X \to Q(X)$ such that

$$D(Tx,Ty) \le cd(x,y)$$

for each $x, y \in X$, where $c \in (0,1]$. Then there exists a fuzzy point $x^* \in X$ such that $\{x^*\} \subset Tx^*$, where $\{x\}$ denotes the characteristic function of x, that is a function on X into $[0,1]$ such that $\{x\} = 1$ and $\{y\} = 0$ for any $x, y \in X$ with $y \neq x$.

Identifying fuzzy fixed point results to at least a certain $\alpha \in (0,1]$, a generalization of the fixed point for fuzzy map in the Heilpern sense was studied by Estruch and Vidal. Their result is given below:

Theorem 1.4. [20]. Let $\alpha \in (0,1]$ and (X,d) be a complete metric space. Let T be a fuzzy map from X into $Q(X)$ satisfying the condition that

$$D_\alpha(Tx,Ty) \le cd(x,y)$$

for each $x, y \in X$, where $c \in (0,1]$. Then there exists fixed fuzzy point $x^* \in X$ for $\alpha \in [0,1]$ such that $x^*_\alpha \subset Tx^*_\alpha$.

Although the study of fuzzy fixed point results of fuzzy maps is progressing rapidly, fixed point results of fuzzy maps defined on a fuzzy set endowed with dynamic membership grades has not being studied.

Our interest in this article is to introduce $\mathbb{T}-$Relative fuzzy maps and investigate $\mathbb{T}-$Relative fuzzy fixed points of the map. Examples are given to explain the concepts introduced and results obtained. It is also shown that the results of Nadler, Heilpern and Estruch and Vidal and others in literature can be derived from our results.

2. Main Result

2.1. **Preliminary Definitions.** In this section, we define the concepts of approximate quantities based on the concepts of $\mathbb{T}-$Relative fuzzy sets and level sets of $\mathbb{T}-$Relative fuzzy sets of Osawaru et al.[2]. Also for such approximate quantities of $\mathbb{T}-$Relative fuzzy sets of a metric space, we define some metric of $\mathbb{T}-$Relative fuzzy sets of a metric space. Some results of the metrics are established.
We also define the concept of $\mathbb{T}-$Relative fuzzy map as a map on any set X into a collection of approximate quantities of the $\mathbb{T}-$Relative fuzzy sets of Y.

Definition 2.1. (**Approximate Quantities of $\mathbb{T}-$Relative Fuzzy Sets**). Let (X, d) be a metric space and $W(X)_{\mathbb{T}}$ (respectively $W(X)_t$) a collection of $\mathbb{T}-$Relative fuzzy sets of X with respect to $\mathbb{T}$ (respectively any $t \in \mathbb{T}$). Then a subcollection of $W(X)_{\mathbb{T}}$ denoted $\pi(X)_{\mathbb{T}}$ (respectively $\pi(X)_t$) is called a subfamily of approximate quantities if for any $R \in \pi(X)_{\mathbb{T}}$ (respectively $R \in \pi(X)_t$), we have that each $R(\phi)_{\mathbb{T}}$ (respectively $R(\phi)_t$) is nonempty and compact, where $\phi : X \times \mathbb{T} \to (0, 1]$.

Definition 2.2. ($\phi - Space$ **of $\mathbb{T}-$Relative Fuzzy Sets**). Let (X, d) be a metric space, $\phi : X \times \mathbb{T} \to (0, 1]$ and $\pi(X)_t$ (respectively $\pi(X)_{\mathbb{T}}$) approximate quantities of

$\mathbb{T}-$Relative fuzzy sets R and $R^{'}$ of X. Then the $\phi - Space$ of the $\mathbb{T}-$Relative fuzzy sets $R, R^{'}$ with respect to $t \in \mathbb{T}$ (respectively all $t \in \mathbb{T}$) is defined as

$$p_{\phi}(R, R^{'})_t = \inf_{x \in R(\phi)_t, y \in R^{'}(\phi)_t} d(x, y) \text{ and } p(R, R^{'})_t = \sup_{\phi} p_{\phi}(R, R^{'})_t.$$

(respectively, $p_{\phi}(R, R^{'})_{\mathbb{T}} = \inf_{x \in R(\phi)_{\mathbb{T}}, y \in R^{'}(\phi)_{\mathbb{T}}} d(x, y)$ and $p(R, R^{'})_{\mathbb{T}} = \sup_{\phi} p_{\phi}(R, R^{'})_{\mathbb{T}}$).

Definition 2.3. ($\phi - diameter$ **of $\mathbb{T}-$Relative Fuzzy Sets**). Let (X, d) be a metric space, $\phi : X \times \mathbb{T} \to (0, 1]$ and $\pi(X)_t$ (respectively $\pi(X)_{\mathbb{T}}$) approximate quantities of $\mathbb{T}-$Relative fuzzy sets R and $R^{'}$ of X. Then the $\phi - distance$ of the $\mathbb{T}-$Relative fuzzy sets $R, R^{'}$ with respect to $t \in \mathbb{T}$ (respectively all $t \in \mathbb{T}$) is defined as

$$\delta_{\phi}(R, R^{'})_t = \sup_{x \in R(\phi)_t, y \in R^{'}(\phi)_t} d(x, y) \text{ and } \delta(R, R^{'})_t = \inf_{\phi} \delta_{\phi}(R, R^{'})_t.$$

(respectively, $\delta_{\phi}(R, R^{'})_{\mathbb{T}} = \sup_{x \in R(\phi)_{\mathbb{T}}, y \in R^{'}(\phi)_{\mathbb{T}}} d(x, y)$ and $\delta(R, R^{'})_{\mathbb{T}} = \inf_{\phi} \delta_{\phi}(R, R^{'})_{\mathbb{T}}$).

Definition 2.4. (**ϕ-Hausdorff metric of $\mathbb{T}-$Relative Fuzzy Sets**). Let (X, d) be a metric space and $\phi : X \times \mathbb{T} \to (0, 1]$. The ϕ-Hausdorff metric on $W_{cb}(X)$ induced by d is defined as

$$H_{\phi}(R, R^{'})_{\mathbb{T}} = \max\{\sup_{x \in R(\phi)_{\mathbb{T}}} p_{\phi}(x, R^{'})_{\mathbb{T}}, \sup_{y \in R^{'}(\phi)_{\mathbb{T}}} p_{\phi}(y, R)_{\mathbb{T}}\}$$

for all $R, R^{'} \in W_{cb}(X)$, where $W_{cb}(X)$ denotes the family of $\mathbb{T}-$Relative fuzzy sets with nonempty closed bounded $\phi_{\mathbb{T}}$-level sets of X and

$$p_{\phi}(x, R^{'})_{\mathbb{T}} = \inf\{d(x, b) : b \in R^{'}(\phi)_{\mathbb{T}}\}$$

for all $x \in X$.

(respectively, $H_{\phi}(R, R^{'})_t = \max\{\sup_{x \in R(\phi)_t} p_{\phi}(x, R^{'})_t, \sup_{y \in R^{'}(\phi)_t} p_{\phi}(y, R)_t\}$ for all $R, R^{'} \in W_{cb}(X)$, where $W_{cb}(X)$ denotes the family of $\mathbb{T}-$Relative fuzzy sets with nonempty closed bounded ϕ_t-level sets of X and

$$p_{\phi}(x, R^{'})_t = \inf\{d(x, b) : b \in R^{'}(\phi)_t\}$$

for all $x \in X$).

Definition 2.5. ($\phi-Distance$ **of** $\mathbb{T}-$**Relative Fuzzy Sets**). Let (X, d) be a metric space, $\phi : X \times \mathbb{T} \to (0, 1]$ and $\pi(X)_t$ (respectively $\pi(X)_{\mathbb{T}}$) approximate quantities of $\mathbb{T}-$Relative fuzzy sets R and $R^{'}$ of X. Then the $\phi - distance$ of the $\mathbb{T}-$Relative fuzzy sets $R, R^{'}$ with respect to $t \in \mathbb{T}$ (respectively all $t \in \mathbb{T}$) is defined as

$$D_\phi(R, R^{'})_t = H(R(\phi)_t, R^{'}(\phi)_t) \text{ and } D(R, R^{'})_t = \sup_{\phi} D_\phi(R, R^{'})_t.$$

(respectively $D_\phi(R, R^{'})_{\mathbb{T}} = H(R(\phi)_{\mathbb{T}}, R^{'}(\phi)_{\mathbb{T}})$ and $D(R, R^{'})_{\mathbb{T}} = \sup_\phi D_\phi(R, R^{'})_{\mathbb{T}}$).

Remark 2.1. *For* $\alpha^{'} : X \to (0, 1]$ *and* $\alpha \in (0, 1]$, *if*

(i) *for all* $x \in X$, *we have that* $\phi(x, t) = \alpha^{'}(x)$ *for all* $t \in \mathbb{T}$ *then* $p_\phi(R, R^{'})_{\mathbb{T}} = p_{\alpha^{'}}(R, R^{'})_{\mathbb{T}}$, $\delta_\phi(R, R^{'})_{\mathbb{T}} = \delta_{\alpha^{'}}(R, R^{'})_{\mathbb{T}}$, $H_\phi(R, R^{'})_{\mathbb{T}} = H_{\alpha^{'}}(R, R^{'})_{\mathbb{T}}$ *(respectively for any* $t \in \mathbb{T}$ *then* $p_\phi(R, R^{'})_t = p_{\alpha^{'}}(R, R^{'})_t$, $\delta_\phi(R, R^{'})_t = \delta_{\alpha^{'}}(R, R^{'})_t$, $H_\phi(R, R^{'})_t = H_{\alpha^{'}}(R, R^{'})_t$*).*

(ii) *for all* $x \in X$, *we have that* $\phi(x, t) = \alpha$ *for all* $t \in \mathbb{T}$ *then* $p_\phi(R, R^{'})_{\mathbb{T}} = p_\alpha(R, R^{'})_{\mathbb{T}}$, $\delta_\phi(R, R^{'})_{\mathbb{T}} = \delta_\alpha(R, R^{'})_{\mathbb{T}}$, $H_\phi(R, R^{'})_{\mathbb{T}} = H_\alpha(R, R^{'})_{\mathbb{T}}$ *(respectively for any* $t \in \mathbb{T}$ *then* $p_\phi(R, R^{'})_t = p_\alpha(R, R^{'})_t$, $\delta_\phi(R, R^{'})_t = \delta_\alpha(R, R^{'})_t$, $H_\phi(R, R^{'})_t = H_\alpha(R, R^{'})_t$*).*

(iii) *Since* $R(\phi)_{\mathbb{T}} \subset R(\phi)_t$ *for any* $t \in \mathbb{T}$ *then* $p(R, R^{'})_{\mathbb{T}} \leq p(R, R^{'})_t$, $\delta(R, R^{'})_{\mathbb{T}} \leq \delta(R, R^{'})_t$, $D(R, R^{'})_{\mathbb{T}} \leq D(R, R^{'})_t$ *for any* $t \in \mathbb{T}$.

Definition 2.6. ($\mathbb{T}-$**Relative Fuzzy Point**). Let X be any set. Then the $\phi - \mathbb{T}-$Relative fuzzy point with respect to $\mathbb{T}$ denoted $x_{\phi_{\mathbb{T}}}$ is a $\mathbb{T}-$Relative fuzzy set with respect to $\mathbb{T}$ with membership function defined as $x_{\phi_{\mathbb{T}}}(x, t) = \phi(x, t)$ and $x_{\phi_{\mathbb{T}}}(w, t) = 0$ if $x \neq w \in X$ for all $t \in \mathbb{T}$, where $\phi : X \times \mathbb{T} \to (0, 1]$.

Remark 2.2. (i) If for all $x \in X$ we have that $\phi(x, t) = \alpha^{'}(x)$ for all $t \in \mathbb{T}$ and $x_{\phi_{\mathbb{T}}}(x, t) = \alpha^{'}(x) \in (0, 1]$ for all $t \in \mathbb{T}$ then the $\phi - \mathbb{T}-$Relative fuzzy point is the $\alpha^{'}-$ fuzzy point.

(ii) If for all $x \in X$ we have that $\phi(x,t) = \alpha$ for all $t \in \mathbb{T}$ and $x_{\phi_{\mathbb{T}}}(x,t) = \alpha \in (0,1]$ for all $t \in \mathbb{T}$ then the $\phi - \mathbb{T}-$Relative fuzzy point is the $\alpha-$ fuzzy point of Estruch and Vidal [20].

(iii) If for all $x \in X$ we have that $\phi(x,t) = 1$ for all $t \in \mathbb{T}$ and $x_{\phi_{\mathbb{T}}}(x,t) = 1$ for all $t \in \mathbb{T}$ then the $\phi - \mathbb{T}-$Relative fuzzy point is the fuzzy point of Heilpern [1].

The following lemmas are significant in the proof of the results in the next section of this article.

Lemma 2.1. Let (X,d) be a metric space, $\pi(X)_{\mathbb{T}}$ a subfamily of approximate quantities and $x_{\phi_{\mathbb{T}}}$, a $\mathbb{T}-$Relative fuzzy point for all $t \in \mathbb{T}$. Let $R \in \pi(X)_{\mathbb{T}}$. Then $x_{\phi_{\mathbb{T}}} \subset R$, iff $p_\phi(x,R)_{\mathbb{T}} = 0$ for each $\phi(x,t) \in (0,1]$ and all $t \in \mathbb{T}$.

Proof. Suppose $x_{\phi_{\mathbb{T}}} \subset R$, then we have that $x \in R(\phi)_{\mathbb{T}}$ for each $\phi \in [0,1]$ and all $t \in \mathbb{T}$. Recall that $p_\phi(R,R^{'})_{\mathbb{T}} = \inf_{x\in R(\phi)_{\mathbb{T}}, y \in R^{'}(\phi)_{\mathbb{T}}} d(x,y)$ where $R, R^{'}$ are $\mathbb{T}-$Relative fuzzy set of approximate quantities with respect to $\mathbb{T}$ and $R(\phi)_{\mathbb{T}}, R^{'}(\phi)_{\mathbb{T}}$ the ϕ-level sets with respect to $\mathbb{T}$ of R and $R^{'}$ respectively. So $p_\phi(x,R)_{\mathbb{T}} = \inf_{y \in R(\phi)_{\mathbb{T}}} d(x,y) = 0$ as $x \in R(\phi)_{\mathbb{T}}$.

Conversely, suppose $p_\phi(x,R)_{\mathbb{T}} = 0$. So by Definition 3.2, $p_\phi(x,R)_{\mathbb{T}} = \inf_{y \in R(\phi)_{\mathbb{T}}} d(x,y) = 0$. Thus $x = y$ and since $y \in R(\phi)$ then $x \in R(\phi)_{\mathbb{T}}$. So $x_{\phi_{\mathbb{T}}} \subset R$. The proof is complete. □

Lemma 2.2. Let (X,d) be a metric space, $\pi(X)_{\mathbb{T}}$ a subfamily of approximate quantities. Then for $x,y \in X$ and $R \in \pi(X)_{\mathbb{T}}$, we have $p_\phi(x,R)_{\mathbb{T}} \leq d(x,y) + p_\phi(y,R)_{\mathbb{T}}$ where $\phi : X \times \mathbb{T} \to (0,1]$.

Proof. By Definition 3.2 and the triangle inequality we have that

$$p_\phi(x,R)_{\mathbb{T}} = \inf_{x^* \in R(\phi)_{\mathbb{T}}} d(x,x^*) \leq \inf_{x^* \in R(\phi)_{\mathbb{T}}} (d(x,y) + d(y,x^*)) = d(x,y) + p_\phi(y,R)_{\mathbb{T}}$$

Therefore $p_\phi(x,R)_{\mathbb{T}} \leq d(x,y) + p_\phi(y,R)_{\mathbb{T}}$ and the proof is complete. □

Lemma 2.3. Let (X,d) be a metric space,$\pi(X)_{\mathbb{T}}$ a subfamily of approximate quantities. Then for each $R,R^{'}\in\pi(X)_{\mathbb{T}}$, we have $p_{\phi}(x,R^{'})_{\mathbb{T}}\leq D_{\phi}(R,R^{'})_{\mathbb{T}}$ if $x_{\phi_{\mathbb{T}}}\subset R$ where $\phi:X\times\mathbb{T}\rightarrow(0,1]$.

Proof. From Definition 3.2 and 3.5,

$$p_{\phi}(x,R^{'})_{\mathbb{T}} \;=\; \inf_{x^{*}\in R^{'}(\phi)_{\mathbb{T}}} d(x,x^{*})\leq \sup_{x\in R(\phi)_{\mathbb{T}}}\inf_{x^{*}\in R^{'}(\phi)_{\mathbb{T}}} d(x,x^{*})\leq D_{\phi}(R,R^{'})_{\mathbb{T}}$$

The proof is complete. □

We now define the concepts of $\mathbb{T}-$Relative fuzzy mapping as a generalization of the fuzzy mapping and the set-valued map of Nadler, Level sets of $\mathbb{T}-$Relative fuzzy maps, Fixed Points of $\mathbb{T}-$Relative Fuzzy Maps, $\mathbb{T}-$Relative Fuzzy Contraction Map and $\phi-\mathbb{T}-$Relative fuzzy Picard iterative scheme.

Definition 2.7. (**$\mathbb{T}-$Relative Fuzzy Mapping**). The map $T_{\mathbb{T}}:X\rightarrow\pi(Y)_{\mathbb{T}}$ (respectively $T_t:X\rightarrow\pi(Y)_t$) such that $T_{\mathbb{T}}(x)\in\pi(Y)_{\mathbb{T}}$ (respectively $T_t(x)\in\pi(Y)_t$) for all $t\in\mathbb{T}$ (respectively for any $t\in\mathbb{T}$) and $x\in X$, where X is any set and (Y,d) a metric space, is called the $\mathbb{T}-$Relative fuzzy map with respect to $\mathbb{T}$ (respectively for any $t\in\mathbb{T}$).

Remark 2.3.
(i) The $\mathbb{T}-$Relative fuzzy map T_t for any $t\in\mathbb{T}$ is the subset of the product $X\times Y$ such that $T_t(x,y)\subset[0,1]$ is membership growth grade of y in $T_t(x)$ for any $t\in\mathbb{T}$.
(ii) The $\mathbb{T}-$Relative fuzzy map $T_{\mathbb{T}}$ is a $\mathbb{T}-$Relative fuzzy subset on $X\times Y$ with a $\mathbb{T}-$Relative membership function $T_{\mathbb{T}}x(y,t)$ for all $t\in\mathbb{T}$. The function value of $T_{\mathbb{T}}x(y,t)$ is the $\mathbb{T}-$Relative membership grade of y in $T_{\mathbb{T}}x$ for all $t\in\mathbb{T}$.

Definition 2.8. (**Level Sets of $\mathbb{T}-$Relative Fuzzy Maps**). Let (X,d) be a metric space, and $T_{\mathbb{T}}:X\rightarrow W(X)_{\mathbb{T}}$ a $\mathbb{T}-$Relative fuzzy self map of X and $\phi:X\times\mathbb{T}\rightarrow(0,1]$.

Then the $\phi_{\mathbb{T}}$-level sets for the $\mathbb{T}-$Relative fuzzy map $T_{\mathbb{T}}$ at any $x \in X$ is defined as

$$T_{\mathbb{T}}x(\phi) = \{y \in X : \mu_{T_{\mathbb{T}}x}(y,t) \geq \phi(y,t), \text{ for all } t \in \mathbb{T}\}.$$

Definition 2.9. (**Fixed Points of $\mathbb{T}-$Relative Fuzzy Maps**). Let (X,d) be a metric space and $T_{\mathbb{T}} : X \to W(X)_{\mathbb{T}}$ a $\mathbb{T}-$Relative fuzzy self map of X, where $W(X)_{\mathbb{T}}$ is a collection of $\mathbb{T}-$Relative fuzzy sets of (X,d) for all $t \in \mathbb{T}$. Then the $\phi-\mathbb{T}-$Relative fuzzy point with respect to $\mathbb{T}$ denoted $x_{\phi_{\mathbb{T}}}$ is called a $\phi-\mathbb{T}-$Relative fuzzy fixed point of $T_{\mathbb{T}}$ if $x_{\phi_{\mathbb{T}}} \subset T_{\mathbb{T}}x$ (alternatively $x \in T_{\mathbb{T}}x(\phi)$) for all $t \in \mathbb{T}$, where $\phi : X \times \mathbb{T} \to (0,1]$.

Definition 2.10. ($\mathbb{T}-$**Relative Fuzzy Contraction Map**). Let (X,d) be a metric space, $\pi(X)_{\mathbb{T}}$ be a family of approximate quantities and $\phi : X \times \mathbb{T} \to (0,1]$. Then the $\mathbb{T}-$Relative fuzzy map $T_{\mathbb{T}} : X \to \pi(X)_{\mathbb{T}}$ with respect to $\mathbb{T}$ is said to be a $\phi-\mathbb{T}-$Relative fuzzy contraction map if for any $x, y \in X$,

$$D_{\phi}(T_{\mathbb{T}}x, T_{\mathbb{T}}y)_{\mathbb{T}} \leq ad(x,y)$$

where for all $x \in X$, $\phi(x,t) \in [0,1]$ for all $t \in \mathbb{T}$ and $a \in [0,1)$.

Definition 2.11. Let (X,d) be a metric space, $\pi(X)_{\mathbb{T}}$ be a family of approximate quantities of $\mathbb{T}-$Relative fuzzy sets of X, $\phi : X \times \mathbb{T} \to (0,1]$ and $T_{\mathbb{T}} : X \to \pi(X)_{\mathbb{T}}$ a $\mathbb{T}-$Relative fuzzy map with respect to $\mathbb{T}$. Then

(i) $F_T(\phi) = \{x^* \in X : x^* \in T_{\mathbb{T}}x^*(\phi)\}$ is the set of $\phi-\mathbb{T}-$Relative fuzzy fixed points of $T_{\mathbb{T}}$ for all $t \in \mathbb{T}$.

(ii) for any $x_0 \in X$, the sequence $\{x_n\}_{n=0}^{\infty}$ defined by $x_{n+1} \in T_{\mathbb{T}}x_n(\phi)$ where $n \geq 0$, is called the $\phi-\mathbb{T}-$Relative fuzzy Picard iterative scheme if and only if for each $x \in X$ and any $y \in T_{\mathbb{T}}x$ the sequence $\{x_n\}_{n=0}^{\infty}$ converges to a fixed point of $T_{\mathbb{T}}$ with $x_0 = x$ and $x_1 = y$.

Thus the sequence $\{x_n\}_{n=0}^{\infty}$ is called sequence of successive approximations of a $\phi - \mathbb{T}-$Relative fuzzy defined by the $\mathbb{T}-$Relative fuzzy operator $T_{\mathbb{T}}$ with starting values (x, y).

2.2. **Fixed Point Results.** We now state and prove the Banach contraction principle in the context of $\mathbb{T}-$Relative fuzzy set. The results are shown to generalize existing fixed point results of fuzzy maps in literature.

Theorem 2.1. Let X be a complete metric space and $T_{\mathbb{T}} : X \to \pi(X)_{\mathbb{T}}$ a $\phi - \mathbb{T}-$Relative fuzzy contraction map. Then

(i) there exists $\phi - \mathbb{T}-$Relative fuzzy fixed point $x^* \in X$ such that $x^* \in T_{\mathbb{T}}x^*(\phi)$.

(ii) $T_{\mathbb{T}}$ has a unique $\phi-\mathbb{T}-$Relative fuzzy fixed point $x^*_{\phi_{\mathbb{T}}}$ if $d(x^*, y^*) \leq p_\phi(T_{\mathbb{T}}x^*, T_{\mathbb{T}}y^*)_{\mathbb{T}}$ for any two points $x^*, y^* \in F_T(\phi)$.

(ii) the $\mathbb{T}-$Relative fuzzy Picard iterative scheme, $x_{n+1} \in T_{\mathbb{T}}x_n(\phi)$ converges strongly to $x^*_{\phi_{\mathbb{T}}}$ of $T_{\mathbb{T}}$

Proof. To show (i), suppose $x_0 \in X$ such that $T_{\mathbb{T}}x_0 \in \pi(X)_{\mathbb{T}}$ and $x_1 \in T_{\mathbb{T}}x_0(\phi)$ for all $x \in X$ and $t \in \mathbb{T}$. Then there is $x_2 \in X$ such that $x_2 \in T_{\mathbb{T}}x_1(\phi)$ for all $x \in X$ and $t \in \mathbb{T}$. Then

$$d(x_1, x_2) \leq p_\phi(Tx_0, Tx_1)_{\mathbb{T}} \leq D_\phi(Tx_0, Tx_1)_{\mathbb{T}} \leq ad(x_0, x_1).$$

Also there is $x_3 \in X$ such that $x_3 \in T_{\mathbb{T}}x_2(\phi)$ for all $x \in X$ and $t \in \mathbb{T}$ so that

$$d(x_2, x_3) \leq p_\phi(Tx_1, Tx_2)_{\mathbb{T}} \leq D_\phi(Tx_1, Tx_2)_{\mathbb{T}} \leq ad(x_2, x_1)$$

giving

$$d(x_2, x_3) \leq ad(x_1, x_2). \tag{2.1}$$

By substitution, we have

$$d(x_2, x_3) \leq a^2 d(x_0, x_1).$$

Continuing in this manner we have that for any natural number n

$$d(x_n, x_{n+1}) \leq a^n d(x_0, x_1). \tag{2.2}$$

To show that any sequence $\{x_n \in X\}$ is Cauchy in X, let $n, m \in \mathbb{N}$ with $m > n$, then

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ &\leq a^n d(x_0, x_1) + a^{n+1} d(x_0, x_1) + \cdots + a^{m-1} d(x_0, x_1) \\ &= a^{n+n+1+\cdots+m-1} d(x_0, x_1) \to 0 \text{ as } n \to \infty \end{aligned}$$

Thus $\{x_n\}$ is a Cauchy sequence.
As a complete metric space there is a limit $x^* \in X$ for any sequence in X. By Lemmas 3.2 and 3.3

$$\begin{aligned} p_\phi(x^*, T_{\mathbb{T}}x^*)_{\mathbb{T}} &\leq d(x^*, x_n) + p_\phi(x_n, T_{\mathbb{T}}x^*)_{\mathbb{T}} \leq d(x^*, x_n) + D_\phi(T_{\mathbb{T}}x_{n-1}, T_{\mathbb{T}}x^*)_{\mathbb{T}} \\ &\leq d(x_n, x^*) + ad(x_{n-1}, x^*) \\ &\to 0 \text{ as } n \to \infty \end{aligned}$$

Therefore $x^*_{\phi_{\mathbb{T}}} \subset T_{\mathbb{T}}x^*$ by Lemma 3.2.
To show (ii), we assume it is not unique i.e there is $y^*_{\phi_{\mathbb{T}}}$, with $x^* \neq y^*$ such that $y^*_{\phi_{\mathbb{T}}} \subset T^{'} y^*$ also. Then

$$\begin{aligned} d(x^*, y^*) &\leq p_\phi(T_{\mathbb{T}}x^*, T_{\mathbb{T}}y^*)_{\mathbb{T}} \\ &\leq D_\alpha(T_{\mathbb{T}}x^*, T_{\mathbb{T}}y^*) \\ &\leq ad(x^*, y^*) \end{aligned}$$

So $(1-a)d(x^*, y^*) \leq 0$ implying that $d(x^*, y^*) \leq 0$. Therefore $x^* = y^*$.
To show (iii),

$$\begin{aligned} d(x_{n+1}, x^*) &\leq p_\phi(T_{\mathbb{T}}x_n, T_{\mathbb{T}}x^*) \\ &\leq D_\phi(T_{\mathbb{T}}x_n, T_{\mathbb{T}}x^*) \\ &\leq ad(x_n, x^*) \end{aligned}$$

Thus $\lim_{n\to\infty} d(x_n, x^*) = 0$. Therefore $\{x_n^*\}_{n=0}^{\infty}$ converges strongly to a $\phi-\mathbb{T}-$Relative fuzzy fixed point x^* as $(1-a)d(x_n, x^*) \to 0$ as $n \to \infty$. The proof is complete. □

Theorem 2.1 above generalize, among others, the following results:

Corollary 2.1. Let X be a complete metric space and $T_t : X \to \pi(X)_t$ a $\phi_{\mathbb{T}}-$Relative fuzzy contraction map. Then there exist ϕ-$\mathbb{T}-$Relative fuzzy fixed point $x^* \in X$ such that $x^* \in T_t x^*(\phi)_t$.

Proof. The proof follows from the proof of Theorem 2.1 above if $\mathbb{T} = t$. □

Remark 2.4. The existence of ϕ-fixed $\mathbb{T}-$Relative fuzzy point of $T_{\mathbb{T}}$ implies existence of ϕ-fixed $\mathbb{T}-$Relative fuzzy point of T_t for any $t \in \mathbb{T}$ but the converse is not true.

Corollary 2.2. (**Theorem 3.2 of Estruch and Vidal, 2001**). Let $\alpha \in (0, 1]$ and X be a complete metric space and T be a fuzzy map from X into $Q(X)$ satisfying the condition that

$$D_\alpha(Tx, Ty) \leq cd(x, y)$$

for each $x, y \in X$ and $c \in (0, 1]$. Then there exists fixed fuzzy point $x^* \in X$ for any $\alpha \in [0, 1]$ such that $x^*_\alpha \subset Tx^*$.

Proof. The proof follows from Theorem 2.1 above if for all $x \in X$, we have that $\phi(x, t) = \alpha$ for all $t \in \mathbb{T}$, $\pi(X)_{\mathbb{T}} = Q(X)$ and the $\mathbb{T}-$Relative fuzzy sets are fuzzy sets. □

Corollary 2.3. (**Theorem 2.1 of Heilpern, 1981**). Let X be a complete metric space and $T : X \to Q(X)$ such that

$$D(Tx, Ty) \leq cd(x, y)$$

for each $x, y \in X$ and $c \in (0, 1]$. Then there exists a fuzzy point $x^* \in X$ such that $\{x^*\} \subset Tx^*$.

Proof. The proof follows from Theorem 2.1 if for all $x \in X$, we have that $\phi(x, t) = 1$ for all $t \in \mathbb{T}$, $\pi(X)_{\mathbb{T}} = Q(X)$ and the $\mathbb{T}-$Relative fuzzy sets are fuzzy sets. □

Corollary 2.4. (**Nadler, 1969**). Let X be a complete metric space. Let $T : X \to 2^X$ such that

$$H(Tx, Ty) \leq cd(x, y)$$

for each $x, y \in X$ and $c \in (0, 1]$. Then there exists a fixed point $x^* \in X$ such that $x^* \in Tx^*$.

Proof. The proof follows from Theorem 2.1 if for all $x \in X$, we have that $\phi(x, t) = 1$ for all $t \in \mathbb{T}$ and $\pi(X)_{\mathbb{T}} = 2^X$. □

The next results shows the relationship that exists between fixed points of T_t for each $t \in \mathbb{T}$ and the fixed point of $T_{\mathbb{T}}$.

Theorem 2.2. Let X be a complete metric space and $x^*_{\alpha_{t_i}}$ the $\alpha - \mathbb{T}-$Relative fuzzy fixed points of the $\mathbb{T}-$Relative fuzzy maps $T_{t_i} : X \to \pi(X)_{t_i}$ for each $t_i \in \mathbb{T}$, $i = 1, 2, \ldots, n$, $n \in \mathbb{N}$. Suppose $x^*_{\alpha_{\mathbb{T}}}$ is the $\alpha - \mathbb{T}-$Relative fuzzy fixed point of $T_{\mathbb{T}} : X \to \pi(X)_{\mathbb{T}}$. Then $\cap_{i=1}^n x^*_{\alpha_{t_i}} = x^*_{\alpha_{\mathbb{T}}}$.

Proof. Let $p^*_\alpha \in \cap_{i=1}^n x^*_{\alpha_{t_i}}$ for any $\alpha \in (0, 1]$. Then $p^*_\alpha \subset T_{t_i}$ for all $t_i \in \mathbb{T}$. This implies that $p^* \in T_{t_i}(\alpha)$ for all $t_i \in \mathbb{T}$. So $p^* \in \cap_{i=1}^n T_{t_i}(\alpha)$. But $\cap_{i=1}^n T_{t_i}(\alpha) = T_{\mathbb{T}}(\alpha)$. Thus $p^*_\alpha \subset T_{\mathbb{T}}$. We have that $\cap_{i=1}^n x^*_{\alpha_{t_i}} \subset x^*_{\alpha_{\mathbb{T}}}$.
Similarly, we can prove that $x^*_{\alpha_{\mathbb{T}}} \subset \cap_{i=1}^n x^*_{\alpha_{t_i}}$ and the proof is complete. □

Next, we prove that a sequence of fixed points of a $\mathbb{T}-$Relative fuzzy map converge for a sequence of α_n with $n \geq 1$ converging to an $\alpha > 0$.

Theorem 2.3. Let X be a complete metric space and $x^*_{\alpha_{n_\mathbb{T}}}$ the $\alpha_n - \mathbb{T}-$Relative fuzzy fixed points of the $\mathbb{T}-$Relative fuzzy map $T_\mathbb{T} : X \to \pi(X)_\mathbb{T}$ for each $n \geq 1$ with $n \in \mathbb{N}$. If α_n is a nondecreasing sequence converging to $\alpha > 0$, then

$$x^*_{\alpha_\mathbb{T}} = \cap_{n\geq 1} x^*_{\alpha_{n_\mathbb{T}}}.$$

Proof. Since each $x^*_{\alpha_{n_\mathbb{T}}}$ is the $\alpha_n - \mathbb{T}-$Relative fuzzy fixed points of $T_\mathbb{T}$ for each $n \in \mathbb{N}$, then $x^* \in T_\mathbb{T} x^*(\alpha_n)$ for each $n \in \mathbb{N}$. Now $T_\mathbb{T} x^*(\alpha) = \cap_{n\geq 1} T_\mathbb{T} x^*(\alpha_n)$ as each $T_\mathbb{T} x^*(\alpha_n) \in \pi(X)_\mathbb{T}$ and α_n is a nondecreasing sequence converging to $\alpha > 0$. So $x^*_{\alpha_\mathbb{T}} = \cap_{n\geq 1} x^*_{\alpha_{n_\mathbb{T}}}$ as $x^* \in T_\mathbb{T} x^*(\alpha_n)$ for each $n \in \mathbb{N}$. The proof is complete. □

The following result is immediate from Theorem 2.3 above.

Corollary 2.5. Let X be a complete metric space and $x^*_{\alpha_{n_t}}$ the $\alpha_n - \mathbb{T}-$Relative fuzzy fixed points of the $\mathbb{T}-$Relative fuzzy map $T_t : X \to \pi(X)_t$ for each $n \geq 1$ with $n \in \mathbb{N}$. If α_n is a nondecreasing sequence converging to $\alpha > 0$, then

$$x^*_{\alpha_t} = \cap_{n\geq 1} x^*_{\alpha_n t}.$$

Proof. The proof follows from the proof of Theorem 2.3 above for $\mathbb{T} = t$. The proof is complete. □

Lastly, we prove the convergence of a sequence of fixed points for a sequence of $\mathbb{T}-$Relative fuzzy maps. The property of uniform convergence of a sequence of functions is a requirement.

Theorem 2.4. Let X be a complete metric space and $x^*_{\alpha_{t_i}}$ the $\alpha - \mathbb{T}-$Relative fuzzy fixed points of a sequence of $\mathbb{T}-$Relative fuzzy maps $T_{t_i} : X \to \pi(X)_{t_i}$ for each $t_i \in \mathbb{T}$, with $t_i \leq t_{i+1}$, $i \in \mathbb{N}$. Then $x^*_{\alpha_{t_i}} \to x^*_\alpha$ as $t_i \to \sup\{\mathbb{T}\}$ if T_{t_i} converges uniformly to a c-contraction $T : X \to \pi(X)$ with $F_T(\alpha) = \{x^*_\alpha\}$ where $\alpha \in (0, 1]$.

Proof. Let $\epsilon > 0$ and choose a natural number I such that $t_i \geq t_I$ then by the uniform convergence of T_{t_i} to T we have

$$\sup_{x\in X} D_\alpha(T_{t_i}x, Tx) < \epsilon(1-c) < \epsilon$$

where c is the constant coefficient. Then for all $t_i \geq t_I$ we have

$$\begin{aligned} d(x^*_{\alpha_{t_i}}, x^*_\alpha) &\leq \sup_{x\in X} D_\alpha(T_{t_i}x^*_{\alpha_{t_i}}, Tx^*_\alpha) \\ &\leq \sup_{x\in X} D_\alpha(T_{t_i}x^*_{\alpha_{t_i}}, Tx^*_{\alpha_{t_i}}) + \sup_{x\in X} D_\alpha(Tx^*_{\alpha_{t_i}}, Tx^*_\alpha) \\ &< \epsilon(1-c) + c\sup_{x\in X} d(x^*_{\alpha_{t_i}}, x^*_\alpha) \\ &< \epsilon + c\sup_{x\in X} d(x^*_{\alpha_{t_i}}, x^*_\alpha) \end{aligned}$$

which gives

$$d(x^*_{\alpha_{t_i}}, x^*_\alpha) < \epsilon \text{ for all } t_i \geq t_I$$

Thus $x^*_{\alpha_{t_i}} \to x^*_\alpha$ as $t_i \to \sup\{\mathbb{T}\}$. The proof is complete. □

2.3. **Examples.** In this section some examples are given to illustrate some of our introduced concepts and results.

The example below shows the computation of the metrics on a collection of approximate quantities using the Definitions 2.2-2.5. It verifies Remark 2.1(iii) also.

Example 2.1. Suppose on $X = \{x : 0.5 \leq x \leq 10\}$, we define $\mathbb{T}-$Relative fuzzy sets R and R' of the real numbers close to 1 relative to the interval $[1,4]$ as $\mu_X(x,t) = exp(-1.2(x-1)^{2\sigma(t)})$ and $\mu'_Y(y,t) = \frac{1}{|\sigma(t)+y|}$ for all $x, y \in \mathbb{R}$ respectively where $\mathbb{R}$ is equipped with the usual metric. Let $W(X) = \{R, R'\}$. If $\alpha = 0.4$, then $R(0.4) = [0.4, 1.2]$ and $R'^{[}(0.4) = [0.8, 5.2]$. So

(i)

$$p_{0.4}(R, R') = \inf_{y\in R'(0.4), x\in R(0.4)} d(x,y) = \inf_{y\in[0.8,5.2], x\in[0.4,1.2]} | x - y |=| 1.2 - 0.8 |= 0.4$$

(ii)

$$\delta_{0.4}(R, R^{'}) = \sup_{y\in R^{'}(0.4), x\in R(0.4)} d(x, y) = \sup_{y\in[0.8,5.2], x\in[0.4,1.2]} \mid x - y \mid = \mid 5.2 - 0.4 \mid = 4.8$$

(iii)

$$\begin{aligned} D_{0.4}(Tx, Ty) &= D_{0.4}(R^{'}, R) = H_{0.4}((R^{'}, R) \\ &= \max\{\sup_{x\in R(0.4)} p_{0.4}(x, R^{'}), \sup_{y\in R^{'}(0.4)} p_{0.4}(y, R)\} \\ &= \max\{\sup_{x\in R(0.4)} \inf_{y\in R^{'}(0.4)} d(x, R^{'}(0.4), \sup_{y\in R^{'}(0.4)} \inf_{x\in R(0.4)} d(y, R(0.4)\} \\ &= \max\{\sup_{x\in R(0.4)} \inf_{y\in[0.8,5.2]} \mid x - [0.8, 5.2] \mid, \sup_{y\in R^{'}(0.4)} \inf_{x\in[0.4,1.2]} \mid y - [0.4, 1.2] \mid\} \\ &= \max\{0.4, 4\} = 4 \end{aligned}$$

The example below shows that as we vary α and t, we have varying $\mathbb{T}-$Relative fuzzy fixed point for a $\mathbb{T}-$Relative fuzzy map.

Example 2.2. Suppose $X = \{x : 0 \le x \le 1\}$ with $d(x, y) = \mid x - y \mid$ for all $x, y \in X$, then (X, d) is a metric space.

Suppose also that $W_\alpha(X)$ is a sub-collection of approximate quantities of $\mathbb{T}-$Relative fuzzy sets R and $R^{'}$ of the real numbers close to 1 relative to the interval $[1, 4]$ with membership functions

$$\mu_X(x, t) = exp(-1.2(x - 0.05)^{2\sigma(t)})$$

and

$$\mu^{'}_X(x, t) = \frac{1}{2 \mid \sigma(t) + x \mid}$$

for all $x \in \mathbb{R}$ and $t \in [1, 4]$ respectively.

Define $T : X \to W_\alpha(X)$ such that $Ty = R$ and $Tz = R^{'}$ for any $y, z \in X$, where

$y \in [0, \frac{1}{2})$ and $z \in [\frac{1}{2}, 1]$.

Thus the $\alpha - \mathbb{T}-$Relative fuzzy fixed point of T is any $x^* \in X$ such that $x^*_\alpha \subset Tx^*$. This is equivalent to finding $x^* \in X$ such that $x^* \in Tx^*(\alpha)$.

(i) Suppose $\alpha = 0.82$, then $R(0.82) = [0, 0.36]$ and $R^{'}(0.82) = \emptyset$ for all $t \in [1, 4]$.

Clearly the set of points of $[0, \frac{1}{2}) \cap Ty(0.82) = [0, \frac{1}{2}) \cap R(0.82) = [0, \frac{1}{2}) \cap [0, 0.36] = [0, 0.36]$

Also the set of points of $[\frac{1}{2}, 1] \cap Tz(0.82) = [\frac{1}{2}, 1] \cap R^{'}(0.82) = [\frac{1}{2}, 1] \cap \emptyset = [\frac{1}{2}, 1] \cap \emptyset$ is empty. So the $0.82 - \mathbb{T}-$Relative fuzzy fixed point is the set of points of $[0, 0.36]$.

(ii) Suppose $\alpha = 0.85$ and $t = 2$, then $R(0.82) = [0, 0.73]$ and $R^{'}(0.82) = [0.76, 1]$.

Clearly the set of points of $[0, \frac{1}{2}) \cap Ty(0.82) = [0, \frac{1}{2}) \cap R(0.82) = [0, \frac{1}{2}) \cap [0, 0.73] = [0, \frac{1}{2}]$

Also, the set of points of $[\frac{1}{2}, 1] \cap Tz(0.82) = [\frac{1}{2}, 1] \cap R^{'}(0.82) = [\frac{1}{2}, 1] \cap [0.76, 1] = [0.76, 1]$

Therefore all points $[0, \frac{1}{2}]$ and $[0.76, 1]$ are the $0.82 - \mathbb{T}-$Relative fuzzy fixed points of X for $t = 2$.

(iii) Let $\alpha = 0.7$, then $R(0.7) = [0, 0.51]$ and $R^{'}(0.7) = \emptyset$.

Clearly the set of points of $[0, \frac{1}{2}) \cap Ty(0.45) = [0, \frac{1}{2}) \cap R(0.45) = [0, \frac{1}{2}) \cap [0, 0.51] = [0, \frac{1}{2})$.

Also the set of points of $[\frac{1}{2}, 1] \cap Tz(0.7) = [\frac{1}{2}, 1] \cap R^{'}(0.7) = [\frac{1}{2}, 1] \cap \emptyset$.

Therefore all points $x^* \in [0, 0.5]$ are the $0.7 - \mathbb{T}-$Relative fuzzy fixed points of X.

(iv) Let $\alpha = 0.7$ and $t = 2$, then $R(0.7) = [0, 0.82]$ and $R^{'}(0.7) = [0, 1]$.

Clearly the set of points of $[0, \frac{1}{2}) \cap Ty(0.7) = [0, \frac{1}{2}) \cap R(0.7) = [0, \frac{1}{2}) \cap [0, 0.82] = [0, \frac{1}{2})$.

Also, the set of points of $[\frac{1}{2}, 1] \cap Tz(0.7) = [\frac{1}{2}, 1] \cap R^{'}(0.7) = [\frac{1}{2}, 1] \cap [0, 1] = [\frac{1}{2}, 1]$.

Therefore all points $x^* \in [0, 1]$ are the $0.7 - \mathbb{T}-$Relative fuzzy fixed points of T at $t = 2$.

The next example shows a $\mathbb{T}-$Relative fuzzy map that satisfies the contraction condition of Theorem 2.1 and Corollary 3.1 above for some $\alpha \in (0, 1]$ and $t \in \mathbb{T}$.

Example 2.3. Suppose $X = [0, 1]$, then (X, d) is a metric space with $d(x, y) =\mid x - y \mid$ for all $x, y \in X$. Let $W_\alpha(X)$ be a sub-collection of approximate quantities of $\mathbb{T}-$Relative fuzzy set relative to the interval $[1, 4]$.
Define the $\mathbb{T}-$Relative fuzzy map $T : X \to W_\alpha(X)$ such that

$$Tx(z, t) = \begin{cases} 0 & 0 \leq z < \frac{1}{7} \\ \frac{0.2}{t} & \frac{1}{7} \leq z \leq \frac{x+1}{7} \\ \frac{0.8}{t} & \frac{x+1}{7} < z \leq 1. \end{cases}$$

(i) Suppose $\alpha = 0.2$, then $Tx(0.2) = [\frac{1}{7}, \frac{x+1}{7}]$ for all $t \in [1, 4]$. Then

$$D_\alpha(Tx, Ty) = H(Tx(\alpha), Ty(\alpha)) = \frac{1}{7} \mid x - y \mid$$

Thus the contraction condition holds and so T has $0.2 - \mathbb{T}-$Relative fuzzy fixed point.

(ii) Suppose $\alpha = 0.26$, then $Tx(0.26) = [\frac{x+1}{7}, 1]$ for all $t \in [1, 4]$. Then

$$D_\alpha(Tx, Ty) = H(Tx(\alpha), Ty(\alpha)) = \frac{1}{7} \mid x - y \mid$$

Thus the contraction condition holds and so T has $0.26 - \mathbb{T}-$Relative fuzzy fixed point.

(iii) Suppose $\alpha = 0.1$, then $Tx(0.1) = [\frac{1}{7}, 1]$ for all $t \in [1,4]$. Then

$$D_\alpha(Tx, Ty) = H(Tx(\alpha), Ty(\alpha)) = \frac{1}{7} \mid x - y \mid .$$

Thus the contraction condition holds and so T has $0.1 - \mathbb{T}-$Relative fuzzy fixed point.

(iv) Suppose $\alpha = 0.1$, then $Tx(0.1) = [\frac{x+2}{14}, \frac{x+8}{14}]$ for $t = 2$. Then

$$\begin{aligned} D_\alpha(Tx, Ty) &= H(Tx(\alpha), Ty(\alpha)) = \mid \frac{x+2}{14} - \frac{y+8}{14} \mid . \\ &\leq \mid \frac{x-y-6}{14} \mid < \mid \frac{x-y}{14} \mid = \frac{1}{14} d(x,y). \end{aligned}$$

Thus the contraction condition holds and so T has $0.1-$fuzzy fixed point at $t = 2$.

The following example illustrates Remark 2.4 above.

Example 2.4. Let $X = [a, b]$ with the metric $d(x, y) = \mid x - y \mid$ for all $x, y \in X$. Let $\alpha \in \frac{1}{4}$ and $\mathbb{T} = \{3, 5\}$. Define $T : X \to W_\alpha(X)$ by

$$Tx(z,t) = \begin{cases} \frac{1}{t} & z = a \\ \frac{\alpha}{2t} & z \in (a,b) \\ \frac{\alpha}{t} & z = b. \end{cases}$$

So $Tx(0.2)_{\{3,5\}} = \{a\}$, $Tx(0.3)_{\{3,5\}} = \emptyset$, $Tx(\frac{1}{40})_{\{3,5\}} = [a, b]$, $Tx(\frac{1}{24})_{\{3,5\}} = \{a, b\}$, $Tx(0.3)_3 = \{a\}$, $Tx(0.3)_5 = \emptyset$, $Tx(\frac{1}{24})_3 = [a, b]$. Clearly, $a_{\alpha_{\{3,5\}}}$ is $\alpha - \mathbb{T}-$Relative fuzzy fixed point, a_{α_3} and a_{α_5} are also $\alpha - \mathbb{T}-$Relative fuzzy fixed point .
Now $Tx(0.3)_{\{3,5\}} = \emptyset$ even though $Tx(0.3)_3 = \{a\}$. Thus Remark 2.4 holds.

The following example illustrates Theorem 2.1 above.

Example 2.5. Let $\mathbb{T} = \{1, \frac{3}{4}, \frac{1}{2}, \frac{1}{4}\}$, $a, b, c \in X$, $\alpha \in (0, 1]$ and $T_{\mathbb{T}} : X \to \pi(X)_{\mathbb{T}}$ such that

$$T_{\mathbb{T}}a(x,t) = \begin{cases} t & x = a \\ \frac{t\alpha}{2} & x = b \\ t\alpha & x = c. \end{cases}$$

Thus for each $t_i \in \mathbb{T}$ we have $T_{t_i} : X \to \pi(X)_{t_i}$ such that

$$T_{t_i}a(x,t) = \begin{cases} t_i & x = a \\ \frac{t_i\alpha}{2} & x = b \\ t_i\alpha & x = c. \end{cases}$$

Suppose $\alpha = \frac{1}{2}$ then $T_{\mathbb{T}}a(1) = \emptyset$ but $T_1 a(1) = \{a\}$ even though $T_{\frac{3}{4}}a(1) = T_{\frac{1}{2}}a(1) = T_{\frac{1}{4}}a(1) = \emptyset$.
$T_{\mathbb{T}}a(\frac{3}{4}) = \emptyset$ but $T_1 a(\frac{3}{4}) = T_{\frac{3}{4}}a(\frac{3}{4}) = \{a\}$ even though $T_{\frac{1}{2}}a(\frac{3}{4}) = T_{\frac{1}{4}}a(\frac{3}{4}) = \emptyset$.
Also $T_{\mathbb{T}}a(\frac{3}{4}) = \emptyset$ but $T_1 a(\frac{1}{2}) = \{a, c\}$, $T_{\frac{3}{4}}a(\frac{1}{2}) = T_{\frac{1}{2}}a(\frac{1}{2})\{a\}$ even though $T_{\frac{1}{4}}a(\frac{1}{2}) = \emptyset$.
But $T_{\mathbb{T}}a(\frac{1}{4}) = \{a\}$, $T_1 a(\frac{1}{4}) = \{a, b, c\}$, $T_{\frac{3}{4}}a(\frac{1}{4}) = T_{\frac{1}{2}}a(\frac{1}{4}) = \{a, c\}$ and $T_{\frac{1}{4}}a(\frac{1}{4}) = \{a\}$.
Thus $T_{\mathbb{T}}a(\frac{1}{2}) = \{a\} = \cap_{i=1}^{4} T_{t_i}a(\frac{1}{4})$. Therefore Theorem 3.3 above holds.

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