Jordan Journal of Mathematics and Statistics (JJMS) 13(1), 2020, pp 89 - 109

CONFORMAL RICCI SOLITONS ON 3-DIMENSIONAL TRANS-SASAKIAN MANIFOLD

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Abstract. In this paper we have studied and obtained results on Conformal Ricci solitons in 3-dimensional trans-Sasakian manifold M satisfying $R(\xi, X).B = 0$,$B(\xi, X).S = 0$, $S(\xi, X).R = 0$,$R(\xi, X).\overline{P} = 0$ and $\overline{P}(\xi, X).S = 0$, where B and $\overline{P}$ are C-Bochner and Pseudo-projective curvature tensor respectively.

1. Introduction

The concept of Ricci flow and the proof of its existence was introduced by Hamilton [9] in 1982. This concept was developed to answer Thurston's geometric conjecture which says that each closed three manifold admits a geometric decomposition. Hamilton also [9] classified all compact manifolds with positive curvature operator in dimension four. The Ricci flow equation is given by

$$\frac{\partial g}{\partial t} = -2S \tag{1.1}$$

on a compact Riemannian manifold M with Riemannian metric g.

A self-similar solution to the Ricci flow [9], [15] is called a Ricci soliton [10] if it moves only by a one parameter family of diffeomorphism and scaling. The Ricci

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1991 *Mathematics Subject Classification.* 53D15,53A30,53C25.

Key words and phrases. Conformal Ricci soliton, trans-Sasakian manifold, $\alpha-$Sasakian manifold.

Received: Sept. 27, 2018 Accepted: Dec. 9, 2018 .

soliton equation is given by

$$\pounds_V g + 2S = 2\lambda g, \tag{1.2}$$

where $\pounds_V$ is the Lie derivative, S is Ricci tensor, g is Riemannian metric, V is a vector field and λ is a scalar. The Ricci soliton is said to be shrinking as λ is positive, steady as λ is zero and expanding as λ is negative.
In [14] and [16] authors have studied various types of Ricci Soliton in their papers.

The concept of conformal Ricci flow [7]was developed by A.E. Fischer during 2003-2004 which is a variation of the classical Ricci flow equation that modifies the unit volume constraint of that equation to a scalar curvature constraint. The conformal Ricci flow on M is defined by the equation [7]

$$\frac{\partial g}{\partial t} + 2(S + \frac{g}{n}) = -pg \tag{1.3}$$

and $r(g) = -1$,
where M is considered as a smooth closed connected oriented n-manifold, p is a scalar non-dynamical field(time dependent scalar field), $r(g)$ is the scalar curvature of the manifold and n is the dimension of manifold.

The notion of conformal Ricci soliton equation was introduced by N. Basu and A. Bhattacharyya [2] in 2015 and the equation is given by

$$\pounds_V g + 2S = [2\lambda - (p + \frac{2}{n})]g, \tag{1.4}$$

where λ is constant.
The equation is the generalization of the Ricci soliton equation and it also satisfies the conformal Ricci flow equation.

In the paper [6],the authors studied a 3-dimensional trans-Sasakian manifold M,which admits Conformal Ricci soliton with the vector field V is pointwise collinear with the vector field ξ and then they also proved under the same condition almost Conformal Ricci soliton reduces to Conformal Ricci soliton.They showed if it admits Conformal gradient shrinking Ricci soliton then the manifold becomes Einstein.
In paper [1] authors studied various curvature properties on Kenmotsu manifold.
So motivated from these two papers we have established some interesting results on 3-dimensional trans-Sasakian manifold admitting Conformal Ricci soliton.Here we have studied 3-dimensional trans-Sasakian manifold admitting Conformal Ricci soliton and satisfying the conditions $R(\xi, X).B = 0$,$B(\xi, X).S = 0$, $S(\xi, X).R = 0$,$R(\xi, X).\overline{P} = 0$ and $\overline{P}(\xi, X).S = 0$, where B and $\overline{P}$ are C-Bochner and Pseudo-projective curvature tensor respectively.

2. Preliminaries

Let M be a connected almost contact metric manifold with an almost contact metric structure (ϕ, ξ, η, g) where ϕ is a $(1, 1)$ tensor field, ξ is a vector field, η is a 1-form and g is the compatible Riemannian metric such that

$$\phi^2(X) = -X + \eta(X)\xi, \eta(\xi) = 1, \eta \circ \phi = 0, \phi\xi = 0, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.2}$$

$$g(X, \phi Y) = -g(\phi X, Y), \tag{2.3}$$

$$g(X, \xi) = \eta(X), \tag{2.4}$$

for all vector fields $X, Y \in \chi(M)$.

An almost contact metric structure (ϕ, ξ, η, g) on M is called a trans-Sasakian structure [13], if $(M \times R, J, G)$ belongs to the class W_4 [8], where J is the almost complex structure on $M \times R$ defined by $J(X, f\frac{d}{dt}) = (\phi X - f\xi, \eta(X)\frac{d}{dt})$ for all vector fields X on M and smooth functions f on $M \times R$. It can be expressed by the condition [4],

$$(\nabla_X \phi)Y = \alpha(g(X,Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X), \tag{2.5}$$

for some smooth functions α, β on M and we say that the trans-Sasakian structure is of type (α, β). From the above expression we can write

$$\nabla_X \xi = -\alpha\phi X + \beta(X - \eta(X)\xi), \tag{2.6}$$

$$(\nabla_X \eta)Y = -\alpha g(\phi X, Y) + \beta g(\phi X, \phi Y). \tag{2.7}$$

For a 3-dimensional trans-Sasakian manifold the following relations hold [6]:

$$2\alpha\beta + \xi\alpha = 0, \tag{2.8}$$

$$S(X,\xi) = (2(\alpha^2 - \beta^2) - \xi\beta)\eta(X) - X\beta - (\phi X)\alpha, \tag{2.9}$$

$$\begin{aligned} S(X,Y) &= (\frac{r}{2} + \xi\beta - (\alpha^2 - \beta^2))g(X,Y) - (\frac{r}{2} + \xi\beta - 3(\alpha^2 - \beta^2))\eta(X)\eta(Y) \\ &- (Y\beta + (\phi Y)\alpha)\eta(X) - (X\beta + (\phi X)\alpha)\eta(Y), \end{aligned} \tag{2.10}$$

where S denotes the Ricci tensor of type $(0,2)$, r is the scalar curvature of the manifold M and α, β are smooth functions on M.

For $\alpha, \beta =$ constant the following relations hold [6]:

$$S(X,Y) = (\frac{r}{2} - (\alpha^2 - \beta^2))g(X,Y) - (\frac{r}{2} - 3(\alpha^2 - \beta^2))\eta(X)\eta(Y), \tag{2.11}$$

$$S(X,\xi) = 2(\alpha^2 - \beta^2)\eta(X), \tag{2.12}$$

$$R(X,Y)\xi = (\alpha^2 - \beta^2)[\eta(Y)X - \eta(X)Y], \tag{2.13}$$

$$R(\xi,X)Y = (\alpha^2 - \beta^2)[g(X,Y)\xi - \eta(Y)X], \tag{2.14}$$

$$R(\xi,X)\xi = (\alpha^2 - \beta^2)[\eta(X)\xi - X], \tag{2.15}$$

$$\eta(R(X,Y)Z) = (\alpha^2 - \beta^2)[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)], \tag{2.16}$$

where R is the Riemannian curvature tensor.

$$QX = (\frac{r}{2} - (\alpha^2 - \beta^2))X - (\frac{r}{2} - 3(\alpha^2 - \beta^2))\eta(X)\xi, \tag{2.17}$$

where Q is the Ricci operator given by $S(X,Y) = g(QX,Y)$.

Again,

$$\begin{aligned}(\pounds_\xi g)(X,Y) &= (\nabla_\xi g)(X,Y) - \alpha g(\phi X,Y) + 2\beta g(X,Y) - 2\beta\eta(X)\eta(Y) \\ &- \alpha g(X,\phi Y) \\ &= 2\beta g(X,Y) - 2\beta\eta(X)\eta(Y).\end{aligned}$$

(2.18)

Putting the above value in the conformal Ricci soliton equation (1.4) and taking $n = 3$ we get,

$$\begin{aligned}S(X,Y) &= \frac{1}{2}[2\lambda - (p + \frac{2}{3})]g(X,Y) - \frac{1}{2}[2\beta g(X,Y) - 2\beta\eta(X)\eta(Y)] \\ &= [\lambda - \frac{1}{2}(p + \frac{2}{3}) - \beta]g(X,Y) + \beta\eta(X)\eta(Y).\end{aligned} \tag{2.19}$$

From the above equation, we have,

$$QX = [\lambda - \frac{1}{2}(p + \frac{2}{3}) - \beta]X + \beta\eta(X)\xi, \tag{2.20}$$

$$S(X,\xi) = [\lambda - \frac{1}{2}(p + \frac{2}{3})]\eta(X), \tag{2.21}$$

$$r = 3[\lambda - \frac{1}{2}(p + \frac{2}{3}) - \beta] + \beta. \tag{2.22}$$

Thus we can state the following proposition.

Proposition 2.1 : If a 3-dimensional trans-Sasakian manifold admits conformal Ricci soliton, then the manifold becomes an η-Einstein manifold.

3. Conformal Ricci Soliton in a 3-dimensional trans-Sasakian Manifold Satisfying $R(\xi, X).B = 0$

Bochner introduced a *Kähler* analogue of the Weyl conformal curvature tensor by purely formal considerations, which is now well known as the Bochner curvature tensor [5]. A geometric meaning of the Bochner curvature tensor is given by Blair in [3] by using the Boothby-Wangs fibration. In 1969, Matsumoto and *Chūman* [12] constructed the notion of C-Bochner curvature tensor in a Sasakian manifold and studied its several properties.

The C-Bochner curvature tensor [11] B in M is defined by,

$$\begin{aligned} B(X,Y)Z = R(X,Y)Z &+ \frac{1}{6}[g(X,Z)QY - S(Y,Z)X - g(Y,Z)QX \\ &+ S(X,Z)Y + g(QX,Z)Q\phi Y - S(\phi Y,Z)\phi X - g(\phi Y,Z)Q\phi X \\ &+ S(\phi X,Z)\phi Y + 2S(\phi X,Y)\phi Z \\ &+ 2g(\phi X,Y)Q\phi Z + \eta(Y)\eta(Z)QX - \eta(Y)S(X,Z)\xi \\ &+ \eta(X)S(Y,Z)\xi - \eta(X)\eta(Z)QY] \\ &- \frac{D+2}{6}[g(\phi X,Z)\phi Y - g(\phi Y,Z)\phi X \\ &+ 2g(\phi X,Y)\phi Z] \\ &+ \frac{D}{6}[\eta(Y)g(X,Z)\xi - \eta(Y)\eta(Z)X \end{aligned} \tag{3.1}$$

$$+\eta(X)\eta(Z)Y-\eta(X)g(Y,Z)\xi]$$
$$-\frac{D-4}{6}[g(X,Z)Y-g(Y,Z)X],$$

where $D=\frac{r+2}{4}$. Taking $Z=\xi$, we get,

$$\begin{aligned}
B(X,Y)\xi &= R(X,Y)\xi+\frac{1}{6}[g(X,\xi)QY-S(Y,\xi)X-g(Y,\xi)QX \\
&+ S(X,\xi)Y+g(QX,\xi)Q\phi Y-S(\phi Y,\xi)\phi X-g(\phi Y,\xi)Q\phi X \\
&+ S(\phi X,\xi)\phi Y+2S(\phi X,Y)\phi\xi \\
&+ 2g(\phi X,Y)Q\phi\xi+\eta(Y)\eta(\xi)QX-\eta(Y)S(X,\xi)\xi \\
&+ \eta(X)S(Y,\xi)\xi-\eta(X)\eta(\xi)QY] \\
&- \frac{D+2}{6}[g(\phi X,\xi)\phi Y-g(\phi Y,\xi)\phi X \\
&+ 2g(\phi X,Y)\phi\xi] \\
&+ \frac{D}{6}[\eta(Y)g(X,\xi)\xi-\eta(Y)\eta(\xi)X \\
&+ \eta(X)\eta(\xi)Y-\eta(X)g(Y,\xi)\xi] \\
&- \frac{D-4}{6}[g(X,\xi)Y-g(Y,\xi)X].
\end{aligned}$$

Now using (2.13),(2.19),(2.20) in the above equation, we get,

$$B(X,Y)\xi=[(\beta^2-\alpha^2)+\frac{1}{6}(\lambda-\frac{1}{2}(p+\frac{2}{3}))+\frac{4}{6}][\eta(X)Y-\eta(Y)X]. \tag{3.2}$$

From (3.1) we have,

$$\eta(B(X,Y)Z=\eta(R(X,Y)Z)+\frac{1}{6}[g(X,Z)\eta(QY)-S(Y,Z)\eta(X)$$
$$-g(Y,Z)\eta(QX)+S(X,Z)\eta(Y)+g(\phi X,Z)\eta(Q\phi Y)$$

$$
\begin{aligned}
&- S(\phi Y, Z)\eta(\phi X) - g(\phi Y, z)\eta(Q\phi X) + S(\phi X, Z)\eta(\phi Y)\\
&+ 2S(\phi X, Y)\eta(\phi Z) + 2g(\phi X, Y)\eta(Q\phi Z) + \eta(Y)\eta(Z)\eta(QX)\\
&- \eta(Y)S(X, Z)\eta(\xi) + \eta(X)S(Y, Z)\eta(\xi) - \eta(X)\eta(Z)\eta(QY)]\\
&- \frac{D+2}{6}[g(\phi X, Z)\eta(\phi Y) - g(\phi Y, Z)\eta(\phi X)\\
&+ 2g(\phi X, Y)\eta(\phi Z)]\\
&+ \frac{D}{6}[\eta(Y)g(X, Z)\eta(\xi) - \eta(Y)\eta(Z)\eta(X)\\
&+ \eta(X)\eta(Z)\eta(Y) - \eta(X)g(Y, Z)\eta(\xi)]\\
&- \frac{D-4}{6}[g(X, Z)\eta(Y) - g(Y, Z)\eta(X)].
\end{aligned}
\tag{3.3}
$$

Using (2.16),(2.19),(2.20),(2.21) in above equation and taking $\eta(\xi)$=1, we get,

$$
\begin{aligned}
\eta(B(X, Y)Z = {}& (\beta^2 - \alpha^2)[g(X, Z)\eta(Y) - g(Y, Z)\eta(X)]\\
&+ \frac{1}{6}[(\lambda - \frac{1}{2}(p + \frac{2}{3}))g(X, Z)\eta(Y)\\
&- (\lambda - \frac{1}{2}(p + \frac{2}{3}))g(Y, Z)\eta(X)]\\
&+ [g(X, Z)\eta(Y) - g(Y, Z)\eta(X)][\frac{D}{6} - \frac{D-4}{6}].
\end{aligned}
$$

Simplifying,we get,

$$
\eta(B(X, Y)Z) = [(\beta^2 - \alpha^2) + \frac{1}{6}(\lambda - \frac{1}{2}(p + \frac{2}{3})) + \frac{4}{6}][g(X, Z)\eta(Y) - g(Y, Z)\eta(X)]. \tag{3.4}
$$

We assume that the condition $R(\xi, X).B = 0$, then we have,

$$
\begin{aligned}
&R(\xi, X)B(Y, Z)W - B(R(\xi, X)Y, Z)W - B(Y, R(\xi, X)Z)W\\
&\qquad - B(Y, Z)R(\xi, X)W = 0.
\end{aligned}
\tag{3.5}
$$

Using (2.14) in (3.5), we get,

$$\eta(B(Y,Z)W)X - g(B(Y,Z)W,X)\xi + g(X,Y)B(\xi,Z)W - \eta(Y)B(X,Z)W + g(X,Z)B(Y,\xi)W - \eta(Z)B(Y,X)W + g(X,W)B(Y,Z)\xi - \eta(W)B(Y,Z)X = 0. \tag{3.6}$$

By taking an inner product with ξ and using (2.1), we get,

$$\eta(B(Y,Z)W)\eta(X) - g(B(Y,Z)W,X) + g(X,Y)\eta(B(\xi,Z)W) - \eta(Y)\eta(B(X,Z)W) + g(X,Z)\eta(B(Y,\xi)W) - \eta(Z)\eta(B(Y,X)W) + g(X,W)\eta(B(Y,Z)\xi) - \eta(W)\eta(B(Y,Z)X) = 0. \tag{3.7}$$

By using (3.2) and (3.4) in (3.7), we get,

$$[(\beta^2-\alpha^2) + \frac{1}{6}(\lambda - \frac{1}{2}(p+\frac{2}{3})) + \frac{4}{6}][g(Y,W)g(Z,X) - g(Z,W)g(Y,X)] - g(B(Y,Z)W,X) = 0. \tag{3.8}$$

Now using (3.1) in (3.8), we get,

$$\begin{aligned}
&[(\beta^2-\alpha^2) + \frac{1}{6}(\lambda - \frac{1}{2}(p+\frac{2}{3})) + \frac{4}{6}][g(Y,W)g(Z,X) - g(Z,W)g(Y,X)]\\
&-g(R(Y,Z)W,X) - \frac{1}{6}[g(Y,W)S(Z,X) - S(Z,W)g(Y,X)\\
&-g(Z,W)S(Y,X) + S(Y,W)g(Z,X) + g(\phi Y,W)S(\phi Z,X)\\
&-S(\phi Z,W)g(\phi Y,X) - g(\phi Z,W)S(\phi Y,X) + S(\phi Y,W)g(\phi Z,X)\\
&+2S(\phi Y,Z)g(X,\phi W) + 2g(\phi Y,Z)S(X,\phi W)\\
&+\eta(W)\eta(Z)S(Y,X) - \eta(X)\eta(Z)S(Y,W) + \eta(Y)\eta(X)S(Z,W)
\end{aligned}$$

$$-\eta(W)\eta(Y)S(Z,X)] + \frac{D+2}{6}[g(\phi Y,W)g(\phi Z,X) - g(\phi Z,W)g(\phi Y,X)$$
$$+2g(\phi Y,Z)g(\phi W,X)] - \frac{D}{6}[\eta(X)\eta(Z)g(Y,W) - \eta(W)\eta(Z)g(Y,X)$$
$$+\eta(W)\eta(Y)g(Z,X) - \eta(Y)\eta(X)g(Z,W)]$$
$$+\frac{D-4}{6}[g(Y,W)g(Z,X) - g(Z,W)g(X,Y)] = 0. \tag{3.9}$$

Taking $X = Y = e_i$ and summing over $i = 1, 2, 3$,w here $\{e_i\}$ is an orthonormal basis of $T_p(M)$ and by using (2.19), (2.20), (2.21) and on simplification, we get,

$$S(Z,W) \tag{3.10}$$
$$= [\frac{r}{6} + (\lambda - \frac{1}{2}(p+\frac{2}{3}) - \beta) - 2\{(\beta^2-\alpha^2) + \frac{1}{6}(\lambda - \frac{1}{2}(p+\frac{2}{3})) + \frac{4}{6}\}$$
$$-\frac{4D-2}{6}]g(Z,W)$$
$$+[\frac{4D+6}{6} - \frac{r}{6} - (\lambda - \frac{1}{2}(p+\frac{2}{3}) - \beta) + \frac{1}{3}(\lambda - \frac{1}{2}(p+\frac{2}{3}))]\eta(Z)\eta(W).$$

Putting $Z = W = \xi$ and the value of D in (3.10) and by using (2.19) and (2.22), we get,

$$[\lambda - \frac{1}{2}(p+\frac{2}{3})] - 2(\alpha^2-\beta^2) = \frac{\beta}{3}. \tag{3.11}$$

As for Conformal Ricci soliton λ is constant,so if we use equation (3.5) of [6] in (3.11), we get,

$$\beta = 0. \tag{3.12}$$

Then from (3.11) we get,

$$\lambda = 2\alpha^2 + \frac{1}{2}(p+\frac{2}{3}). \tag{3.13}$$

So we can state the following theorem:

Theorem 3.1. *If a 3-dimensional trans-Sasakian manifold satisfies $R(\xi,X).B = 0$ and admits Conformal Ricci soliton, then the nature of the soliton behaves as:*

(a)If $p = -\frac{2}{3} - 4\alpha^2$, *then* $\lambda = 0$ *and the Ricci soliton is steady.*

(b)If $p > -\frac{2}{3} - 4\alpha^2$, *then* $\lambda > 0$ *and the Ricci soliton is shrinking.*

(c)If $p < -\frac{2}{3} - 4\alpha^2$, *then* $\lambda < 0$ *and the Ricci soliton is expanding.*

Also the manifold reduces to a 3-dimensional α*-Sasakian manifold.*

4. Conformal Ricci Soliton in a 3-dimensional trans-Sasakian Manifold Satisfying $B(\xi, X).S = 0$

The condition $B(\xi, X).S = 0$ implies that,

$$S(B(\xi, X)Y, Z) + S(Y, B(\xi, X)Z) = 0. \tag{4.1}$$

By using the expression of $S(X, Y)$ from (2.19) in (4.1), we get,

$$\begin{aligned}[\lambda - \frac{1}{2}(p + \frac{2}{3}) - \beta]g(B(\xi, X)Y, Z) + \beta\eta(Z)\eta(B(\xi, X)Y) \\ + [\lambda - \frac{1}{2}(p + \frac{2}{3}) - \beta]g(Y, B(\xi, X)Z) + \beta\eta(Y)\eta(B(\xi, X)Z) = 0.\end{aligned} \tag{4.2}$$

The above equation can be written after taking $[\lambda - \frac{1}{2}(p + \frac{2}{3}) - \beta]$ common from first and third components and β from second and fourth components as,

$$\begin{aligned}[\lambda - \frac{1}{2}(p + \frac{2}{3}) - \beta][g(B(\xi, X)Y, Z) + g(Y, B(\xi, X)Z)] \\ = -\beta[\eta(Z)\eta(B(\xi, X)Y) + \eta(Y)\eta(B(\xi, X)Z)].\end{aligned} \tag{4.3}$$

By using (3.1) and (3.4) in (4.3), we get,

$$\begin{aligned}[\beta\{(\beta^2 - \alpha^2) + \frac{1}{6}(\lambda - \frac{1}{2}(p + \frac{2}{3})) + \frac{4}{6}\} - \frac{1}{6}\{\lambda - \frac{1}{2}(p + \frac{2}{3})\}\{\lambda - \frac{1}{2}(p + \frac{2}{3}) - \beta\}] \\ \times [2\eta(X)\eta(Y)\eta(Z) - (g(X, Z)\eta(Y) + g(X, Y)\eta(Z))] = 0.\end{aligned} \tag{4.4}$$

Now if we put $X = Y = \xi$ in (4.4),then the equation is identically satisfied and we can not get the value for λ.

So taking $X = Y = e_i$ and summing over $i = 1, 2, 3$, where $\{e_i\}$ is an orthonormal basis of $T_p(M)$ and also taking the condition $\eta(Z) \neq 0$, we get,

$$\beta\{(\beta^2-\alpha^2)+\frac{1}{6}(\lambda-\frac{1}{2}(p+\frac{2}{3}))+\frac{4}{6}\}-\frac{1}{6}\{\lambda-\frac{1}{2}(p+\frac{2}{3})\}\{\lambda-\frac{1}{2}(p+\frac{2}{3})-\beta\}=0. \tag{4.5}$$

Now if we use equation (3.5) of [6] in (4.5), where λ is constant for Conformal Ricci soliton, then we get,

$$\lambda = \frac{1}{2}(p+\frac{2}{3}) - \frac{\beta}{2} \pm \sqrt{4\beta + \frac{\beta^2}{4}}. \tag{4.6}$$

So we can state the following theorem:

Theorem 4.1. *If a 3-dimensional trans-Sasakian manifold satisfies $B(\xi, X).S = 0$ and admits Conformal Ricci soliton, then the nature of the soliton behaves as:*
(a)If $p = -\frac{2}{3} + \beta \mp 2\sqrt{4\beta + \frac{\beta^2}{4}}$, then $\lambda = 0$ and the Ricci soliton is steady.
(b)If $p > -\frac{2}{3} + \beta \mp 2\sqrt{4\beta + \frac{\beta^2}{4}}$, then $\lambda > 0$ and the Ricci soliton is shrinking.
(c)If $p < -\frac{2}{3} + \beta \mp 2\sqrt{4\beta + \frac{\beta^2}{4}}$, then $\lambda < 0$ and the Ricci soliton is expanding.

5. Conformal Ricci Soliton in a 3-dimensional trans-Sasakian Manifold Satisfying $S(\xi, X).R = 0$

Using the following equation,

$$\begin{aligned} S((X,\xi).R)(U,V)W &= ((X \wedge_S \xi).R)(U,V)W \\ &= (X \wedge_S \xi)R(U,V)W + R((X \wedge_S \xi)U,V)W \\ &+ R(U,(X \wedge_S \xi)V)W + R(U,V)(X \wedge_S \xi)W, \end{aligned} \tag{5.1}$$

where the endomorphism $X \wedge_S Y$ is defined by,

$$(X \wedge_S Y)Z = S(Y,Z)X - S(X,Z)Y, \tag{5.2}$$

we have ,

$$\begin{aligned} S((X,\xi).R)(U,V)W &= S(\xi,R(U,V)W)X - S(X,R(U,V)W)\xi \\ &+ S(\xi,U)R(X,V)W - S(X,U)R(\xi,V)W \\ &+ S(\xi,V)R(U,X)W - S(X,V)R(U,\xi)W \\ &+ S(\xi,W)R(U,V)X - S(X,W)R(U,V)\xi. \end{aligned}$$

(5.3)

Now by using the condition $S(\xi, X).R = 0$ and the equations (2.19), (2.21), we get,

$$\begin{aligned} (5.4)\quad & [\lambda - \frac{1}{2}(p+\frac{2}{3})]\eta(R(U,V)W)X - \{[\lambda - \frac{1}{2}(p+\frac{2}{3}) - \beta]g(X,R(U,V)W) \\ & +\beta\eta(X)\eta(R(U,V)W)\}\xi + [\lambda - \frac{1}{2}(p+\frac{2}{3})]\eta(U)R(X,V)W - \{[\lambda - \frac{1}{2}(p+\frac{2}{3}) - \beta]g(X,U) \\ & +\beta\eta(X)\eta(U)\}R(\xi,V)W + [\lambda - \frac{1}{2}(p+\frac{2}{3})]\eta(V)R(U,X)W - \{[\lambda - \frac{1}{2}(p+\frac{2}{3}) - \beta]g(X,V) \\ & +\beta\eta(X)\eta(V)\}R(U,\xi)W + [\lambda - \frac{1}{2}(p+\frac{2}{3})]\eta(W)R(U,V)X - \{[\lambda - \frac{1}{2}(p+\frac{2}{3}) - \beta]g(X,W) \\ & + \beta\eta(X)\eta(W)\}R(U,V)\xi = 0. \end{aligned}$$

By taking inner product with ξ and using (2.13), (2.14), (2.15) and (2.16), we get,

$$\begin{aligned} (5.5)\quad & [\lambda - \frac{1}{2}(p+\frac{2}{3}) - \beta](\alpha^2-\beta^2)\eta(X)[g(V,W)\eta(U) - g(U,W)\eta(V)] \\ & - [\lambda - \frac{1}{2}(p+\frac{2}{3}) - \beta]g(X,R(U,V)W) + [\lambda - \frac{1}{2}(p+\frac{2}{3})](\alpha^2-\beta^2)[g(V,W)\eta(U)\eta(X) \\ & - g(U,W)\eta(V)\eta(X) + g(V,X)\eta(W)\eta(U) - g(U,X)\eta(W)\eta(V)] \\ & - [\lambda - \frac{1}{2}(p+\frac{2}{3})\beta](\alpha^2-\beta^2)[g(X,U)g(V,W) - g(X,U)\eta(W)\eta(V) \\ & + g(X,V)\eta(W)\eta(U) - g(X,V)g(U,W)] - \beta(\alpha^2-\beta^2)[\eta(X)\eta(U)g(V,W) \\ & - \eta(X)\eta(V)g(U,W)] = 0. \end{aligned}$$

Taking $X = Y = e_i$ and summing over $i = 1, 2, 3$,where $\{e_i\}$ is an orthonormal basis of $T_p(M)$, we get,

$$[\lambda - \frac{1}{2}(p + \frac{2}{3})](\alpha^2 - \beta^2)[\eta(V)\eta(W) - g(V, W)] - [\lambda - \frac{1}{2}(p + \frac{2}{3}) - \beta]S(V, W) + [\lambda - \frac{1}{2}(p + \frac{2}{3})](\alpha^2 - \beta^2)[g(V, W) - 3\eta(V)\eta(W)] = 0. \tag{5.6}$$

Putting $V = W = \xi$, we get,

$$[\lambda - \frac{1}{2}(p + \frac{2}{3})][(\lambda - \frac{1}{2}(p + \frac{2}{3}) - \beta) + 2(\alpha^2 - \beta^2)] = 0. \tag{5.7}$$

Now as for Conformal Ricci soliton λ is constant, so if we use equation (3.5) of [6] in (5.7), then we get,

$$[\lambda - \frac{1}{2}(p + \frac{2}{3})][2(\lambda - \frac{1}{2}(p + \frac{2}{3})) - \beta] = 0. \tag{5.8}$$

This implies that, either $\lambda = \frac{1}{2}(p + \frac{2}{3})$ or $\lambda = \frac{\beta+p+\frac{2}{3}}{2}$.
So we can state the following theorem:

Theorem 5.1. *If a 3-dimensional trans-Sasakian manifold satisfies* $S(\xi, X).R = 0$ *and admits Conformal Ricci soliton, then the nature of the soliton behaves as:*
Case-I:When λ *depends only on* p.
(a)If $p = -\frac{2}{3}$,*then* $\lambda = 0$ *and the Ricci soliton is steady.*
(b)If $p > -\frac{2}{3}$,*then* $\lambda > 0$ *and the Ricci soliton is shrinking.*
(c)If $p < -\frac{2}{3}$,*then* $\lambda < 0$ *and the Ricci soliton is expanding.*
Case-II:When λ *depends on both* p *and* β.
(d)If $\beta + p = -\frac{2}{3}$,*then* $\lambda = 0$ *and the Ricci soliton is steady.*
(e)If $\beta + p > -\frac{2}{3}$,*then* $\lambda > 0$ *and the Ricci soliton is shrinking.*
(f)If $\beta + p < -\frac{2}{3}$,*then* $\lambda < 0$ *and the Ricci soliton is expanding.*

6. Conformal Ricci Soliton in a 3-dimensional trans-Sasakian Manifold Satisfying $R(\xi, X).\overline{P} = 0$

The Pseudo-projective curvature tensor $\overline{P}$ in M is defined by,

$$\begin{aligned}\overline{P}(X,Y))Z &= aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y] \\ &- \frac{r}{3}(\frac{a}{2} + b)[g(Y,Z)X - g(X,Z)Y],\end{aligned} \tag{6.1}$$

where $a, b \neq 0$ are constants.

Taking $Z = \xi$ in (6.1) and using (2.13), (2.19), (2.21), we get,

$$\overline{P}(X,Y)\xi = [a(\alpha^2 - \beta^2) + b\{\lambda - \frac{1}{2}(p + \frac{2}{3})\} - \frac{r}{3}(\frac{a}{2} + b)][\eta(Y)X - \eta(X)Y]. \tag{6.2}$$

Now using (2.16), (2.19), (2.20) and (2.21) in (6.1), we get,

$$\begin{aligned}\eta(\overline{P}(X,Y)Z) &= [a(\alpha^2 - \beta^2) + b\{\lambda - \frac{1}{2}(p + \frac{2}{3}) - \beta\} - \frac{r}{3}(\frac{a}{2} + b)] \\ &\times [g(Y,Z)\eta(X) - g(X,Z)\eta(Y)].\end{aligned} \tag{6.3}$$

We assume that the condition $R(\xi, X).\overline{P} = 0$, then we have,

$$R(\xi,X)\overline{P}(U,V)W - \overline{P}(R(\xi,X)U,V)W - \overline{P}(U,R(\xi,X)V)W - \overline{P}(U,V)R(\xi,X)W = 0. \tag{6.4}$$

Now using (2.14) in (6.4) and taking the condition $\alpha^2 \neq \beta^2$, we get,

$$\begin{aligned}&g(X,\overline{P}(U,V)W)\xi - \eta(\overline{P}(U,V)W)X - g(X,U)\overline{P}(\xi,V)W + \eta(U)\overline{P}(X,V)W \\ &- g(X,V)\overline{P}(U,\xi)W + \eta(V)\overline{P}(U,X)W - g(X,W)\overline{P}(U,V)\xi + \eta(W)\overline{P}(U,V)X \\ &= 0.\end{aligned} \tag{6.5}$$

By taking inner product with ξ, we get,

$$(6.6)\quad g(X,\overline{P}(U,V)W)-\eta(\overline{P}(U,V)W)\eta(X)-g(X,U)\eta(\overline{P}(\xi,V)W)$$
$$+\eta(U)\eta(\overline{P}(X,V)W)-g(X,V)\eta(\overline{P}(U,\xi)W)+\eta(V)\eta(\overline{P}(U,X)W)$$
$$-g(X,W)\eta(\overline{P}(U,V)\xi)+\eta(W)\eta(\overline{P}(U,V)X)=0.$$

Now by using (6.2) and (6.3) in (6.6), we get,

$$(6.7)\quad g(X,\overline{P}(U,V)W)-[a(\alpha^2-\beta^2)+b\{\lambda-\frac{1}{2}(p+\frac{2}{3})-\beta\}-\frac{r}{3}(\frac{a}{2}+b)]$$
$$\times[g(V,W)g(X,U)-g(U,W)g(X,V)]=0.$$

Using (6.1)in (6.7), we get,

(6.8)
$$ag(X,R(U,V)W)+b[\{\lambda-\frac{1}{2}(p+\frac{2}{3})-\beta\}\{g(V,W)g(X,U)-g(U,W)g(X,V)\}$$
$$+\beta\{\eta(V)\eta(W)g(X,U)-\eta(U)\eta(W)g(X,V)\}]-[a(\alpha^2-\beta^2)+b\{\lambda-\frac{1}{2}(p+\frac{2}{3})-\beta\}]$$
$$\times[g(V,W)g(X,U)-g(U,W)g(X,V)]=0.$$

Taking $X=U=e_i$ and summing over $i=1,2,3$,where $\{e_i\}$ is an orthonormal basis of $T_p(M)$ and on simplification, we get,

$$(6.9)\quad aS(V,W)+b[2\{\lambda-\frac{1}{2}(p+\frac{2}{3})-\beta\}g(V,W)+2\beta\eta(V)\eta(W)]$$
$$-2[a(\alpha^2-\beta^2)+b\{\lambda-\frac{1}{2}(p+\frac{2}{3})-\beta\}]g(V,W)=0.$$

Putting $V=W=\xi$ in (6.9) and using (2.21),(2.22), we get,

$$(6.10)\qquad a[\lambda-\frac{1}{2}(p+\frac{2}{3})-2(\alpha^2-\beta^2)]+2b\beta=0.$$

Now if we use equation (3.5) of [6] in (6.10), where λ is constant for Conformal Ricci soliton, then we get,

$$\beta = 0[\because b \neq 0]. \tag{6.11}$$

Then from (6.10) we get,

$$\lambda = 2\alpha^2 + \frac{1}{2}(p + \frac{2}{3})[\because a \neq 0]. \tag{6.12}$$

So we can state the following theorem:

Theorem 6.1. *If a 3-dimensional trans-Sasakian manifold satisfies $R(\xi, X).\overline{P} = 0$ and admits Conformal Ricci soliton, then the nature of the soliton behaves as:*
(a)If $p = -\frac{2}{3} - 4\alpha^2$, then $\lambda = 0$ and the Ricci soliton is steady.
(b)If $p > -\frac{2}{3} - 4\alpha^2$, then $\lambda > 0$ and the Ricci soliton is shrinking.
(c)If $p < -\frac{2}{3} - 4\alpha^2$, then $\lambda < 0$ and the Ricci soliton is expanding.
Also the manifold reduces to a 3-dimensional α-Sasakian manifold.

7. Conformal Ricci Soliton in a 3-dimensional trans-Sasakian Manifold Satisfying $\overline{P}(\xi, X).S = 0$

The condition $\overline{P}(\xi, X).S = 0$ implies that,

$$S(\overline{P}(\xi, X)Y, Z) + S(Y, \overline{P}(\xi, X)Z) = 0. \tag{7.1}$$

Now using (2.19) in (7.1), we get,

$$\begin{aligned}[\lambda - \frac{1}{2}(p + \frac{2}{3}) - \beta][g(\overline{P}(\xi, X)Y, Z) + g(Y, \overline{P}(\xi, X)Z)]& \\ + \beta[\eta(Z)\eta(\overline{P}(\xi, X)Y) + \eta(Y)\eta(\overline{P}(\xi, X)Z)] &= 0.\end{aligned} \tag{7.2}$$

Using (6.1),(6.2) in (7.2) and taking the condition $\beta \neq 0$, we get,

$$[a(\alpha^2 - \beta^2) - \frac{r}{3}(\frac{a}{2} + b)] \times [2\eta(X)\eta(Y)\eta(Z) - g(X, Z)\eta(Y) - g(X, Y)\eta(Z)] = 0. \tag{7.3}$$

Now if we put $X = Y = \xi$ in (7.3),then the equation is identically satisfied and we can not get the value for λ.

So taking $X = Y = e_i$ and summing over $i = 1, 2, 3$,where $\{e_i\}$ is an orthonormal basis of $T_p(M)$ and using (2.22) and also taking the condition $\eta(Z) \neq 0$, we get,

$$a[(\alpha^2 - \beta^2) - \frac{1}{2}\{\lambda - \frac{1}{2}(p + \frac{2}{3}) - \beta\}] - b[\lambda - \frac{1}{2}(p + \frac{2}{3}) - \beta] + \frac{a\beta}{6} + \frac{b\beta}{3} = 0. \tag{7.4}$$

As for Conformal Ricci soliton λ is constant, so if we use equation (3.5) of [6] in (7.4), we get,

$$\lambda = \frac{2\beta}{3b}(a + 2b) + \frac{1}{2}(p + \frac{2}{3}). \tag{7.5}$$

So we can state the following theorem:

Theorem 7.1. *If a 3-dimensional trans-Sasakian manifold satisfies $\overline{P}(\xi, X).S = 0$ and admits Conformal Ricci soliton, then the nature of the soliton behaves as:*
(a)If $p = -\frac{2}{3} - \frac{4\beta}{3b}(a + 2b)$, then $\lambda = 0$ and the Ricci soliton is steady.
(b)If $p > -\frac{2}{3} - \frac{4\beta}{3b}(a + 2b)$, then $\lambda > 0$ and the Ricci soliton is shrinking.
(c)If $p < -\frac{2}{3} - \frac{4\beta}{3b}(a + 2b)$, then $\lambda < 0$ and the Ricci soliton is expanding.

8. Example of a 3-dimensional trans-Sasakian manifold

In this section we construct an example of a 3-dimensional trans-Sasakian manifold as given in [6]. To construct this, we consider the three dimensional manifold $M = \{(x, y, z) \in R^3 : z \neq 0\}$ where (x, y, z) are the standard coordinates in R^3. The vector fields

$$e_1 = e^{-z}(\frac{\partial}{\partial x} - y\frac{\partial}{\partial z}), e_2 = e^{-z}\frac{\partial}{\partial y}, e_3 = \frac{\partial}{\partial z}$$

are linearly independent at each point of M. Let g be the Riemannian metric defined by,

$$g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1,$$

$$g(e_1, e_2) = g(e_2, e_3) = g(e_3, e_1) = 0.$$

Let η be the 1-form which satisfies the relation

$$\eta(e_3) = 1.$$

Let ϕ be the $(1, 1)$ tensor field defined by $\phi(e_1) = e_2, \phi(e_2) = -e_1, \phi(e_3) = 0$. Then we have,

$$\phi^2(Z) = -Z + \eta(Z)e_3,$$

$$g(\phi Z, \phi W) = g(Z, W) - \eta(Z)\eta(W),$$

for any $Z, W \in \chi(M^3)$. Thus for $e_3 = \xi, (\phi, \xi, \eta, g)$ defines an almost contact metric structure on M. Now, after calculating we have,

$$[e_1, e_3] = e_1, [e_1, e_2] = ye^{-z}e_2 + e^{-2z}e_3, [e_2, e_3] = e_2.$$

The Riemannian connection ∇ of the metric is given by the Koszul's formula,

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \\ &- g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]). \end{aligned}$$

(8.1)

By Koszul's formula we get,

$$\nabla_{e_1} e_1 = -e_3, \nabla_{e_2} e_1 = -ye^{-z}e_2 - \frac{1}{2}e^{-2z}e_3, \nabla_{e_3} e_1 = -\frac{1}{2}e^{-2z}e_2,$$

$$\nabla_{e_1} e_2 = \frac{1}{2}e^{-2z}e_3, \nabla_{e_2} e_2 = ye^{-z}e_1 - e_3, \nabla_{e_3} e_2 = \frac{1}{2}e^{-2z}e_1,$$

$$\nabla_{e_1} e_3 = e_1 - \frac{1}{2}e^{-2z}e_2, \nabla_{e_2} e_3 = \frac{1}{2}e^{-2z}e_1 + e_2, \nabla_{e_3} e_3 = 0.$$

From the above we have found that $\alpha = \frac{1}{2}e^{-2z}, \beta = 1$ and it can be easily shown that $M^3(\phi, \xi, \eta, g)$ is a trans-Sasakian manifold.

On this trans-Sasakian manifold we can easily verify our results.

Acknowledgement

We would like to thank the editor and the referees for valuable suggestions to improve the paper.

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