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$\varphi-$ APPROXIMATE BIPROJECTIVE AND $(\varphi, \psi)-$AMENABLE BANACH ALGEBRAS

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ABSTRACT. We introduce and study the concept of $\varphi-$approximate biprojective and $(\varphi, \psi)-$amenable Banach algebra A, where φ is a continuous homomorphism on A and $\psi \in \Phi_A$. We show that if A is $(\varphi, \psi)-$amenable then there exists a bounded net (m_α) in $(A \hat{\otimes} A)$ such that $\|m_\alpha \cdot \varphi(a) - \psi \circ \varphi(a) \cdot m_\alpha\| \longrightarrow 0$ and $\psi \circ \pi(m_\alpha) \cdot \varphi(a) \longrightarrow \varphi(a)$ for all $a \in A$.

1. INTRODUCTION

Amenable Banach algebra was introduced by Johnson in [10]. He showed that A is amenable Banach algebra if and only if A has a approximate diagonal, that is a bounded net (m_α) in $(A \hat{\otimes} A)$ such that $m_\alpha a - a m_\alpha \longrightarrow 0$ and $\pi(m_\alpha)a \longrightarrow a$ for every $a \in A$. The notion of a biflat and biprojective Banach algebra was introduced by Helemskii [8, 9]. Indeed, A is called biprojective, if there exists a bounded A-bimodule map $\theta : A \longrightarrow A \hat{\otimes} A$ such that $\pi \circ \theta = id_A$.

He considered a Banach algebra to be amenable if A is biflat and has a bounded approximate identity[7, 9]. In fact, A is called biflat if there exists a bounded A-bimodule map $\theta : (A \hat{\otimes} A)^* \longrightarrow A^*$ such that $\theta \circ \pi^* = id_{A^*}$.

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Given a continuous homomorphism φ from A into A, authors in [13, 14] defined and studied φ-derivations and φ-amenability.

Recall that a character on A is a non-zero homomorphism from A into the scalar field. The set of all characters on A, called the character space of A, is denoted by Φ_A.

Motivated by these considerations, author and M. Lashkarizadeh Bami introduced some generalizations of Helemskii's concepts like φ-biflatness and φ-biproject-ivity, where φ is a continuous homomorphism from A into A [5, 6]. The author states that Banach algebra A is $\varphi-$ biflat($\varphi-$ approximate biprojective) if there exists a bounded A-bimodule map $\theta : A \longrightarrow (A \hat{\otimes} A)^{**}$ $(\theta_\alpha : A \longrightarrow (A \hat{\otimes} A))$ such that $\pi^{**} \circ \theta \circ \varphi$ is the canonical embedding of A into $A^{**}(\pi \circ \theta_\alpha(a) \to \varphi(a))$.

In this paper, we define $(\varphi, \psi)-$amenability Banach algebra A, where φ is a continuous homomorphism on A and $\psi \in \Phi_A$. We prove that A is $(\varphi, \psi)-$amenable if and only if there exists a bounded net $(m_\alpha) \subset A$ such that $\|\varphi(a)m_\alpha - m_\alpha \psi \circ \varphi(a)\| \longrightarrow 0$ and $\psi \circ \varphi(m_\alpha) = 1$ for all α.

We shows that $(l^1)^\sharp$ is not biprojective Banach algebra which is $\varphi-$ biprojective Banach algebra.

First we recall lemma and theorem that we shall need in this paper. The following result can be found in [13].

Lemma 1.1. *Let A be a Banach algebra. Then there exists an A-bimodule homomorphism $\gamma : (A \hat{\otimes} A)^* \longrightarrow (A^{**} \hat{\otimes} A^{**})^*$ such that for any functional $f \in (A \hat{\otimes} A)^*$, elements $\varphi, \psi \in A^{**}$ and nets $(a_\alpha), (b_\beta)$ in A with $w^* - lim_\alpha a_\alpha = \varphi$ and $w^* - lim_\beta b_\beta = \psi$, we have*

$$\gamma(f)(\varphi \otimes \psi) = lim_\alpha lim_\beta f(a_\alpha \otimes b_\beta).$$

Remark: In general, weak convergence implies weak* convergence but the converse is not always true. However, the notions are equivalent if the normed space X is reflexive. Let X be Banach space. The closed unit ball of X, denoted by B_X is defined to be the set $B_X = \{x \in X : \|x\| \leq 1\}$

Theorem 1.1 (Goldstine's Theorem)**.** ([4]) *Let X be a Banach space and $\overline{B_X}$ be a closed unit ball identified as a subset of X^{**} under the canonical embedding. Then B_X is weak* dense in $B_{X^{**}}$*

2. Main Results

In this section we investigate the hereditary properties of $\varphi-$biprojective Banach algebra. The main result (Proposition 2.1) converse the projective tensor product of two Banach algebra.

Let A be a Banach algebra and X, Y be Banach A-bimodules. Then A-bimodule morphism from X to Y is a morphism $\varphi : X \longrightarrow Y$ with

$$\varphi(a \cdot x) = a \cdot \varphi(x), \quad \varphi(x \cdot a) = \varphi(x) \cdot a \ \ (a \in A, \ x \in X)$$

In the next result $\varphi : A \longrightarrow A$ is a homomorphism and I is a closed ideal of A. We define the map $\tilde{\varphi} : A/I \longrightarrow A/I$ by $\tilde{\varphi}(a + I) = \varphi(a) + I$.

Theorem 2.1. *Suppose that A is a $\varphi-$ approximate biprojective Banach algebra. If I is a closed ideal of A, then A/I is $\tilde{\varphi}-$ approximate biprojective.*

Proof. Let $\theta_\alpha : A \longrightarrow (A \hat{\otimes} A)$ be a continuous A-bimodule map such that $\pi \circ \theta_\alpha(a) \to \varphi(a)$. Let $q : A \longrightarrow A/I$ be the quotient map. Define a map $\tilde{\theta}_\alpha : A/I \longrightarrow (A/I \hat{\otimes} A/I)$ by $a+I \mapsto (q \hat{\otimes} q) \circ \theta_\alpha(a)$ $(a \in A)$. We prove that $\tilde{\theta}_\alpha$ is an A/I- bimodul map. Let $a, b, c \in A$. Then we have

$$\begin{aligned}
\tilde{\theta}_\alpha((a+I)(b+I)(c+I)) &= \tilde{\theta}_\alpha(abc+I) \\
&= (q \hat{\otimes} q) \circ \theta_\alpha(abc) \\
&= (q \hat{\otimes} q)(a \cdot \theta_\alpha(b) \cdot c) \\
&= a \cdot (q \hat{\otimes} q)(\theta_\alpha(b) \cdot c \\
&= (a+I) \cdot \tilde{\theta}_\alpha(b+I) \cdot (c+I),
\end{aligned}$$

and also we have

$$\begin{aligned}
\pi_{A/I} \circ \tilde{\theta}_\alpha(a+I) &= \pi_{A/I} \circ (q \hat{\otimes} q) \circ \theta_\alpha(a) \\
&= q \circ \pi_A \circ \theta_\alpha(a) \to q(\varphi(a)) = \tilde{\varphi}(a+I).
\end{aligned}$$

That is, A/I is $\tilde{\varphi}-$approximate biprojective. □

Theorem 2.2. *Suppose that A is a $\varphi-$approximate biprojective Banach algebra. If I is a closed ideal of A with one sided bounded approximate identity and $\varphi(I) \subset I$. Then I is $\varphi|_I-$approximate biprojective.*

Proof. Assume that $\theta_\alpha : A \longrightarrow (A \hat{\otimes} A)$ is a continuous A-bimodule map such that $\pi \circ \theta_\alpha(a) \to \varphi(a)$. Let $\iota : I \hookrightarrow A$ be the inclusion map. Then $\theta_\alpha|_I = \theta_\alpha \circ \iota : I \longrightarrow (A \hat{\otimes} A)$ is I-bimodule homomorphism. If I^3 denotes span $\{abc : \ a, b, c \in I\}^-$, then $I^3 = I$ because of I has a one sided bounded approximate identity and

$$\begin{aligned}
\theta_\alpha|_I &= \theta_\alpha(I) \\
&= \theta_\alpha(I^3) \\
&\subseteq span\{a \cdot \theta_\alpha(b) \cdot c\}^- \\
&\subseteq span\{a \cdot m \cdot c : a, c \in I, m \in A \hat{\otimes} A\}^- \subseteq I \hat{\otimes} I.
\end{aligned}$$

Therefore, for every $a \in I$, we have

$$\begin{aligned} \pi \circ \theta_\alpha|_I(a) &= \pi(\theta_\alpha(a)) \\ &\to \varphi(a). \end{aligned}$$

□

Let A is $\varphi-$approximate biprojective and B is $\psi-$approximate biprojective Banach algebra, then $A \hat{\otimes} B$ is $\varphi \otimes \psi-$approximate biprojective. We now prove a partial converse.

Proposition 2.1. *Let A be a unital Banach algebra and B be a Banach algebra containing a non-zero idempotent b_0. If $A \hat{\otimes} B$ is $\varphi \otimes \psi-$approximate biprojective. Then A is $\varphi-$approximate biprojective.*

Proof. There exists an $A \hat{\otimes} B$-bimodule $\theta_\alpha : A \hat{\otimes} B \longrightarrow (A \hat{\otimes} B)\hat{\otimes}(A \hat{\otimes} B)$ with $\pi_{A\hat{\otimes}B} \circ \theta_\alpha(a \otimes b) \to \varphi \otimes \psi(a \otimes b)$. We consider $A \hat{\otimes} B$ as an $A-$bimodule with the actions given by

$$a_1 \cdot (a_2 \otimes b) = a_1 a_2 \otimes b, \ \ \textit{and} \ \ (a_2 \otimes b) \cdot a_1 = a_2 a_1 \otimes b \quad (a_1, a_2 \in A, \ b \in B)$$

Thus, for every $a_1, a_2 \in A$, we have

$$\begin{aligned} \theta_\alpha(a_1 a_2 \otimes b_0) &= \theta_\alpha((a_1 \otimes b_0)(a_2 \otimes b_0)) \\ &= (a_1 \otimes b_0) \cdot \theta_\alpha((a_2 \otimes b_0)) \\ &= a_1 \cdot (e_A \otimes b_0) \cdot \theta_\alpha((a_2 \otimes b_0)) \\ &= a_1 \cdot \theta_\alpha(a_2 \otimes b_0). \end{aligned}$$

Similarly, we can show a right-module version of this equation. So we have

$$\theta_\alpha(a_1 a_2 \otimes b_0) = a_1 \cdot \theta_\alpha(a_2 \otimes b_0) = \theta_\alpha(a_1 \otimes b_0) \cdot a_2 \quad (a_1, a_2 \in A)$$

Take $f \in \Phi_A$ with $f(b_0) = 1$ and define

$$\rho : (A \hat{\otimes} B)\hat{\otimes}A \hat{\otimes} B \longrightarrow (A \hat{\otimes} A); \quad (a_1 \otimes b_1) \otimes (a_2 \otimes b_2) \mapsto f(b_1 b_2) a_1 \otimes a_2$$

Then ρ is an A-bimodule morphism.
We now define $\tilde{\theta}_\alpha : A \longrightarrow (A \hat{\otimes} A)$ by

$$\tilde{\theta}_\alpha(a) = \rho \circ \theta(a \otimes \psi(b0)) \quad (a \in A).$$

Then $\tilde{\theta}_\alpha$ is an A-bimodule morphism and

$$\pi_A \circ \rho = (id_A \otimes f) \circ \pi_{A\hat{\otimes}B}.$$

Therefore

$$\begin{aligned} \pi_A \circ \tilde{\theta}_\alpha(a) &= \pi_A \circ \rho \circ \theta_\alpha(a \otimes \psi(b0)) \\ &= (id_A \otimes f) \circ \pi_{A\hat{\otimes}B} \circ \theta_\alpha(a \otimes \psi(b0)) \\ &\to \varphi(a). \end{aligned}$$

That is, A is $\varphi-$approximate biprojective. □

We remind that a Banach algebra A is $\varphi-$ approximate biflat if there is a net $\theta_\alpha : A \longrightarrow (A \hat{\otimes} A)^{**} \quad (\alpha \in I)$ of bounded A-bimodule morphisms such that $\pi^{**} \circ \theta_\alpha(a) \to \varphi(a)$.

Proposition 2.2. *Let A be a unital Banach algebra and B be a Banach algebra containing a non-zero idempotent b_0. If $A \hat{\otimes} B$ is $\varphi \otimes \psi-$approximate biflat. Then A is $\varphi-$approximate biflat.*

Proof. There exists an $A \hat{\otimes} B$-bimodule $\theta_\alpha : A \hat{\otimes} B \longrightarrow (A \hat{\otimes} B)\hat{\otimes}(A \hat{\otimes} B)^{**}$ with $\pi^{**}_{A\hat{\otimes}B} \circ \theta_\alpha \to (\varphi \otimes \psi)$. The following proof is similar to that of Proposition 2.1. We now define $\tilde{\theta} : A \longrightarrow (A \hat{\otimes} A)^{**}$ by

$$\tilde{\theta}_\alpha(a) = \rho^{**} \circ \theta_\alpha(a \otimes \psi(b0)) \quad (a \in A).$$

Then $\tilde{\theta}_\alpha$ is an A-bimodule morphism and

$$\pi_A^{**} \circ \tilde{\theta}_\alpha \to \varphi.$$

That is, A is $\varphi-$approximate biflat. $\square$

Suppose that A is a Banach algebra and Λ is a non-empty set. We denote by $M_\Lambda(A)$ the set of $\Lambda \times \Lambda$ matrices $(a_{ij})_{i,j\in\Lambda}$ with entries in A such that

$$\|(a_{ij})\| = \sum_{i,j} \|a_{ij}\|_A < \infty.$$

$M_\Lambda(A)$ is a Banach algebra with matrix multiplication. The matrix units in $M_\Lambda(\mathbb{C})$ are denoted by $e_{i,j}$ so that

$$e_{i,j}e_{k,l} = \delta_{j,k}e_{i,l} \quad (i,j,k,l \in \Lambda),$$

where $\delta_{j,k} = 1$ if $j = k$ and $\delta_{j,k} = 0$ if $j \neq k$. The map

$$\theta : M_\Lambda(A) \longrightarrow (A\hat{\otimes}M_\Lambda(\mathbb{C})) \ \textit{given by} \ (a_{ij}) \mapsto \sum_{i,j} a_{ij} \otimes e_{i,j},$$

is an isometric algebra isomorphism.

Corollary 2.1. *Let A be a unital Banach algebra and let Λ be a non-empty set and also $\varphi = \varphi_0 \otimes \varphi_1$, $\varphi_0 \in Hom(A)$ and $\varphi_1 \in Hom(M_\Lambda(\mathbb{C}))$. Then $M_\Lambda(A)$ is $\varphi-$approximate biprojectiv($\varphi-$approximate biflat) if and only if A is φ_0-approximate biprojective($\varphi-$approximate biflat).*

Proof. Let $M_\Lambda(A)$ be $\varphi-$approximate biprojective($\varphi-$approximate biflat). Since $M_\Lambda(A) = A\hat{\otimes}M_\Lambda(\mathbb{C})$, the result follows from Proposition 2.1 and Proposition 2.2. Conversely, fix $k_0 \in \Lambda$ and define $\theta_\alpha : M_\Lambda(\mathbb{C}) \longrightarrow (M_\Lambda(\mathbb{C}) \hat{\otimes} M_\Lambda(\mathbb{C}))$ by

$$\theta_\alpha(a) = \sum_{i,j\in\Lambda} \varphi_1(a_{ij})e_{i,k_0} \otimes e_{k_0,j} \quad (a = (a_{i,j}) \in M_\Lambda(\mathbb{C})).$$

The sum converges because of $\sum_{i,j} \mid a_{ij} \mid < \infty$. Hence

$$\begin{aligned} \pi_{M_\Lambda(\mathbb{C})} \circ \theta(a) &= \pi_{M_\Lambda(\mathbb{C})} \circ \sum_{i,j\in\Lambda} \varphi_1(a_{ij}) e_{i,k_0} \otimes e_{k_0,j} \\ &\rightarrow \varphi_1(a_{ij}) = \varphi_1(a). \end{aligned}$$

That is, $M_\Lambda(\mathbb{C})$ is $\varphi-$approximate biprojective. Therefore, $M_\Lambda(A)$ $\varphi-$approximate biprojective($\varphi-$approximate biflat).

□

In analogy with the classical case we characterize the second dual of Banach algebra if $\varphi^{**}-$approximate biprojective then its $\varphi-$approximate biflat.

Theorem 2.3. *Suppose that A is a Banach algebra and $\varphi \in Hom(A)$. If A^{**} is $\varphi^{**}-$approximate biprojective. Then A is $\varphi-$approximate biflat.*

Proof. Let $\kappa : A \longrightarrow A^{**}$, $\kappa_1 : A^* \longrightarrow A^{***}$ and $\kappa_* : A^{**} \longrightarrow A^{****}$ denote the natural inclusions, π ($^{**}\pi$, respectively) the product maps on A (A^{**}, respectively) and γ be defined as in Lemma (1.1). Then the following diagram commutes:

$$\begin{array}{ccc} A^* & \xrightarrow{\pi^*} & (A \hat{\otimes} A)^* \\ \kappa_1 \downarrow & & \downarrow \gamma \\ A^{***} & \xrightarrow[{}^{**}\pi^*]{} & (A^{**} \hat{\otimes} A^{**})^* \end{array}$$

for each $a^* \in A^*$, elements $a_1^{**}, a_2^{**} \in A^{**}$ and nets $(a_\alpha), (b_\beta) \subset A$ with $w^* - \lim_\alpha a_\alpha = a_1^{**}, w^* - \lim_\beta b_\beta = a_2^{**}$, we get

$$\begin{aligned}(\gamma(\pi^*(a^*)))(a_1^{**} \otimes a_2^{**}) &= \lim_\alpha \lim_\beta \pi^*(a^*)(a_\alpha \otimes b_\beta)\\ &= \lim_\alpha \lim_\beta a^*(a_\alpha b_\beta)\\ &= w^* - \lim_\alpha w^* - \lim_\beta \kappa(a_\alpha b_\beta)(a^*)\\ &= \kappa_1(a^*)(a_1^{**} a_2^{**})\\ &= \kappa_1(a^*)(^{**}\pi(a_1^{**} \otimes a_2^{**})) = (^{**}\pi^*(\kappa_1(a^*)))(a_1^{**} \otimes a_2^{**}).\end{aligned}$$

Thus $\gamma \circ \pi^* =^{**} \pi^* \circ \kappa_1$. So $\pi^{**} \circ \gamma^* = \kappa_1^* \circ^{**} \pi^{**}$. Since A^{**} is $\varphi^{**}-$approximate biprojective, there is an A-bimodule map $\theta_{0\alpha} : A^{**} \longrightarrow (A^{**} \hat{\otimes} A^{**})$ such that $\pi \circ \theta_{0\alpha} \to \varphi^{**}$. Putting $\theta_\alpha := \gamma^* \circ \theta_{0\alpha} \circ \kappa$, then for each $a \in A$ we have

$$\begin{aligned}\pi^{**} \circ \theta_\alpha(a) &= \pi^{**} \circ \gamma^* \circ \theta_{0\alpha} \circ \kappa(a)\\ &= \kappa_1^* \circ \pi^{**} \circ \theta_{0\alpha} \circ \kappa(a)\\ &= \kappa_1^* \circ \pi^{**} \circ \theta_{0\alpha}(a^{**})\\ &\to \kappa_1^*(\varphi^{**}(a)) = \varphi(a).\end{aligned}$$

That is, A is $\varphi-$approximate biflat. □

Theorem 2.4. *Suppose that A is a $\varphi-$ approximate biflat Banach algebra with one sided bounded approximate identity. If I is a closed ideal of A. Then A/I is $\tilde{\varphi}-$approximate biflat.*

Proof. Let $\theta_\alpha : A \longrightarrow (A \hat{\otimes} A)^{**}$ be a bounded A-bimodule map such that $\lim_\alpha \pi^{**} \circ \theta_\alpha(a) = \varphi(a)$. Let $q : A \longrightarrow A/I$ be the quotient map. Define a map $\tilde{\theta}_\alpha : A/I \longrightarrow (A/I \hat{\otimes} A/I)^{**}$ by $a + I \mapsto (q \hat{\otimes} q)^{**} \circ \theta_\alpha(a)$ $(a \in A)$. If (e_β) is a bounded left approximate identity for A (the right case is similar), then

$$\begin{aligned}\|(q \hat{\otimes} q)^{**}(\theta_\alpha(a))\| &= \lim_\beta \|(q \hat{\otimes} q)^{**}(\theta_\alpha(e_\beta a))\| \\ &= \lim_\beta \|q(a)(q \hat{\otimes} q)^{**}(\theta_\alpha(e_\beta))\| \\ &\leq \|q\|^2 \|\theta_\alpha\| \sup_\beta \|e_\beta\| \|q(a)\|.\end{aligned}$$

And $\tilde{\theta}_\alpha$ is well-defined. We show that $\tilde{\theta}_\alpha$ is an A/I- bimodul map. To do this, choose $a, b, c \in A$, then we have

$$\begin{aligned}\tilde{\theta}_\alpha((a+I)(b+I)(c+I)) &= (q \hat{\otimes} q)^{**} \circ \theta_\alpha(abc) \\ &= (q \hat{\otimes} q)^{**}(a \cdot \theta_\alpha(b) \cdot c) \\ &= a \cdot (q \hat{\otimes} q)^{**}(\theta_\alpha(b) \cdot c \\ &= (a+I) \cdot \tilde{\theta}_\alpha(b+I) \cdot (c+I).\end{aligned}$$

We also have

$$\begin{aligned}\lim_\alpha \pi^{**}_{A/I} \circ \tilde{\theta}_\alpha(a+I) &= \lim_\alpha \pi^{**}_{A/I} \circ (q \hat{\otimes} q)^{**} \circ \theta_\alpha(a) \\ &= \lim_\alpha q^{**} \circ \pi^{**}_A \circ \theta_\alpha(a) \longrightarrow q(\varphi(a)) = \tilde{\varphi}(a+I).\end{aligned}$$

That is, A/I is $\tilde{\varphi}-$approximate biflat. □

The proof of the next result is similar to that of Theorem (2.4) and we omit it.

Theorem 2.5. *Suppose that A is a Banach algebra and $\varphi \in Hom(A)$. If A^{**} is $\varphi^{**}-$ approximate biflat. Then A is $\varphi-$approximate biflat.*

Example 2.1. *Consider* $A = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} : a, b \in \mathbb{C} \right\}$ *under the standard operator norm, we see that A has neither identity nor right approximate identity. Therefore*

A is not $\varphi-$approximate amenable Banach algebra. Put

$$f = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

we define

$$\theta_\alpha\left(\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}\right) = a(f \otimes f).$$

Then for $a \in A$ and $\varphi\left(\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$. we have $\pi \circ \theta_\alpha \to \varphi$. Thus A is $\varphi-$approximate biprojective Banach algebra, but A is not $\varphi-$approximate biflat.

In the next example, we bring a $\varphi-$biprojective Banach algebra which is not biprojective Banach algebra.

Example 2.2. *The Banach algebra l^1 with respect to pointwise product is non-amenable and biprojective Banach algebra [4, Example 4.1.42]. Hence, $(l^1)^\sharp$ (unitization of l^1) not biprojective. If we define $\varphi : (l^1)^\sharp \longrightarrow (l^1)^\sharp$ by $\varphi(a + \lambda e) = \lambda$ for $a \in l^1$and $\lambda \in \mathbb{C}$, then $(l^1)^\sharp$ is $\varphi-$ biprojective and so $\varphi-$ approximate biprojective Banach algebra.*

3. $(\varphi, \psi)-$amenable Banach algebras

We begin this section with the following definition of $(\varphi, \psi)-$amenable Banach algebra. Let $\psi \in \Phi_A$. Then ψ has a unique extension $\tilde{\psi}$ on A^{**} and defined by $\tilde{\psi}(F) = F(\psi)$ for every $F \in A^{**}$.

Definition 3.1. Let A be a Banach algebra and $\varphi \in Hom(A), \psi \in \Phi_A$. Then A is called $(\varphi, \psi)-$amenable if there exists $M \in A^{**}$ such that $M(\psi \circ \varphi) = 1$ and $M(\varphi(a) \cdot f) = M(f \cdot \psi \circ \varphi(a))$ for all $a \in A,\ f \in A^*$.

The next theorem characterizes $(\varphi,\psi)-$amenability of Banach algebra. We are now going to prove the main result in this section.

Theorem 3.1. *Let A be a Banach algebra and $\varphi \in Hom(A), \psi \in \Phi_A$. Then A is $(\varphi,\psi)-$amenable if and only if there exists a bounded net $(m_\alpha) \subset A$ such that $\|\varphi(a)m_\alpha - m_\alpha\psi\circ\varphi(a)\| \longrightarrow 0$ and $\psi\circ\varphi(m_\alpha) = 1$ for all α.*

Proof. There exists $M \in A^{**}$ such that $M(\psi\circ\varphi) = 1,\ M(\varphi(a)\cdot f) = M(f\cdot\psi\circ\varphi(a))$ for $a \in A,\ f \in A^*$. Choose a net (a_α) in A with $a_\alpha \longrightarrow M$ in the w^*-topology of A^{**} and $\|a_\alpha\| \le \|M\|$ for all α. Since $< \psi\circ\varphi, a_\alpha > \longrightarrow < \psi\circ\varphi, M >= 1$, passing to a subnet and replacing (a_α) by $(1/\psi\circ\varphi(a_\alpha))a_\alpha$, we may assume that $\psi\circ\varphi(a_\alpha) = 1$ and $\|a_\alpha\| \le \|M\|+1$ for all α. Consider the product space A^A endowed with the product of norm topologies. If and define a linear map $T : A \longrightarrow A^A$ by $T(b) = \varphi(a)b - b\varphi(a)$, for all $b \in A$.

$$B = \{b \in A : \|b\| \le \|M\| + 1 \ \ and \ \ \psi\circ\varphi(ba) = 1\}$$

Clearly, B is convex and so $T(B)$ is a convex subset of A^A. For every $f \in A^*$, we have

$$\begin{aligned} < f, \varphi(a)a_\alpha - a_\alpha\psi\circ\varphi(a) > &= < f, \varphi(a)a_\alpha > - < f, a_\alpha\psi\circ\varphi(a) > \\ &= < f\cdot\varphi(a), a_\alpha > - < \psi\circ\varphi(a)\cdot f, a_\alpha > \\ &\longrightarrow < M, f\cdot\varphi(a) > - < M, \psi\circ\varphi(a)\cdot f >= 0. \end{aligned}$$

By Theorem 1.1 we can replace weak* convergence in equations by weak convergence and Aapplying Mazur's Theorem, then we obtain a net (m_α) in A such that $\|\varphi(a)m_\alpha - m_\alpha\psi\circ\varphi(a)\| \longrightarrow 0$ and $\psi\circ\varphi(m_\alpha) = 1$.

Conversely, assume that a net (m_α) exists. Let M be a w^*-cluster point of the net (m_α) in A^{**}. Then, $< M, \psi\circ\varphi >= \lim_\alpha < \psi\circ\varphi, m_\alpha >= 1$. For every $a \in A$ and

$f \in A^*$, we get

$$\begin{aligned} < M, f \cdot \varphi(a) > &= \lim_\alpha < f \cdot \varphi(a), a_\alpha >= \lim_\alpha < f, \varphi(a) a_\alpha > \\ &= \lim_\alpha < f, \varphi(a) a_\alpha - a_\alpha \psi \circ \varphi(a) > + \lim_\alpha < f, a_\alpha \psi \circ \varphi(a) > \\ &= \lim_\alpha < \psi \circ \varphi(a) \cdot f, a_\alpha >=< M, \psi \circ \varphi(a) \cdot f > . \end{aligned}$$

□

Definition 3.2. Let A be a Banach algebra and $\varphi \in Hom(A), \psi \in \Phi_A$. An element M of $(A \hat{\otimes} A)^{**}$ is a $(\varphi, \psi)-$ virtual diagonal for A if

i) $\varphi(a) \cdot M = M \cdot \psi \circ \varphi(a) \qquad (a \in A)$.

ii) $\tilde{\psi} \circ \pi^{**}(M) \cdot \varphi(a) = \varphi(a) \qquad (a \in A)$.

Proposition 3.1. *Let A be a Banach algebra and $\varphi \in Hom(A), \psi \in \Phi_A$. Then a Banach algebra A has a $(\varphi, \psi)-$ virtual diagonal if and only if there exists a bounded net (m_α) in $(A \hat{\otimes} A)$ such that $m_\alpha \cdot \varphi(a) - \psi \circ \varphi(a) \cdot m_\alpha \longrightarrow 0$ and $\psi \circ \pi(m_\alpha) \cdot \varphi(a) \longrightarrow \varphi(a)$ for every $a \in A$.*

Proof. Let M be a $\varphi-$ virtual diagonal for A and let (m_α) be a net in $(A \hat{\otimes} A)$ such that $M = w^* - \lim_\alpha m_\alpha$. Then a routine verification shows that for the net (m_α), $m_\alpha \cdot \varphi(a) - \psi \circ \varphi(a) \cdot m_\alpha \longrightarrow 0$ and $\psi \circ \pi(m_\alpha) \cdot \varphi(a) \longrightarrow \varphi(a)$ for every $a \in A$, holds in the $weak^*-$ topology. Following the argument given in the proof of [4, Lemma 2.9.64] we can show that there exists a net (m_β) of convex combinations of (m_α)'s satisfying both conditions.

Conversely, let $(m_\alpha) \subset (A \hat{\otimes} A)$ be a bounded net such that $m_\alpha \cdot \varphi(a) - \psi \circ \varphi(a) \cdot m_\alpha \longrightarrow 0$ and $\psi \circ \pi(m_\alpha) \cdot \varphi(a) \longrightarrow \varphi(a)$ for every $a \in A$. After passing to a subnet if necessary, let $M \in (A \hat{\otimes} A)^{**}$ be a w^*-cluster point of the net (m_α). Since $w^* - \lim m_\alpha \cdot \varphi(a) - \psi \circ \varphi(a) \cdot m_\alpha = 0$, it can easily be shown that $\varphi(a) \cdot M = M \cdot \varphi(a)$, for every $a \in A$. Also the w^*-continuity of π^{**} implies that $\tilde{\psi} \circ \pi^{**}(M) \cdot \varphi(a) = \varphi(a)$ and the proof is complete. □

Finally, we prove the following result related to $(\varphi, \psi)-$ virtual diagonal and $(\varphi, \psi)-$amenability.

Theorem 3.2. *Assume that A is a Banach algebra and $\varphi \in Hom(A), \psi \in \Phi_A$. If A is $(\varphi, \psi)-$amenable, then A has a $(\varphi, \psi)-$ virtual diagonal.*

Proof. Let A be $(\varphi, \psi)-$amenable, then by Theorem(3.1) there exists a bounded net $(m_\alpha) \subset A$ such that $\|\varphi(a)m_\alpha - m_\alpha\psi \circ \varphi(a)\| \longrightarrow 0$ and $\psi \circ \varphi(m_\alpha) = 1$ for all α. Define $a_\alpha = \varphi(m_\alpha) \otimes \varphi(m_\alpha)$, therefore

$$\begin{aligned} \psi \circ \pi(a_\alpha) \cdot \varphi(a) &= \psi \circ \pi(\varphi(m_\alpha) \otimes \varphi(m_\alpha)) \cdot \varphi(a) \\ &= \psi \circ \varphi(m_\alpha)\psi \circ \varphi(m_\alpha) \cdot \varphi(a) \\ &= \varphi(a). \end{aligned}$$

So by proposition (3.1), A has a $(\varphi, \psi)-$ virtual diagonal. □

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