Jordan Journal of Mathematics and Statistics (JJMS) 13(2), 2020, pp 221 - 230

3-DIVISOR CORDIAL LABELING OF SOME JOIN GRAPHS

S. SATHISH NARAYANAN

ABSTRACT. Let G be a (p, q) graph and $2 \leq k \leq p$. Let $f : V(G) \rightarrow \{1, 2, \ldots, k\}$ be a map. For each edge uv, assign the label 1 if either $f(u)$ or $f(v)$ divides the other and 0 otherwise. f is called a k-divisor cordial labeling if $|v_f(i) - v_f(j)| \leq 1$ $i, j \in \{1, 2, ..., k\}$ and $|e_f(0) - e_f(1)| \leq 1$ where $v_f(x)$ denotes the number of vertices labeled with x, where $x \in \{1, 2, \ldots, k\}$, $e_f(i)$ denote the number of edges labeled with i, $i \in \{0, 1\}$. A graph with a k-divisor cordial labeling is called a k-divisor cordial graph. In this paper, we discuss 3-divisor cordial labeling behavior of wheel and $\overline{K_n} + 2K_2$.

1. Introduction

Graphs considered here are finite, undirected and simple. The symbols $V(G)$ and $E(G)$ will denote the vertex set and edge set of a graph G. The order and size of a graph G are respectively denoted by p and q. Let G_1 and G_2 be two graphs with vertex sets V_1 and V_2 and edge sets E_1 and E_2 respectively. Then their *join* $G_1 + G_2$ is the graph whose vertex set is $V_1 \cup V_2$ and edge set is $E_1 \cup E_2 \cup \{uv : u \in V_1 \text{ and } v \in V_2\}$. In 1980, Cahit [1] introduced the cordial labeling of graphs. In [5], Varatharajan, Navananeethakrishnan, and Nagarajan introduced a notion, called divisor cordial labeling and proved the standard graphs such as paths, cycles, wheels, stars and some complete bipartite graphs are divisor cordial. Sathish Narayanan introduced

2000 *Mathematics Subject Classification.* 05C78.

Key words and phrases. cycle, join, wheel, complete graph.

Received: Dec. 5, 2018 Accepted: April 2, 2019 .

the notion of k-divisor cordial labeling and found a 3-divisor cordial labeling of path, cycle, comb and crown graphs in [4]. Let G be a (p, q) graph and $2 \leq k \leq p$. Let $f : V(G) \to \{1, 2, \ldots, k\}$ be a map. For each edge xy, assign the label 1 if either $f(x)$ or $f(y)$ divides the other and 0 otherwise. f is called a k-divisor cordial labeling if $|v_f(i) - v_f(j)| \leq 1$ $i, j \in \{1, 2, ..., k\}$ and $|e_f(0) - e_f(1)| \leq 1$ where $v_f(x)$ denotes the number of vertices labeled with x, where $x \in \{1, 2, \ldots, k\}$, $e_f(i)$ denote the number of edges labeled with i, $i \in \{0, 1\}$. A graph with a k-divisor cordial labeling is called a k-divisor cordial graph. In this paper we studied the 3-divisor cordial labeling behavior of wheel and $\overline{K_n} + 2K_2$. Let x be any real number. Then $\lfloor x \rfloor$ stands for the largest integer less than or equal to x and $\lceil x \rceil$ stands for the smallest integer greater than or equal to x. Terms and definitions not defined here are used in the sense of Harary [3] and Gallian [2].

2. WHEEL GRAPH

The graph $W_n = C_n + K_1$ is called wheel. In a wheel, a vertex of degree 3 is called a rim vertex. A vertex which is adjacent to all the rim vertices is called the central vertex. The edges with one end incident with the rim and the other incident with the central vertex are called spokes. Let $C_n : u_1, u_2, \ldots, u_n, u_1$ be a cycle. Let $V(W_n) = V(C_n) \cup \{u\}$ and $E(W_n) = E(C_n) \cup \{uu_i : 1 \leq i \leq n\}$. It is clear that the order and size of W_n are $n + 1$ and $2n$ respectively.

Theorem 2.1. Wheel W_n is 3-divisor cordial if and only if $n \equiv 1 \pmod 3$.

Proof. The proof is divided into the following three possible cases:

Case 1. $n \equiv 0 \pmod 3$.

Let $n = 3t$, $t \in \mathbb{N}$ and so the number of vertices $p = 3t+1$ and the number of edges $q = 6t$. Suppose there exists a 3-divisor cordial labeling f of W_n, then we have the following possibilities in the vertex condition of f: (i) $v_f(1) = t+1$; $v_f(2) = v_f(3) = t$.

(ii) $v_f(2) = t + 1$; $v_f(1) = v_f(3) = t$. (iii) $v_f(3) = t + 1$; $v_f(1) = v_f(2) = t$. Suppose (i) is true. First we assume that $f(u) = 1$. In this case, the number of edges with the label 1 is at least $n + (t + 1) = 4t + 1$, because the label 1 of the central vertex contributes n to $e_f(1)$ and on the edges of the cycle C_n, there are t vertices labeled with 1. So at least $t + 1$ edges of the cycle C_n are labeled with 1, a contradiction to the edge condition of 3-divisor cordial labeling. Consider the case that $f(u) = 2$. It is clear that, to get the label 0 for an edge of C_n, the end vertices of that edge must receive the labels 3 and 2. So the maximum possible edges with the label 0 is obtained by assigning the labels 3 and 2 successively to the vertices of C_n with the available number of labels 3, 2. If so, the number of edges of W_n with the label 0 is at most $(2t - 2) + t = 3t - 2$, a contradiction to the edge condition of f. A similar argument shows that, if $f(u) = 3$, then $e_f(1) - e_f(0) \geq 4$. Consider the case that (ii) is true. Suppose the label of central vertex is 1, then the number of edges with the label 1 is at least $n + t = 4t$, a contradiction to our assumption that f satisfies the edge condition of 3-divisor cordial cordial labeling. Assume the case that the central vertex receives the label 2. As in the previous discussion, the number of edges of W_n with the label 0 is at most $(2t - 1) + t = 3t - 1$, a contradiction. Similarly we can prove that if the central vertex is labeled by 3, then $e_f(1) - e_f(0) \geq 2$. We can prove that the case (iii) is also invalid with a similar argument of (ii).

Case 2. $n \equiv 1 \pmod 3$.

Let $n = 3t + 1$, $t \in \mathbb{N}$. Put the label 2 to the central vertex u. The consecutive rim vertices, namely $u_1, u_2, \ldots, u_t$ are labeled by 1. Then the remaining vertices $u_{t+1}, u_{t+2}, \ldots, u_{3t+1}$ are labeled by 3, 2 in the order 3, 2, 3, 2, …, 2, 3. Note that the vertex u_n is labeled by 3. Assume that g is the above mentioned vertex labeling. One can easily check that the number of edges with the label 1 is $(t + 1) + t + t = 3t + 1$ and that of the edges with the label 0 is $2t + (t + 1) = 3t + 1$. Also $v_g(1) = t$;

$v_g(2) = v_g(3) = t+1$. It follows that g is a 3-divisor cordial labeling of W_n if $n \equiv 1 \pmod 3$.

Case 3. $n \equiv 2 \pmod 3$.

Let $n = 3t+2$, $t \in \mathbb{N}$. If possible, let h be a 3-divisor cordial labeling of W_n. Here, the vertex condition is $v_h(1) = v_h(2) = v_h(3) = t+1$. Suppose the central vertex is labeled by 1, then there are at least $(3t+2)+(t+1) = 4t+3$ edges labeled by 1, this forces a contradiction to the edge condition of h. It remains to show that, by symmetry, if we put the label 3 or 2 to the central vertex, h can not be a 3-divisor cordial labeling. Without loss of generality assume that 2 is the label of u. As in case 1, maximum number of edges with the label 0 is obtained by arranging the labels 3, 2 to the successive vertices of C_n. If it is so, number of edges with the label zero is at most $2t+(t+1) = 3t+1$, a contradiction to the edge condition of h. This shows that W_n does not allow a 3-divisor cordial labeling when $n \equiv 2 \pmod 3$. □

Example 2.1. A 3-divisor cordial labeling of W_{10} is given in Figure 1.

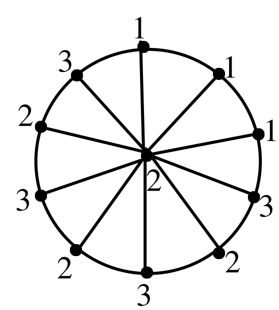

FIGURE 1

3. THE GRAPH $\overline{K_n} + 2K_2$

Let $V(\overline{K_n}+2K_2) = \{u, v, x, y\} \cup \{u_i : 1 \le i \le n\}$ and $E(\overline{K_n}+2K_2) = \{uu_i, vu_i, xu_i, yu_i : 1 \le i \le n\} \cup \{uv, xy\}$. The order and size of this graph are $n+4$ and $4n+2$ respectively.

Theorem 3.1. $\overline{K_n} + 2K_2$ is 3-divisor cordial iff $n = 1, 4$.

Proof. For $n = 1, 4$, the graphs given in Figure 2, establish that $\overline{K_1} + 2K_2$ and $\overline{K_4} + 2K_2$ are 3-divisor cordial.

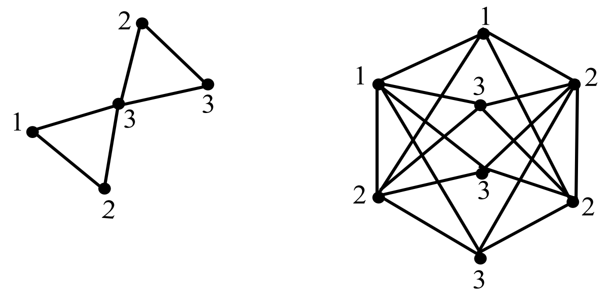

FIGURE 2

Conversely suppose f is a 3-divisor cordial labeling of $\overline{K_n} + 2K_2$ where $n \neq 1$ and $n \neq 4$. If all the vertices of $\{u, v, x, y\}$ are labeled by 1, then $e_f(1) = 4n + 2$, a contradiction. Suppose any three vertices of $\{u, v, x, y\}$ are labeled by 1. Without loss of generality assume that $f(u) = f(v) = f(x) = 1$ and $f(y) = 2$ or 3. Since u is adjacent to n vertices other than v, the n edges incident with u contributes n to $e_f(1)$. Similarly the edges incident with v and x contributes n to $e_f(1)$. Also the edges uv and xy receives the label 1. Suppose the vertex y recieves the label 2. Clearly y is adjacent to atleast $\left\lfloor \frac{n}{3} \right\rfloor$ vertices of $\overline{K_n}$ which are labeled with 1 and atleast $\left\lfloor \frac{n}{3} \right\rfloor$ vertices of $\overline{K_n}$ which are labeled with 2. It follows that $e_f(1) \geq 3n + 2\left\lfloor \frac{n}{3} \right\rfloor + 2$, a contradiction. The same is true if 3 is the label of y. In a similar argument shows that if any two vertices of $\{u, v, x, y\}$ are labeled by 1, then $e_f(1) \geq 2n + 2 + 4\left\lfloor \frac{n}{3} \right\rfloor$, a contradiction. Now we consider the following cases.

Case 1. $n \equiv 0 \pmod 3$.

Let $n = 3t$, $t \in \mathbb{N}$. Note that $p = 3t + 4$ and $q = 12t + 2$. The 3-divisor cordial labeling f should satisfy any one of the following vertex conditions. (i) $v_f(1) = t + 2$;

$v_f(2) = v_f(3) = t + 1$. (ii) $v_f(2) = t + 2$; $v_f(1) = v_f(3) = t + 1$. (iii) $v_f(3) = t + 2$; $v_f(1) = v_f(2) = t + 1$. Consider the following possible subcases.

Subcase 1. $f(u) = 1$, $f(v) = 2$, $f(x) = f(y) = 3$.

If f satisfy the vertex condition (i), then $e_f(1) = 9t + 3$ and $e_f(0) = 3t - 1$. It follows that $e_f(1) - e_f(0) = 6t + 4$, a contradiction. Suppose f satisfy the condition (ii), then $e_f(1) = 9t+1$ and $e_f(0) = 3t+1$. Here $e_f(1) - e_f(0) = 6t$, a contradiction. If (iii) is true, then $e_f(1) = 9t+2$ and $e_f(0) = 3t$. This shows that $e_f(1) - e_f(0) = 6t+2$, a contradiction. The case $f(u) = 1$, $f(v) = 3$, $f(x) = f(y) = 2$ gives a similar result.

Subcase 2. $f(u) = 1$, $f(v) = f(x) = 2$, $f(y) = 3$.

First we consider the vertex condition (i). In this case, $e_f(1) = 9t+2$ and $e_f(0) = 3t$. Hence $e_f(1) - e_f(0) = 6t + 2$. This is impossible. For the condition (ii), we have $e_f(1) = 9t + 1$ and $e_f(0) = 3t + 1$. This implies $e_f(1) - e_f(0) = 6t$, a contradiction. If (iii) is true, then $e_f(1) = 9t$ and $e_f(0) = 3t + 2$. Here $e_f(1) - e_f(0) = 6t - 2$, an impossibility to the edge condition of f. Similar result is obtained when $f(u) = 1$, $f(v) = f(y) = 3$ and $f(x) = 2$.

Subcase 3. $f(u) = 1$, $f(v) = f(x) = f(y) = 2$.

Suppose the condition (i) is true, then $e_f(1) = 9t - 1$ and $e_f(0) = 3t + 3$. Thus $e_f(1) - e_f(0) = 6t + 4$, a contradiction. For the condition (ii), we get a similar . If (iii) is the vertex condition of f, then $e_f(1) = 9t - 4$ and $e_f(0) = 3t + 6$. This gives $e_f(1) - e_f(0) = 6t - 10$, a contradiction to the edge condition of f. Similarly, we can prove that f is not a 3-divisor cordial labeling if $f(u) = 1$, $f(v) = f(x) = f(y) = 3$.

Subcase 4. $f(u) = 2$, $f(v) = f(x) = f(y) = 3$.

If possible (i) is the vertex condition of f, then $e_f(1) = 8t + 3$ and $e_f(0) = 4t - 1$. This implies $e_f(1) - e_f(0) = 4t + 4$, a contradiction. Suppose (ii) is true, then $e_f(1) = 8t$ and $e_f(0) = 4t+2$. In this case $e_f(1) - e_f(0) = 4t-2$, a contradiction. For the case when (iii) is the vertex condition of f, we get $e_f(1) = 8t + 2$ and $e_f(0) = 4t$.

It follows that $e_f(1) - e_f(0) = 4t + 2$, a contradiction. Similarly we can prove that if $f(u) = 3$, $f(v) = f(x) = f(y) = 2$, then f is not a 3-divisor cordial labeling.

Subcase 5. $f(u) = f(v) = f(x) = f(y) = 2$.

Assume that f satisfies the vertex condition (i). Here $e_f(1) = 8t - 2$ and $e_f(0) = 4t+4$. This gives $e_f(1)-e_f(0) = 4t-6$, a contradiction. Suppose the vertex condition (ii) is true, then $e_f(1) = 8t-2$ and $e_f(0) = 4t+4$. Hence $e_f(1)-e_f(0) = 4t-6$. This is impossible. For the case when (iii) is true, we get $e_f(1) = 8t-6$ and $e_f(0) = 4t+8$. Thus $e_f(1) - e_f(0) = 4t - 4$, a contradiction to our assumption. Similarly we can prove that f is not a 3-divisor cordial if $f(u) = f(v) = f(x) = f(y) = 3$.

Subcase 6. $f(u) = f(v) = 2$, $f(x) = f(y) = 3$.

Consider the case that f satisfy the vertex condition (i). In this case, $e_f(1) = 8t+6$ and $e_f(0) = 4t - 4$. It follows that $e_f(1) - e_f(0) = 4t + 10$. If f satisfy the vertex condition (ii), then $e_f(1) = 8t + 4$ and $e_f(0) = 4t - 2$. Here $e_f(1) - e_f(0) = 4t + 6$, a contradiction. If possible (iii) is true, then $e_f(1) = 8t + 4$ and $e_f(0) = 4t - 2$. Here $e_f(1) - e_f(0) = 4t + 6$, a contradiction.

Subcase 7. $f(u) = f(x) = 2$, $f(v) = f(y) = 3$.

Suppose f satisfy the vertex condition (i), then $e_f(1) = 8t + 4$ and $e_f(0) = 4t - 2$. In this case $e_f(1) - e_f(0) = 4t + 6$, a contradiction. If (ii) is the vertex condition of f, then $e_f(1) = 8t + 2$ and $e_f(0) = 4t$. This implies $e_f(1) - e_f(0) = 4t + 2$, a contradiction. For the case when (iii) is true, $e_f(1) = 8t + 2$ and $e_f(0) = 4t$, we get $e_f(1) - e_f(0) = 4t + 2$, a contradiction.

Case 2. $n \equiv 1 \pmod 3$.

Let $n = 3t + 1$ and $t \geq 2$. Note that $p = 3t + 5$ and $q = 12t + 6$. The possible vertex conditions of f are given by (i) $v_f(1) = t + 1$; $v_f(2) = v_f(3) = t + 2$. (ii) $v_f(2) = t + 1$; $v_f(1) = v_f(3) = t + 2$. (iii) $v_f(3) = t + 1$; $v_f(1) = v_f(2) = t + 2$. Consider the following subcases.

Subcase 1. $f(u) = 1$, $f(v) = 2$, $f(x) = f(y) = 3$.

Suppose f satisfy the vertex condition (i), then $e_f(1) = 9t + 4$ and $e_f(0) = 3t + 2$. Here $e_f(1) - e_f(0) = 6t + 2$, a contradiction. If the vertex condition (ii) is true, then $e_f(1) = 9t + 6$ and $e_f(0) = 3t$. It follows that $e_f(1) - e_f(0) = 6t + 6$, a contradiction. For the vertex condition (iii), we have $e_f(1) = 9t + 5$ and $e_f(0) = 3t + 1$. This implies $e_f(1) - e_f(0) = 6t + 4$, a contradiction.

Subcase 2. $f(u) = 1$, $f(v) = f(x) = 2$, $f(y) = 3$.

Suppose f satisfy the vertex condition (i), then $e_f(1) = 9t + 3$ and $e_f(0) = 3t + 3$. It follows that $e_f(1) - e_f(0) = 6t$, a contradiction. If the condition (ii) is true, then $e_f(1) = 9t + 4$ and $e_f(0) = 3t + 2$. This implies $e_f(1) - e_f(0) = 6t + 2$, a contradiction. For the case when the vertex condition (iii) is true, then $e_f(1) = 9t+5$ and $e_f(0) = 3t + 1$. Hence $e_f(1) - e_f(0) = 6t + 4$, a contradiction.

Subcase 3. $f(u) = 1$, $f(v) = f(x) = f(y) = 2$.

Consider the case if f satisfy the vertex condition (i). Here $e_f(1) = 9t$ and $e_f(0) = 3t + 6$. It follows that $e_f(1) - e_f(0) = 6t - 6$, a contradiction. Suppose (ii) is true, we get a similar result. For the case if f satisfy the vertex condition (iii), then $e_f(1) = 9t + 3$ and $e_f(0) = 3t + 3$. Thus $e_f(1) - e_f(0) = 6t$, a contradiction.

Subcase 4. $f(u) = 2$, $f(v) = f(x) = f(y) = 3$.

Consider the vertex condition (i). Here $e_f(1) = 8t + 3$ and $e_f(0) = 4t + 3$. This implies $e_f(1) - e_f(0) = 4t$, a contradiction. If f satisfy the vertex condition (ii), then $e_f(1) = 8t+6$ and $e_f(0) = 4t$. Hence $e_f(1) - e_f(0) = 4t + 6$, a contradiction. Suppose the condition (iii) is true, then $e_f(1) = 8t + 4$ and $e_f(0) = 4t + 2$. This shows that $e_f(1) - e_f(0) = 4t + 2$. This is impossible.

Subcase 5. $f(u) = f(v) = f(x) = f(y) = 2$.

If possible f satisfies the vertex condition (i), f $e_f(1) = 8t - 2$ and $e_f(0) = 4t + 8$. Hence $e_f(1) - e_f(0) = 4t - 10$. This is a contradiction. Suppose the vertex condition (ii) is true, then $e_f(1) = 8t-2$ and $e_f(0) = 4t+8$. This implies $e_f(1)-e_f(0) = 4t-10$, a contradiction. For the case when (iii) is true, we get $e_f(1) = 8t+2$ and $e_f(0) = 4t+4$. In this case $e_f(1) - e_f(0) = 4t - 2$, a contradiction.

Subcase 6. $f(u) = f(v) = 2$, $f(x) = f(y) = 3$.

If f satisfy the vertex condition (i), then $e_f(1) = 8t + 6$ and $e_f(0) = 4t$. This shows that $e_f(1) - e_f(0) = 4t + 6$, a contradiction. Suppose (ii) is the required vertex condition of f, then $e_f(1) = 8t + 8$ and $e_f(0) = 4t - 2$. It follows that $e_f(1) - e_f(0) = 4t + 10$, a contradiction. Assume that (iii) is the vertex condition of f, then $e_f(1) = 8t + 8$ and $e_f(0) = 4t - 2$. Thus $e_f(1) - e_f(0) = 4t + 10$. This is impossible.

Subcase 7. $f(u) = f(x) = 2$, $f(v) = f(y) = 3$.

If possible f satisfy the vertex condition (i) is true, then $e_f(1) = 8t + 4$ and $e_f(0) = 4t + 2$. This implies $e_f(1) - e_f(0) = 4t + 2$, a contradiction. Suppose (ii) is true, then $e_f(1) = 8t+6$ and $e_f(0) = 4t$. Hence $e_f(1)-e_f(0) = 4t+6$, a contradiction. Suppose the vertex condition (iii) is true, $e_f(1) = 8t + 6$ and $e_f(0) = 4t$. It follows that $e_f(1) - e_f(0) = 4t + 6$, a contradiction.

Case 3. $n \equiv 2 \pmod 3$.

Let $n = 3t + 1$ and $t \in \mathbb{N}$. Here $p = 3t + 6$ and $q = 12t + 10$. In this case the only possible vertex condition is $v_f(1) = v_f(2) = v_f(3) = t + 2$.

Subcase 1. $f(u) = 1$, $f(v) = 2$, $f(x) = f(y) = 3$.

Here $e_f(1) = 9t + 8$ and $e_f(0) = 3t + 2$. It follows that $e_f(1) - e_f(0) = 6t + 6$, a contradiction.

Subcase 2. $f(u) = 1$, $f(v) = f(x) = 2$, $f(y) = 3$.

In this case $e_f(1) = 9t + 7$ and $e_f(0) = 3t + 3$. This implies $e_f(1) - e_f(0) = 6t + 4$. This is impossible.

Subcase 3. $f(u) = 1$, $f(v) = f(x) = f(y) = 2$.

The edge conditions of this subcase are $e_f(1) = 9t+4$ and $e_f(0) = 3t+6$. It follows that $e_f(1) - e_f(0) = 6t - 2$, a contradiction.

Subcase 4. $f(u) = 2$, $f(v) = f(x) = f(y) = 3$.

Here $e_f(1) = 8t + 7$ and $e_f(0) = 4t + 3$. This shows that $e_f(1) - e_f(0) = 4t + 4$, a contradiction.

Subcase 5. $f(u) = f(v) = f(x) = f(y) = 2$.

In this case, $e_f(1) = 8t + 2$ and $e_f(0) = 4t + 8$. This implies $e_f(1) - e_f(0) = 4t - 6$, a contradiction.

Subcase 6. $f(u) = f(v) = 2$, $f(x) = f(y) = 3$.

Here $e_f(1) = 8t + 10$ and $e_f(0) = 4t$. It follows that $e_f(1) - e_f(0) = 4t + 10$, a contradiction.

Subcase 7. $f(u) = f(x) = 2$, $f(v) = f(y) = 3$.

In this case, $e_f(1) = 8t + 8$ and $e_f(0) = 4t + 2$. Hence $e_f(1) - e_f(0) = 4t + 6$, a contradiction.

Hence $\overline{K_n} + 2K_2$ is 3-divisor cordial if and only if $n = 1, 4$. □

References

[1] I. Cahit, Cordial graphs: a weaker version of graceful and harmonious graphs, *Ars combin.,* **23** (1987) 201-207.

[2] J. A. Gallian, A Dynamic survey of graph labeling, *The Electronic Journal of Combinatorics*, (2017) # Ds6.

[3] F. Harary, Graph theory, *Narosa Publishing house*, New Delhi (2001).

[4] S.Sathish Narayanan, 3-divisor cordial labeling of graphs, *Ars combinatoria*, accepted for publication.

[5] R.Varatharajan, S.Navanaeethakrishnan, and K.Nagarajan, Divisor cordial graphs, *Internat. J. Math. Combin.,* **4** (2011) 15-25.

Department of Mathematics,

Sri Ram Nallamani Yadava College of Arts and Science,

Kodikurichi, Tenkasi-627804, India.

Email address: `sathishrvss@gmail.com`