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GENERALIZED NORMAL SUBGROUPS

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Abstract. In this paper, we generalize the concept of normal subgroups to $\mathcal{N}_c$-normal subgroups with respect to the variety of all nilpotent groups of class at most c, $(c \geq 1)$. We state some properties of $\mathcal{N}_c$-normal subgroups. Also we determine $\mathcal{N}_2$-normal subgroups and $\mathcal{N}_3$-normal subgroups of Q_{4n}, D_{2n}, SD_{2^n} and $\mathcal{N}_c$-normal subgroups of $SL(2, F)$.

1. Introduction

A variety of groups is the class of all groups satisfying a fixed system of identity relations, or laws, $v(x_1, ..., x_n) = 1$ where v runs through the non-empty set V (laws) of the free group F.

Let $\mathcal{V}$ be a variety of groups defined by the set of laws V. There exist two important subgroups associated to a given group G with respect to a variety $\mathcal{V}$,

$$V(G) = \langle v(g_1, g_2, ..., g_r)|\ g_i \in G, 1 \leq i \leq r, v \in V\rangle$$

which is called the verbal subgroup of G, and

$$V^*(G) = \{g \in G|\ v(g_1, g_2, ..., g_ig, ..., g_r) = v(g_1, g_2, ..., gg_i, ..., g_r)$$

$$= v(g_1, g_2, ..., g_r)\ |g_i \in G, 1 \leq i \leq r, v \in V\}$$

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which is called the marginal subgroup of G (See also [1], [2] ,[3] and [4]).

The upper central series (or ascending central series) of a group G is the sequence of subgroups

$$1 = Z_0(G) \triangleleft Z_1(G) \triangleleft ... \triangleleft Z_i(G) \triangleleft Z_{i+1}(G) \triangleleft ...,$$

where each successive group is defined by:

$$Z_{i+1}(G) = \{x \in G \mid \forall y \in G, [x, y] = x^{-1}y^{-1}xy \in Z_i(G)\}$$

and is called the ith center of G. In this case, $Z_1(G)$ is the center of G, and for each successive group, the factor group $Z_{i+1}(G)/Z_i(G)$ is the center of $G/Z_i(G)$. This series is called an upper central series of quotients.

The lower central series (or descending central series) of a group G is the sequence of subgroups

$$G = \gamma_1(G) \geq \gamma_2(G) \geq ... \geq \gamma_i(G) \geq ...,$$

in which $\gamma_i(G) = [G, \gamma_{i-1}(G)]$, $i > 1$. Notice that $\gamma_i(G)/\gamma_{i+1}(G)$ lies in the center of $G/\gamma_{i+1}(G)$ and that each $\gamma_i(G)$ is fully invariant in G. Also $\gamma_2(G) = [G, G] = G'$ is the derived subgroup of G.

Let $\mathcal{N}_c$ be the variety of nilpotent groups of class at most c, where c is a natural number. In this variety, $V(G) = \gamma_{c+1}(G)$ denotes $(c+1)^{th}$-term of lower central series of G and $V^*(G) = Z_c(G)$ is $(c+1)^{th}$-term of upper central series of G (See also [5], [6]).

The concept of a normal subgroup of a group G is one of the most important concepts and applied subjects in group theory. In this paper, we generalize this concept to $\mathcal{N}_c$-normal subgroup.

In Section 2, we define the concept of $\mathcal{N}_c$-normal subgroup with respect to the variety $\mathcal{N}_c$ $(c \geq 1)$ which is a new definition of normal subgroups and we will discuss some properties of it.

In Section 3, we determine $\mathcal{N}_2$-normal subgroups and $\mathcal{N}_3$-normal subgroups of Q_{4n}, D_{2n}, SD_{2^n} and $\mathcal{N}_c$-normal subgroups of $SL(2,F)$, for $c \geq 2$.

2. $\mathcal{N}_c$-Normal Subgroups

We begin with the definition of $\mathcal{N}_c$-normal subgroups of a group.

Definition 2.1. Let G be a group, a subgroup H of G is called $\mathcal{N}_c$-normal subgroup if for all $g \in G$ and $h \in H$, we have $g^{-1}hg \in HZ_{c-1}(G)$ or equivalency $HZ_{c-1}(G) \lhd G$ and we denote it by $H \lhd_{\mathcal{N}_c} G$.

Clearly, if $H \lhd_{\mathcal{N}_c} G$, then $H \lhd_{\mathcal{N}_d} G$, for $d \geq c$.
Every normal subgroup is $\mathcal{N}_c$-normal, but there exist $\mathcal{N}_c$-normal subgroups which are not normal. For instance, consider the dihedral group $D_8 = \langle a, b | a^4 = b^2 = (ab)^2 = 1\rangle$ and a subgroup H of D_8. We have $HZ(D_8) = Z(D_8)$ or D_8 or $[D_8 : HZ(D_8)] = 2$. It shows that $HZ(D_8) \lhd D_8$. Thus $H \lhd_{\mathcal{N}_2} D_8$. Therefore $H \lhd_{\mathcal{N}_c} D_8$, for $c \geq 2$. Where the subgroups $< b >, < ab >, < a^2b >$ and $< a^3b >$ of D_8 are not normal in D_8.

In the following, we state some useful and elementary results of $\mathcal{N}_c$-normal subgroups.

Theorem 2.1. *If $G \in \mathcal{N}_c$, then every subgroup of G is $\mathcal{N}_d$-normal, for $d \geq c$.*

Proof. It is enough to show that $H \lhd_{\mathcal{N}_c} G$, for every subgroup H of G. We know $G = Z_c(G)$, hence for all $g \in G$ and $h \in H$ we have $g^{-1}hg = h[h,g] \in HZ_{c-1}(G)$, since $G' \leq Z_{c-1}(G)$. Thus $H \lhd_{\mathcal{N}_c} G$. □

Lemma 2.1. *Let G be a group and H, K subgroups of G, where $Z_{c-1}(G) \subseteq Z_{c-1}(K)$ and $H \lhd_{\mathcal{N}_c} G$, then $H \cap K \lhd_{\mathcal{N}_c} K$ and $HK \leq G$.*

Proof. Let $k \in K$ and $x \in H \cap K$, then by assumption and Dedekind's Modular Law we have $k^{-1}xk \in (HZ_{c-1}(G)) \cap K = (H \cap K)Z_{c-1}(G) \subseteq (H \cap K)Z_{c-1}(K)$. It implies that $H \cap K \lhd_{\mathcal{N}_c} K$. Since $Z_{c-1}(G) \subseteq K$, we can easily show that $HK \leq G$. □

Consider the dihedral group $D_{48} = \langle a, b \mid a^{24} = b^2 = (ab)^2 = 1\rangle$. $H = \langle a^8, b\rangle$ and $K = \langle a^2, b\rangle$ are subgroups of it, clearly $Z_2(D_{48}) = Z_2(K) =< a^6 >$ and $H \lhd_{\mathcal{N}_3} D_{48}$. Hence $H = H \cap K \lhd_{\mathcal{N}_3} K$.

Remark 1. i) Let G be a group and H, K subgroups of G, where $H \lhd_{\mathcal{N}_c} G$, $H \subseteq K$ and $Z_{c-1}(G) \subseteq Z_{c-1}(K)$, then $H \lhd_{\mathcal{N}_c} K$.

ii) Let $H, K \lhd_{\mathcal{N}_c} G$, where $Z_{c-1}(G) \subseteq K$, then $H \cap K \lhd_{\mathcal{N}_c} G$.

iii) Let $\{N_i\}_{i=1}^{n}$ be a family of $\mathcal{N}_c$-normal subgroups of a group G, where $Z_{c-1}(G) \subseteq N_i$, for $i \geq 2$, then by part (ii), $\cap_{i=1}^{n} N_i \lhd_{\mathcal{N}_c} G$.

iv) Let G be a group and N a normal subgroup of G, where $N \leq H \leq G$ and $N \cap \gamma_c(G) = 1$, then $H \lhd_{\mathcal{N}_c} G$ if and only if $H/N \lhd_{\mathcal{N}_c} G/N$, for $c \geq 2$.

Theorem 2.2. *Let G and H be groups and $\alpha : G \longrightarrow H$ be a homomorphism. If $K \lhd_{\mathcal{N}_c} G$, then $\alpha(K) \lhd_{\mathcal{N}_c} \alpha(G)$. In particular, if α is an isomorphism, then $K \lhd_{\mathcal{N}_c} G$ if and only if $\alpha(K) \lhd_{\mathcal{N}_c} H$.*

Proof. It is obvious. □

Clearly, if G is a group and H a subgroup of G, then $H \lhd_{\mathcal{N}_c} G$ if and only if $HZ_{c-1}(G)/Z_{c-1}(G)$ is invariant under all inner automorphisms of $G/Z_{c-1}(G)$.

Remark 2. If a group G does not have nontrivial $\mathcal{N}_c$-normal subgroups, then G is simple. Now, assume that G is simple, if G is abelian, then G does not have nontrivial $\mathcal{N}_c$-normal subgroups. If $Z(G) < G$, then $Z(G) = 1$. Therefore $Z_c(G) = 1$, for $c \geq 2$. Hence for every subgroup H of G, $H \lhd G$ if and only if $H \lhd_{\mathcal{N}_c} G$, so a group G is simple if and only if G does not have nontrivial $\mathcal{N}_c$-normal subgroups.

3. Computing $\mathcal{N}_c$-Normal Subgroups of Some Groups

At first, we determine $\mathcal{N}_2$-normal subgroups and $\mathcal{N}_3$-normal subgroups of the generalized quaternion groups with the presentation

$$Q_{4n} = \langle a, b \mid a^n = b^2 \ , \ a^{2n} = 1 \ , \ b^{-1}ab = a^{-1} \rangle.$$

Since $Q_4 \cong Z_4$, hence every subgroup of Q_4 is $\mathcal{N}_c$-normal, for $c \geq 1$.

Theorem 3.1. *(i) A subgroup H of Q_{4n} is normal if and only if it is $\mathcal{N}_2$-normal, for $n \geq 2$.*

(ii) If $n \geq 3$ and odd, then the subgroup H of Q_{4n} is normal if and only if it is $\mathcal{N}_c$-normal, for $c \geq 2$.

Proof. (i) Clearly, if H is normal, then H is $\mathcal{N}_2$-normal. Now we assume that H is $\mathcal{N}_2$-normal. If $|H|$ is even, then the unique element of Q_{4n} of order 2, a^n, belongs to H, hence $Z(Q_{4n}) =< a^n > \leq H$ and $HZ(Q_{4n}) = H$, thus H is normal. If $|H|$ is odd, then $H \leq < a >$, since $|a^i b| = 4$, for $0 \leq i < 2n$, hence H is normal.

(ii) We know $Q_{4n}/Z(Q_{4n}) \cong D_{2n}$, so $Z_2(Q_{4n})/Z(Q_{4n}) = Z(Q_{4n}/Z(Q_{4n})) \cong Z(D_{2n})$, since $Z(D_{2n}) = 1$ we have $Z_2(Q_{4n}) = Z(Q_{4n})$, thus $Z_c(Q_{4n}) = Z(Q_{4n})$, for $c \geq 2$. Hence by part (i), $H \lhd Q_{4n}$ if and only if $H \lhd_{\mathcal{N}_c} Q_{4n}$, for $c \geq 2$. □

Clearly, every subgroup of Q_8 is $\mathcal{N}_c$-normal, for $c \geq 1$. We know $Q_{16} \in \mathcal{N}_3$, hence every subgroup of Q_{16} is $\mathcal{N}_c$-normal, for $c \geq 3$.

Theorem 3.2. *If $n \geq 6$ and even, then Q_{4n} has $\mathcal{N}_3$-normal subgroups (of order n) which are not $\mathcal{N}_2$-normal if and only if $n = 4(2k+1)$, for natural number k.*

Proof. Let $G = Q_{4n}$ and $H \lhd_{\mathcal{N}_3} G$ but $H \ntriangleleft_{\mathcal{N}_2} G$, so $H \ntriangleleft G$, $|H|$ is even and

$Z(G) \leq H \cap Z_2(G)$. We know $Z_2(G) =< a^{\frac{n}{2}} >$ and $|Z_2(G)| = 4$, hence

$|H \cap Z_2(G)| = 2$ or 4. $|H \cap Z_2(G)| \neq 4$ since $HZ_2(G) \lhd G$ and $H \ntriangleleft G$, so $|H \cap Z_2(G)| = 2$, it implies that $H \cap Z_2(G) = Z(G)$ and $|HZ_2(G)| = \dfrac{|H|.4}{2} = 2|H|$.

We have $HZ_2(G) \lhd G$ and $HZ_2(G)$ is not cyclic, hence $|G : HZ_2(G)| = 2$, $\dfrac{4n}{2|H|} = 2$, thus $|H| = n$. We conclude that H is one of the subgroups of the form $\langle a^4, a^i b\rangle$, $0 \leq i \leq 3$. Since $Z(G) \leq H$ and $Z_2(G) \nleq H$ we have $4|n$ and $4 \nmid \dfrac{n}{2}$, hence $n = 4(2k+1)$, for natural number k.

Conversely, if $n = 4(2k+1)$, then the subgroups $H_i = \langle a^4, a^i b\rangle$, $0 \leq i \leq 3$ are of order n and they are not $\mathcal{N}_2$-normal, but they are $\mathcal{N}_3$-normal.

Since $Z_2(G) \cap H_i =< a^n >$ we have $|H_i Z_2(G)| = \dfrac{n.4}{2} = 2n$, thus $H_i Z_2(G) \lhd G$ or equivalency $H_i \lhd_{\mathcal{N}_3} G$. □

Now we investigate $\mathcal{N}_2$-normal subgroups of the dihedral groups D_{2n} with the presentation

$$\langle a, b \mid a^n = b^2 = (ab)^2 = 1\rangle.$$

If n is odd, then every subgroup H of D_{2n} is normal if and only if it is $\mathcal{N}_c$-normal, for $c \geq 2$.

Clearly, every subgroup of D_4 is $\mathcal{N}_c$-normal, for $c \geq 1$. We know $D_8 \in \mathcal{N}_2$, hence every subgroup of D_8 is $\mathcal{N}_c$-normal, for $c \geq 2$.

Theorem 3.3. *If $n \geq 6$ and even, then D_{2n} has $\mathcal{N}_2$-normal subgroups (of order $\dfrac{n}{2}$) which are not normal if and only if $n = 4(2k+1)$, for natural number k.*

Proof. Let $G = D_{2n}$ and $H \lhd_{\mathcal{N}_2} G$ but $H \ntriangleleft G$, then $HZ(G) \lhd G$ and $HZ(G) \neq H$, therefore $Z(G) \nleq H$. By the structure of the subgroups of D_{2n} that are not normal, we have $H = \langle a^d, a^r b\rangle$ where $d|n$, $0 \leq r < d$ and $d \neq 1, 2$. Since $Z(G) \nleq H$ we have $a^{\frac{n}{2}} \notin H$, so $d \nmid \dfrac{n}{2}$, $Z(G) \cap H = 1$ and $|HZ(G)| = \dfrac{4n}{d}$. Thus $\dfrac{4n}{d} = n$, that is $d = 4$. We obtain $4|n$ and $4 \nmid \dfrac{n}{2}$, hence $n = 4(2k+1)$, for natural number k.

Conversely, if $n = 4(2k+1)$, then the subgroups $H_i = \langle a^4, a^i b\rangle$, where $0 \leq i \leq 3$ are of order $\dfrac{n}{2}$ and they are not normal. Now we show that they are $\mathcal{N}_2$-normal. We have $Z(G) =< a^{\frac{n}{2}} >$, clearly $Z(G) \cap H_i = 1$, for $0 \leq i \leq 3$, so $|H_i Z(G)| = n$, it implies that $H_i Z(G) \lhd G$ or equivalency $H_i \lhd_{\mathcal{N}_2} G$. □

We determine the $\mathcal{N}_3$-normal subgroups of D_{2n}, where n is even.

If $n = 2k$ and k is odd, then $Z_2(D_{2n}) = Z(D_{2n})$, hence the subgroup H of D_{2n} is $\mathcal{N}_3$-normal if and only if it is $\mathcal{N}_2$-normal.

Every subgroup of D_{16} is $\mathcal{N}_c$-normal, for $c \geq 3$, since $D_{16} \in \mathcal{N}_3$.

Theorem 3.4. *If $n = 2k$, $k \geq 6$ and even, then D_{2n} has $\mathcal{N}_3$-normal subgroups (of order $\frac{n}{2}$ and $\frac{n}{4}$) which are not $\mathcal{N}_2$-normal if and only if $n = 8(2t+1)$, for natural number t.*

Proof. If $G = D_{2n}$, $H \lhd_{\mathcal{N}_3} G$ and $H \ntriangleleft_{\mathcal{N}_2} G$, then $HZ_2(G) \lhd G$ and $HZ(G) \ntriangleleft G$, therefore $HZ_2(G) \neq HZ(G)$. Clearly $Z_2(G) \cap H < Z_2(G)$. Since $Z_2(G) =< a^{\frac{n}{4}} >$ and $|Z_2(G)| = 4$ we have $|Z_2(G) \cap H| = 1$ or 2.

Let $|Z_2(G) \cap H| = 1$. Since H is not normal we have $H = \langle a^d, a^r b \rangle$, where $d|n$, $0 \leq r < d$ and $|H| = \frac{2n}{d}$, so $|HZ_2(G)| = \frac{8n}{d} = n$, therefore $d = 8$ and $8|n$. But $8 \nmid \frac{n}{2}$, hence $n = 8(2t+1)$, for natural number t.

Let $|Z_2(G) \cap H| = 2$. Then $Z_2(G) \cap H =< a^{\frac{n}{2}} >$, so $|HZ_2(G)| = \frac{4n}{d}$ and $d = 4$, thus $4|n$ and $4|\frac{n}{2}$ but $4 \nmid \frac{n}{4}$, so $n = 8(2t+1)$, for natural number t.

Conversely, if $n = 8(2t+1)$, then the subgroups $H_i = \langle a^4, a^i b \rangle$, where $0 \leq i \leq 3$ are of order $\frac{n}{2}$. We know $H_i \ntriangleleft G$ and $4|\frac{n}{2}$, so $a^{\frac{n}{2}} \in H_i$ and $Z(G) \leq H_i$, thus $H_i \ntriangleleft_{\mathcal{N}_2} G$. Now we prove that $H_i \lhd_{\mathcal{N}_3} G$. Clearly $H_i \cap Z_2(G) =< a^{\frac{n}{2}} >$, hence $|H_i Z_2(G)| = n$, as desired. Also the subgroups $K_i = \langle a^8, a^i b \rangle$ are of order $\frac{n}{4}$, where $0 \leq i \leq 7$, also $|K_i \cap Z(G)| = 1$ and $|K_i Z(G)| = \frac{n}{2}$. We have $K_i Z(G) \ntriangleleft G$, hence $K_i \ntriangleleft_{\mathcal{N}_2} G$, but it is easy to show that $|K_i Z_2(G)| = n$, hence $K_i \lhd_{\mathcal{N}_3} G$. □

In the following, we investigate the semidihedral groups SD_{2^n} with the presentation

$$\langle a, b \mid a^2 = 1\ ,\ b^{2^{n-1}} = 1\ ,\ a^{-1}ba = b^{2^{n-2}-1}\rangle.$$

We know $SD_8 = C_2 \times C_4$, hence every subgroup of it is $\mathcal{N}_c$-normal, for $c \geq 1$.

Theorem 3.5. *If $n \geq 4$ is a natural number, then the subgroup H of SD_{2^n} is normal if and only if it is $\mathcal{N}_2$-normal.*

Proof. Let $G = SD_{2^n}$. If H is normal, then it is $\mathcal{N}_2$-normal. Suppose that $H \lhd_{\mathcal{N}_2} G$, then $HZ(G) \lhd G$. We know $|Z(G)| = 2$, therefore $|H \cap Z(G)| = 1$ or 2.

If $|H \cap Z(G)| = 2$, then $Z(G) \leq H$ and $HZ(G) = H$, hence $H \lhd G$. Now we assume that $|H \cap Z(G)| = 1$, by the structure of the normal subgroups of SD_{2^n} we have $HZ(G) \leq < b >$ or $[G : HZ(G)] = 2$.

If $HZ(G) \leq < b >$, then $H \leq < b >$, hence $H \lhd G$ as $< b >$ is a normal cyclic subgroup of G. We claim that the case $[G : HZ(G)] = 2$ doesn't happen.

If $[G : HZ(G)] = 2$, then $|H| = 2^{n-2}$, hence $H =< b^2 >$ or $H = \langle ab^i, b^4\rangle$, for $0 \leq i \leq 3$. Clearly, $Z(G) \leq H$ thus $|H \cap Z(G)| = 2$, which is a contradiction. □

We know $SD_{16} \in \mathcal{N}_3$, hence every subgroup of SD_{16} is $\mathcal{N}_c$-normal, for $c \geq 3$.

Lemma 3.1. *If $n \geq 5$ is a natural number, then a subgroup H of SD_{2^n} is normal if and only if it is $\mathcal{N}_3$-normal.*

Proof. It is enough to show that if H is a $\mathcal{N}_3$-normal subgroup of $G = SD_{2^n}$, then it is normal. Let $H \lhd_{\mathcal{N}_3} G$. Then $HZ_2(G) \lhd G$, since $|Z_2(G)| = 4$. Now three cases may happen:

Case 1: $|H \cap Z_2(G)| = 4$, thus $Z_2(G) \leq H$ and $H \lhd G$.

Case 2: $|H \cap Z_2(G)| = 2$, since $HZ_2(G) \lhd G$, we have $HZ_2(G) \leq < b >$ or $[G : HZ_2(G)] = 2$.

If $HZ_2(G) \leq < b >$, then $H \leq < b >$ and $H \lhd G$. We claim that the case $[G : HZ_2(G)] = 2$ doesn't happen. If $[G : HZ_2(G)] = 2$, then $|H| = 2^{n-2}$ so $H =< b^2 >$ or $H = \langle ab^i, b^4 \rangle$, for $0 \leq i \leq 3$. As $n \geq 5$ we have $Z_2(G) \leq H$, hence $|Z_2(G) \cap H| = 4$, which is a contradiction.

Case 3: $|H \cap Z_2(G)| = 1$, clearly if $HZ_2(G) \leq < b >$, then $H \lhd G$. It is easy to check that the case $[G : HZ_2(G)] = 2$ doesn't happen. □

Finally, we determine $\mathcal{N}_c$-normal subgroups of the special linear group $SL(2, F)$, where F is a finite field.

Theorem 3.6. *Let F be a finite field, then a subgroup H of $SL(2, F)$ is normal if and only if it is $\mathcal{N}_2$-normal.*

Proof. Since $SL(2, 2) \cong S_3$, the proof is clear. If $G = SL(2, 3)$, then $|G| = 24$ and G has two nontrivial normal subgroups, one of them is $Z(G)$ (the only subgroup of order 2) and another is isomorphic to Q_8(the only subgroup of order 8). Let $H \lhd_{\mathcal{N}_2} G$ or equivalency $HZ(G) \lhd G$, then one of the following cases happens:
Case 1: $|HZ(G)| = 2$, hence $HZ(G) = Z(G)$, thus $H \lhd G$.
Case 2: $|HZ(G)| = 8$. We have $|H \cap Z(G)| =$1 or 2, hence $|H| = 4$ or 8. $|H|$ can't be 4, since every subgroup of order 4 contains $Z(G)$, therefore $HZ(G) = H$, thus $H \lhd G$.
Case 3: $|HZ(G)| = 24$, we don't have the subgroup of order 12, hence $H = G$, as desired.
Let $G = SL(2, F)$ where $|F| \geq 4$ and $H \lhd_{\mathcal{N}_2} G$, hence $HZ(G) \lhd G$. Therefore $HZ(G) \leq Z(G)$ or $HZ(G) = G$. If $HZ(G) \leq Z(G)$, then $H \leq Z(G)$ so $H \lhd G$. If $HZ(G) = G$, then since $|Z(G)| = 2$ we have $H \cap Z(G) = 1$ or $H \cap Z(G) = Z(G)$, therefore $[G : H] = 2$ or $G = H$, as desired. □

By above theorem, we have the following corollary.

Corollary 3.1. *If $G = SL(2, F)$, where $|F| \geq 3$, then a subgroup H of G is normal if and only if it is $\mathcal{N}_c$-normal, for $c \geq 2$.*

References

[1] Ph. Hall, Verbal and marginal subgroups, *J. Reine Angew. Math.* **182** (1940), 156-157.

[2] N. S. Hekster, Varieties of groups and isologisms, *J. Austral. Math. Soc. Ser. A.* **46** (1989), no. 1, 22-60.

[3] C. R. Leedham-Green and S. McKay, Baer-invariants, isologism, varietal laws and homology, *Acta Math.* **137** (1976), 99-150.

[4] H. Neumann, Varieties of groups, Springer-Verlag, (1967).

[5] D. J. S. Robinson, A Course in the Theory of Groups, 2^{nd} ed. Springer -Verlag, Berlin, New york, (1996).

[6] J. S. Rose, A Course on Group Theory, Cambridge University Press, (1978).

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