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STATISTICAL INFERENCE FOR THE LOMAX DISTRIBUTION UNDER PARTIALLY ACCELERATED LIFE TESTS WITH PROGRESSIVELY TYPE-II CENSORING WITH BINOMIAL REMOVAL

R. ZAMAN [(1)], P. NASIRI [(2)] AND A. SHADROKH [(3)]

Abstract. In this paper a step-stress Partially Accelerated Life Test (SSPALT) is obtained for Lomax distribution under progressive Type II censoring with random removals, assuming that the number of units removed at each failure time has a binomial distribution. The maximum likelihood estimators ($MLEs$) are derived using the expectation-maximization (EM) algorithm. The Confidence intervals for the model parameters are constructed. SSPALT plan is used to minimize the Generalized Asymptotic Variance (GAV) of the ML estimators of the model parameters. We explain the performance of our procedures using a simulation study.

1. Introduction

Lomax distribution was introduced by Lomax for the first time [1] to model the business failure data. It has been used in areas of statistical modelling such as reliability theory and economics, too. The Lomax distribution has been under consideration for years.

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The most important scheme in life-time experiments is the progressive Type-II censored sampling scheme. For more information on the subject of progressively censoring, see [2] and [3]. Under this scheme, n units are placed on a life test at time zero, and m failures are going to be observed. Now at the time of the first failure, R_1 of surviving units are randomly selected and removed from the experiment, and so on. Finally, at the time of the m^{th} failure, all the remaining $R_m = n - R_1 - R_2 - ... - R_{m-1} - m$ surviving units are removed from the experiment. In the progressive Type-II censored sampling scheme, $R_1, R_2, ..., R_m$ are all prefixed. In many reliability experiments, the pattern of removal is random at each failure [4]. We suppose that any test unit being withdrawn from the life test is independent of the others but with the same removal probability (as a binomial parameter), p. The number of test units dropped out at each failure time has a binomial distribution. Statistical inference on the parameters of some distributions under progressive Type-II censoring with random removals and with binomial removals has been studied by several researchers [5][6][7][8][9].

The life tests of products with high reliability need longer time period. Then, the accelerated life tests (ALT) or partially accelerated life tests (PALT) are used to shorten the time period. Actually, with the help of these methods, all or some of test units go under stress to obtain failure units, faster.

In an ALT, the acceleration factor is presumed as a known value or there is a known mathematical model which identifies the relationship between lifetime and stress conditions. In some situations, such life-stress relationship is unknown and cannot be presumed. Then, in this condition, PALTs are the superior criterion for performing life test to estimate the acceleration factor and parameters of the life distribution. The ALT was first introduced by Chernoff and Bessler [10][11].

The stress can be applied in various ways, one of these methods is step-stress in which test units go under stress at a pre-determined time [12]. Several authors have studied ALT and PALT under step-stress based on censored data [13][14][15][16].

Asgharzadeh and Valiollahi [17] obtained the estimation of the scale parameter of the Lomax distribution under progressive censoring. Helu and et al. [18] studied estimation of the parameter of the Lomax distribution under progressive censoring using EM algorithm. In reference to the literature, there is no research work about optimum partially accelerated life test plans for the Lomax distribution under the progressive type-II censoring scheme with random removals. In this paper, we consider progressively Type-II censored data from the Lomax distribution with binomial removals. Comparing to progressive censoring, the proposed censoring scheme provides a more general method. The progressive type-II censoring leads to the efficient use of time and costs associated with the testing procedure. In section 2, the maximum likelihood estimators (MLEs) for the scale parameter and acceleration factor using the expectation-maximization (EM) are obtained. Moreover, the confidence intervals obtained by the sampling distribution of the mentioned parameters of the MLEs are also presented. The optimization of test plans is explained in section 3. For illustrating the theoretical results, the simulation studies are explained in section 4. Finally, conclusions are presented in section 5.

2. Estimation of parameters

Suppose that the life distribution, T, has a Lomax distribution with shape parameter k and scale parameter β, with the probability density function (pdf) and the cumulative distribution function (cdf) given by

$$
\begin{aligned}
f_T(t) &= k\beta(1+\beta t)^{-(k+1)}, t>0, k>0, \beta>0. \\
F_T(t) &= 1-(1+\beta t)^{-k}, t>0
\end{aligned}
\tag{2.1}
$$

and the failure-rate function is

$$h_T(t) = \frac{k\beta}{(1+\beta t)} \tag{2.2}$$

In the step-stress PALT, the total lifetime, Y, can be written as

$$Y = \begin{cases} T & T \leq \tau \\ \tau + \frac{T-\tau}{\alpha} & T > \tau \end{cases} \tag{2.3}$$

where τ is the stress change time and α is the acceleration factor, $\alpha \geq 1$. So, the probability density function and the cumulative distribution function of Y are respectively,

$$f(y) = \begin{cases} 0 & y \leq 0 \\ f_1(y) = k\beta(1+\beta y)^{-(k+1)} & 0 < y \leq \tau \\ f_2(y) = \alpha k\beta(1+\beta\tau+\alpha\beta(y-\tau))^{-(k+1)} & y > \tau \end{cases} \tag{2.4}$$

$$F(y) = \begin{cases} 0 & y \leq 0 \\ F_1(y) = 1-(1+\beta y)^{-k} & 0 < y \leq \tau \\ F_2(y) = 1-(1+\beta(\alpha(y-\tau)+\tau))^{-k} & y > \tau \end{cases} \tag{2.5}$$

Let n_u and n_a be number of items failed at normal and accelerated condition respectively. The experimental values of the total lifetime Y are expressed by

$y_{(1)} \leq ... \leq y_{(n_u)} \leq \tau \leq y_{(n_u+1)} \leq ... \leq y_{(m)}$

Also suppose that $\delta_{1i} = I(Y_i \leq \tau)$ and $\delta_{2i} = I(\tau < Y_i \leq y_{(m)})$.

Let $Y = (Y_{1:m:n}, Y_{2:m:n}, ..., Y_{m:m:n})$ be a progressively Type-II right censored sample from a life test of size m from a sample of size n, and $\mathbf{R} = (R_1, R_2, ..., R_m)$ be the progressive censoring scheme where lifetimes have a Lomax distribution with pdf as given by (2.1). Then, the likelihood function based on this progressively type-II censored sample is

$$L(y;\alpha,\beta,\delta_{1i},\delta_{2i}\mid \mathbf{R}=\mathbf{r})=\prod_{i=1}^{m}[f_1(y_i)(R_1(y_i))^{r_i}]^{\delta_{1i}}[f_2(y_i)(R_2(y_i))^{r_i}]^{\delta_{2i}} \tag{2.6}$$

where $R_1(y_i)=1-F_1(y_i), R_2(y_i)=1-F_2(y_i)$ and $\mathbf{r}=(r_1,r_2,...,r_m)$.

By substituting (2.4) into (2.6), for the progressive Type-II with determined number of removals $\mathbf{R}=\mathbf{r}$, the conditional likelihood and log-likelihood function are, respectively,

$$L(\beta,\alpha\mid \mathbf{R}=\mathbf{r})=\prod_{i=1}^{m}[k\beta(1+\beta y_i)^{-(k+1)}(1+\beta y_i)^{-kr_i}]^{\delta_{1i}}[\alpha k\beta A_i^{-(k(1+r_i)+1)}]^{\delta_{2i}} \tag{2.7}$$

where $A_i=1+\beta(\tau+\alpha(y_i-\tau))$.

$$\begin{aligned} l(\beta,\alpha\mid \mathbf{R}=\mathbf{r}) &= m\log k+m\log\beta+n_a\log\alpha-(k+1)\sum_{i=1}^{n_u}\log(1+\beta y_i) \\ &- k\sum_{i=1}^{n_u}r_i\log(1+\beta y_i)-(k+1)\sum_{i=1}^{n_a}\log A_i-k\sum_{i=1}^{n_a}r_i\log A_i \end{aligned} \tag{2.8}$$

Suppose that an individual unit is removed from the life test. As we know, it is independent of the other units, but with the same probability p. Then, the number of units removed at each failure time follows a binomial distribution such that

$$P(R_1=r_1)=\binom{n-m}{r_1}p^{r_1}(1-p)^{n-m-r_1}, r_1=0,...,n-m \tag{2.9}$$

and

$$\begin{aligned} P(R_i=r_i\mid R_{i-1}=r_{i-1},...,R_1=r_1) &= \binom{n-m-\sum_{k=1}^{i-1}r_k}{r_i} \\ &. \quad p^{r_i}(1-p)^{n-m-\sum_{k=1}^{i}r_k} \end{aligned} \tag{2.10}$$

where $r_i=0,...,n-m-\sum_{k=1}^{i-1}r_k, i=2,3,...,m-1$.

Moreover, suppose that R_i is independent of Y_i. Then, the joint likelihood function $Y=(Y_1,Y_2,...,Y_m)$ and $\mathbf{R}=(R_1,R_2,...,R_m)$ can be expressed as

$$L(\beta, p; x, r) = L(\beta, x \mid \mathbf{R} = \mathbf{r})P(\mathbf{R} = \mathbf{r}) \tag{2.11}$$

where

$$\begin{aligned} P(\mathbf{R} = \mathbf{r}) &= P(R_{m-1} = r_{m-1} \mid R_{m-2} = r_{m-2}, ..., R_1 = r_1) \\ &... \ P(R_2 = r_2 \mid R_1 = r_1)P(R_1 = r_1) \end{aligned} \tag{2.12}$$

By substituting (2.9) and (2.10) into (2.12), we have

$$\begin{aligned} P(\mathbf{R} = \mathbf{r}) &= \frac{(n-m)!}{\prod_{i=1}^{m-1} r_i!(n-m-\sum_{i=1}^{m-1} r_i)!} \\ &. \ p^{\sum_{i=1}^{m-1} r_i}(1-p)^{(m-1)(n-m)-\sum_{i=1}^{m-1}(m-i)r_i} \end{aligned} \tag{2.13}$$

Also, by substituting (2.8) and (2.12) into (2.11), then the likelihood function can be written as

$$L(\beta, \alpha, p; x, r) = CL_1(\beta, \alpha)L_2(p) \tag{2.14}$$

where $C = \frac{(n-m)!k^m}{\prod_{i=1}^{m-1} r_i!(n-m-\sum_{i=1}^{m-1} r_i)!}$.

$$L_1(\beta, \alpha) = \prod_{i=1}^{m}[\beta(1+\beta y_i)^{-(k+1)}(1+\beta y_i)^{-kr_i}]^{\delta_{1i}}[\alpha\beta A_i^{-(k(1+r_i)+1)}]^{\delta_{2i}} \tag{2.15}$$

$$L_2(p) = p^{\sum_{i=1}^{m-1} r_i}(1-p)^{(m-1)(n-m)-\sum_{i=1}^{m-1}(m-i)r_i} \tag{2.16}$$

In the following, the maximum likelihood estimates of the parameters β, p and α are obtained based on progressively type-II censored sample with binomial removals. The log-likelihood function of L_1 is given by

$$\begin{aligned} l_1(\beta, \alpha) &= m\log\beta + n_a\log\alpha - (k+1)\sum_{i=1}^{n_u}\log(1+\beta y_i) \\ &- \ k\sum_{i=1}^{n_u} r_i\log(1+\beta y_i) - (k+1)\sum_{i=1}^{n_a}\log A_i - k\sum_{i=1}^{n_a} r_i\log A_i \end{aligned} \tag{2.17}$$

In fact, the MLE of α and β can be obtained from the following equations

$$
\begin{aligned}
\frac{\partial l_1(\beta,\alpha)}{\partial \alpha} &= \frac{n_a}{\alpha} - (k+1)\sum_{i=1}^{n_a}\frac{\beta(y_i-\tau)}{A_i} - k\sum_{i=1}^{n_a}\frac{r_i(\beta(y_i-\tau))}{A_i} = 0 \\
\frac{\partial l_1(\beta,\alpha)}{\partial \beta} &= \frac{m}{\beta} - (k+1)\sum_{i=1}^{n_u}\frac{y_i}{1+\beta y_i} - k\sum_{i=1}^{n_u}\frac{r_i y_i}{1+\beta y_i} \\
&- (k+1)\sum_{i=1}^{n_a}\frac{\tau+\alpha(y_i-\tau)}{A_i} - k\sum_{i=1}^{n_a}\frac{r_i(\tau+\alpha(y_i-\tau))}{A_i} = 0 \qquad (2.18)
\end{aligned}
$$

Regarding the formulas of the likelihood equations in (2.18), the numerical methods are used to estimate the parameters. Here, we apply the EM algorithm to estimate of parameters, for more details on the EM algorithm and its applications, the readers are referred to a book by McLachlan an Krishnan [19]. Dempster et al. [20] introduced the EM algorithm for incomplete data sets. Let Y be an incomplete observed data and $Z = (Z_1, Z_2, ..., Z_m)$ with $Z_j = (z_{j1}, z_{j2}, ..., z_{jR_j}), j = 1, ..., m$, be the censored data. We consider the censored data as the missing data. The combination of $(Y, Z) = X$ constitutes the complete data set. Log-likelihood function based on X can be written as

$$
\begin{aligned}
l_c(X;\alpha,\beta) &= n\ln(k\beta) + (n-n_u-\sum_{i=1}^{n_u}R_i)\ln\alpha - (k+1)\sum_{i=1}^{n_u}\ln(1+\beta y_i) \\
&- (k+1)\sum_{i=1}^{n_u}\sum_{j=1}^{R_{n_u}}\ln(1+\beta z_{ij}) - (k+1)\sum_{i=1}^{n_a}\ln A_i - (k+1)\sum_{i=1}^{n_a}\sum_{j=n_u+1}^{R_{n_a}}\ln\eta_{ij} \qquad (2.19)
\end{aligned}
$$

where $\eta_{ij} = 1 + \beta(\tau + \alpha(z_{ij} - \tau))$, and z_{ij} is the removed data.

Then, the MLE of β and α for complete sample of X can be achieved from the following equation

$$\begin{aligned}
\frac{\partial l_c(X;\beta,\alpha)}{\partial\beta} &= \frac{n}{\beta} - (k+1)\sum_{i=1}^{n_u}\frac{y_j}{(1+\beta y_j)} - (k+1)\sum_{i=1}^{n_u}\sum_{j=1}^{R_{n_u}}\frac{z_{ij}}{(1+\beta z_{ij})} \\
&- (k+1)\sum_{i=1}^{n_a}\frac{\tau+\alpha(y_j-\tau)}{A_i} - (k+1)\sum_{i=1}^{n_a}\sum_{j=n_u+1}^{R_{n_a}}\frac{\tau+\alpha(z_{ij}-\tau)}{\eta_{ij}} \\
&= 0 \qquad (2.20)\\
\frac{\partial l_c(X;\beta,\alpha)}{\partial\alpha} &= \frac{n-n_u-\sum_{i=1}^{n_u}R_i}{\alpha} - (k+1)\sum_{i=1}^{n_a}\frac{\beta(y_j-\tau)}{A_i} \\
&- (k+1)\sum_{i=1}^{n_a}\sum_{j=n_u+1}^{R_{n_a}}\frac{\beta(z_{ij}-\tau)}{\eta_{ij}} \\
&= 0 \qquad (2.21)
\end{aligned}$$

There are two steps in the EM algorithm. In the E-step (expectation step), the expected value of the complete log-likelihood $l_c(X;\alpha,\beta)$ has been obtained with respect to the conditional distribution of Z given the observed data $Y_i = y_i$ and the current estimate of the parameter $\beta^{(k-1)}$ at the $(k-1)^{th}$ iteration.

$$\begin{aligned}
\frac{\partial l_c(X;\beta,\alpha)}{\partial\beta} &= \frac{n}{\beta} - (k+1)\sum_{i=1}^{n_u}\frac{y_j}{(1+\beta y_j)} - (k+1)\sum_{i=1}^{n_u}\sum_{j=1}^{R_{n_u}}A_1(\alpha,\beta) \\
&- (k+1)\sum_{i=1}^{n_a}\frac{\beta(y_j-\tau)}{A_i} - (k+1)\sum_{i=1}^{n_a}\sum_{j=n_u+1}^{R_{n_a}}A_2(\alpha,\beta) = 0 \qquad (2.22)\\
\frac{\partial l_c(X;\beta,\alpha)}{\partial\alpha} &= \frac{n-n_u-\sum_{i=1}^{n_u}R_i}{\alpha} - (k+1)\sum_{i=1}^{n_a}\frac{\beta(y_j-\tau)}{A_i} \\
&- (k+1)\sum_{i=1}^{n_a}\sum_{j=n_u+1}^{R_{n_a}}A_3(\alpha,\beta) = 0 \qquad (2.23)
\end{aligned}$$

where

$$\begin{aligned}
A_1(\alpha,\beta) &= E\Big(\frac{z_{ij}}{(1+\beta z_{ij})} \mid Z_{ij} > y_i\Big) \\
&= \frac{1+k\beta y_i(1+\beta y_i)^{-1} - (1+\beta y_i)^k(1+\beta\tau)^{-k}(1+k\beta\tau(1+\beta\tau)^{-1})}{\beta(k+1)} \\
A_2(\alpha,\beta) &= E\Big(\frac{\tau+\alpha(z_{ij}-\tau)}{1+\beta(\tau+\alpha(z_{ij}-\tau))} \mid Z_{ij} > y_i\Big) \\
&= \frac{k}{(1+\beta(\tau+\alpha(y_i-\tau)))^{-k}} \\
&\cdot \Big[\frac{(\tau+\alpha^2-\tau\alpha)(1+\beta(\tau+\alpha^2-\tau\alpha))^{-(k+1)}}{k+1} + \frac{(1+\beta(\tau+\alpha^2-\tau\alpha))^{-k}}{k(k+1)\beta}\Big] \\
A_3(\alpha,\beta) &= E\Big(\frac{\beta(z_{ij}-\tau)}{1+\beta(\tau+\alpha(z_{ij}-\tau))} \mid Z_{ij} > y_i\Big) \\
&= \frac{(1+\beta\tau)^{-k+1}}{\alpha(k-1)(1+\beta(\tau+\alpha(y_i-\tau)))^{-k+1}} \qquad (2.24)
\end{aligned}$$

In the M-step (maximization step), the EM algorithm will maximize $A_1(\beta,\alpha)$ and $A_2(\beta,\alpha)$ with respect to β to give an update value $\beta^{(k)}$ until convergence with an acceptable error occurs

$$\begin{aligned}
\frac{\partial l_c(X,\beta)}{\partial \beta} &= \frac{n}{\beta^{(k+1)}} - (k+1)\sum_{i=1}^{n_u} \frac{y_i}{1+\beta^{(k+1)}y_j} - (k+1)\sum_{i=1}^{n_u} R_i A_1(\alpha^{(k)},\beta^{(k)}) \\
&- (k+1)\sum_{i=1}^{n_a} \frac{\tau+\alpha^k(y_i-\tau)}{1+\beta^{(k+1)}(\tau+\alpha^{(k)}(y_i-\tau))} - (k+1)\sum_{i=1}^{n_a} R_i A_2(\alpha^{(k)},\beta^{(k)}) \\
&= 0 \qquad (2.25)
\end{aligned}$$

Once $\beta^{(k+1)}$ is obtained from equation (2.21), $\alpha^{(k+1)}$ is achieved by solving the following equation.

$$\begin{aligned}
\frac{\partial l_c(X,\beta)}{\partial \alpha} &= \frac{n-n_u-\sum_{i=1}^{n_u} R_i}{\alpha^{(k+1)}} - (k+1)\sum_{i=1}^{n_a} \frac{\beta^{(k+1)}(y_j-\tau)}{1+\beta^{(k+1)}((\tau+\alpha^{(k+1)}(y_j-\tau))} \\
&- (k+1)\sum_{i=1}^{n_a} R_i A_3(\alpha^{(k)},\beta^{(k)}) = 0 \qquad (2.26)
\end{aligned}$$

On the other hand, $L_2(p)$ is only a function based on p, then (2.16) is used to obtain MLE of p. The log-likelihood function of L_2 is

$$\log L_2(p) = \sum_{i=1}^{m-1} r_i \log p + ((m-1)(n-m) - \sum_{i=1}^{m-1}(m-i)r_i) \log(1-p) \tag{2.27}$$

Thus, the MLE of p can be found

$$\hat{p} = \frac{\sum_{i=1}^{m-1} r_i}{(m-1)(n-m) - \sum_{i=1}^{m-1}(m-i-1)r_i} \tag{2.28}$$

2.1. **Interval Estimation.** In this section, the approximate confidence intervals for the parameters (α, β) based on the asymptotic distributions of the MLE are derived. The asymptotic distribution of the MLE $(\hat{\alpha}, \hat{\beta})$ is $((\hat{\alpha} - \alpha), (\hat{\beta} - \beta)) \to N_2(0, I^{-1}(\alpha, \beta))$[21], where $I^{-1}(\alpha, \beta)$ is the variance-covariance matrix of the parameters (α, β) may be written as

$$I^{-1}(\hat{\alpha}, \hat{\beta}) = \begin{bmatrix} -\frac{\partial^2 \log L(\alpha,\beta)}{\partial \alpha^2} & -\frac{\partial^2 \log L(\alpha,\beta)}{\partial \alpha \partial beta} \\ -\frac{\partial^2 \log L(\alpha,\beta)}{\partial \beta \partial \alpha} & -\frac{\partial^2 \log L(\alpha,\beta)}{\partial \beta^2} \end{bmatrix}^{-1}_{(\alpha,\beta)=(\hat{\alpha},\hat{\beta})} = \begin{bmatrix} Var(\hat{\alpha}) & cov(\hat{\alpha}, \hat{\beta}) \\ cov(\hat{\beta}, \hat{\alpha}) & Var(\hat{\beta}) \end{bmatrix}$$

where

$$\begin{aligned} \frac{\partial^2 \log L(\alpha, \beta)}{\partial \alpha^2} &= \frac{-n_a}{\alpha^2} \\ \frac{\partial^2 \log L(\alpha, \beta)}{\partial \beta^2} &= \frac{-m}{\beta^2} + (k+1) \sum_{i=1}^{n_u} \frac{(1+R_i) y_i^2}{(1+\beta y_i)^2} + \sum_{i=1}^{n_u} \frac{y_i^2}{(1+\beta y_i)^2} \\ &+ (k+1) \sum_{i=1}^{n_a} \frac{(1+R_i)(\tau + \alpha(y_i - \tau))^2}{A_i^2} + \sum_{i=1}^{n_a} \frac{(\tau + \alpha(y_i - \tau))^2}{A_i^2} \\ \frac{\partial^2 \log L(\alpha, \beta)}{\partial \alpha \partial \beta} &= \frac{\partial^2 \log L(\alpha, \beta)}{\partial \beta \partial \alpha} = -k \sum_{i=1}^{n_a} \frac{(1-R_i)(y_i - \tau)(A_i - \beta(\tau + \alpha(y_i - \tau)))}{A_i^2} \\ &- \sum_{i=1}^{n_a} \frac{(y_i - \tau)(A_i - \beta(\tau + \alpha(y_i - \tau)))}{A_i^2} \end{aligned} \tag{2.29}$$

The approximate $100(1-\gamma)\%$ confidence intervals of the parameters β and α are derived, respectively,

$$
\begin{aligned}
\hat{\alpha}_{EM} &\pm z_{\frac{\gamma}{2}}\sqrt{var(\hat{\alpha})} \\
\hat{\beta}_{EM} &\pm z_{\frac{\gamma}{2}}\sqrt{var(\hat{\beta})}
\end{aligned}
\tag{2.30}
$$

where $z_{\frac{\gamma}{2}}$ is the upper $\frac{\gamma}{2}$th percentile of the standard normal distribution.

3. Optimum test plans

In step-stress, the analysers try to estimate the mean life time as an important characteristic in reliability theory, preciously. The optimum test plans are essential in enhancing the quality of the statistical inference.

In this section, the optimal stress-change time τ^* is formed based on progressive type-II censoring with binomial removals under different schemes. The optimal stress change time, τ^* is chosen to minimize the generalized asymptotic variance (GAV) of the ML estimators of the model parameters. This method as an optimality criterion is often utilized and given below as the reciprocal of the determinant of the Fisher information matrix $\mathbf{F}$([22]).

$$
GAV(\hat{\alpha},\hat{\beta}) = \frac{1}{|\mathbf{F}|} \tag{3.1}
$$

Where

$$
\begin{aligned}
|\mathbf{F}| &= \Big(\frac{-n_a}{\alpha^2} + (k+1)\sum_{i=1}^{n_a}\frac{\beta^2(y_i-\tau)^2}{A_i^2} + k\sum_{i=1}^{n_a}\frac{R_i\beta^2(y_i-\tau)^2}{A_i^2}\Big) \\
&\times \Big(\frac{-m}{\beta^2} + (k+1)\sum_{i=1}^{n_u}\frac{(1+R_i)y_i^2}{(1+\beta y_i)^2} + \sum_{i=1}^{n_u}\frac{y_i^2}{(1+\beta y_i)^2} \\
&+ (k+1)\sum_{i=1}^{n_a}\frac{(1+R_i)(\tau+\alpha(y_i-\tau))^2}{A_i^2} + \sum_{i=1}^{n_a}\frac{(\tau+\alpha(y_i-\tau))^2}{A_i^2}\Big) \\
&- \Big(k\sum_{i=1}^{n_a}\frac{(1-R_i)(y_i-\tau)(A_i-\beta(\tau+\alpha(y_i-\tau)))}{A_i^2} \\
&+ \sum_{i=1}^{n_a}\frac{(y_i-\tau)(A_i-\beta(\tau+\alpha(y_i-\tau)))}{A_i^2}\Big)^2.
\end{aligned}
$$

We have used a combination of golden section search and successive parabolic interpolation to minimize the generalized asymptotic variance.

4. Numerical examples

In this section, we present a Monte Carlo simulation study using R software to clarify theoretical results discussed in the previous sections. We have considered different sample sizes; $n = 20, 30, 50, 100$, and different effective sample sizes; $m = 10, 15, 20, 30, 40, 50, 60, 70, 80, 90$ using progressively Type-II censoring under binomial removal scheme. With no loss of generality, $k = 2.1$, $\beta = 0.1$, $\alpha = 2.5$, $\tau = 4.5$ and $p = 0.2, 0.5, 0.75$ are taken. Using binomial removal technique, for a given n and m different samples were generated. The MLEs estimates of the unknown parameters were obtained by the methods proposed in section 2 and 3. A comparison between the performance of estimates was done based on the root mean square error ($RMSE$) of the estimates under 10000 replications. In addition, the 95% confidence intervals (CIs) based on the same 10000 replications was computed. Table 1, table 2 and table 3 show the summarized results of simulation study. From the tables, it is concluded that with increasing sample size, the $RMSE$ decreases, as well as the width of confidence intervals. It is observed that at fixed sample size, $RMSE$ is decreased by the increasing of the failure units number.

The optimal test plans are obtained for different values of m, n and p. Table 4 shows the numerical results of stress change time optimal (τ^*) under different situations and the optimal GAV of the MLEs of the model parameters. It indicates that the stress change time optimal and the optimal GAV decrease with the increase of m, for constant n and p. This result is also valid for increasing n. Another conclusion is that the stress change time optimal and the optimal GAV increase with the increase of p, for constant n and m.

To examine the proposed method in real-life data, we recall the example mentioned in section 4 of Wang and Fei ([23]). This example is to achieve all the reliability indices of an electronic device. They randomly selected 100 units of a batch of devices. The failure times are: 32, 54, 59, 86, 117, 123, 213,267, 268, 273, 299, 311, 321, 333,339, 386, 408, 422, 435, 437, 476, 518, 570, 632, 666, 697, 796, 854, 858, 910. The stress, temperature variation, is applied to the remaining units at $\tau = 911$ and the obtained failure times are as follows: 926, 929, 931, 946, 947, 973, 980, 985, 993, 1005, 1010,1016, 1020, 1023, 1026, 1045, 1046, 1059, 1082, 1096. With the assumption of Lomax distribution with k=2.5 and $\beta = 0.001$, the result of Kolmogorov-Smirnov test is D=0.24 with p-value=0.1124. The proposed method with $m = 50$, $p = 0.2$ and progressive censoring scheme $R = c(14, 6, 9, 5, 4, 2, 4, 0, 0, 2, 0, 0, 1, 0, 1, 1, 0, 0, 1, 31^0)$ results in $\hat{\beta} = 0.00086$, $\hat{\alpha} = 3.0359$, $\tau^* = 743$ and $OptimalGAV = 0.00264$ for the stated data-set. Also, for $m = 50$, $p = 0.5$ and progressive censoring scheme $R = c(21, 16, 9, 3, 1, 45^0)$, $\hat{\beta}$ is 0.00089 and $\hat{\alpha}$ is 3.3128 and $\tau^* = 917$, $OptimalGAV = 0.00938$.

TABLE 1. Estimation of α and β, $\alpha = 2.5$, $\beta = 0.1$, $k = 2.1$, $p = 0.2$, $\tau = 4.5$

n	m	$\hat{\alpha}$	$RMSE(\hat{\alpha})$	$CI(\alpha)$	$\hat{\beta}$	$RMSE(\hat{\beta})$	$CI(\beta)$
20	10	1.1029	0.3550	(0.0068,2.2014)	0.0635	0.0307	(0.0164,0.1107)
20	15	1.6875	0.3403	(0.6019,2.7731)	0.0710	0.0223	(0.0262,0.1157)
30	15	1.2823	0.3586	(0.4058,2.1589)	0.0815	0.0281	(0.0292,0.1338)
30	20	1.6911	0.3222	(0.3162,3.0658)	0.0806	0.0183	(0.0159,0.1452)
50	20	1.3316	0.2574	(0.4452,2.2211)	0.1059	0.0285	(0.0288,0.1830)
50	30	1.8590	0.2641	(0.9165,2.8015)	0.0969	0.0231	(0.0443,0.1495)
50	40	2.1541	0.2223	(1.0162,3.2920)	0.0822	0.0089	(0.0121,0.1522)
100	30	1.5533	0.2831	(0.7036,2.4030)	0.1538	0.0398	(0.0808,0.2269)
100	40	1.8915	0.3397	(0.9850,2.7980)	0.1387	0.0177	(0.0462,0.2313)
100	50	2.0879	0.2692	(1.1573,3.0185)	0.1218	0.0126	(0.0346,0.2090)
100	60	2.1936	0.1587	(1.2648,3.1224)	0.1068	0.0099	(0.0228,0.1908)
100	70	2.2463	0.1990	(1.4809,3.0117)	0.0943	0.0098	(0.0553,0.1333)
100	80	2.2710	0.1139	(1.5385,3.0035)	0.0842	0.0036	(0.0502,0.1182)
100	90	2.2808	0.1042	(1.6137,2.9479)	0.0757	0.0055	(0.0443,0.1072)

TABLE 2. Estimation of α and β, $\alpha = 2.5$, $\beta = 0.1$, $k = 2.1$, $p = 0.5$, $\tau = 4.5$

n	m	$\hat{\alpha}$	$RMSE(\hat{\alpha})$	$CI(\alpha)$	$\hat{\beta}$	$RMSE(\hat{\beta})$	$CI(\beta)$
20	10	1.8381	0.5184	(0.0167,3.6596)	0.1086	0.0388	(0.0004,0.2178)
20	15	2.1226	0.4564	(0.6493,3.5959)	0.0846	0.0135	(0.0098,0.1594)
30	15	2.0291	0.4503	(0.4016,3.6567)	0.1162	0.0224	(-0.0353,0.2678)
30	20	2.1591	0.3090	(0.6263,3.6916)	0.0946	0.0113	(0.0115,0.1777)
50	20	2.1078	0.2830	(0.3326,3.8716)	0.1402	0.0144	(-0.3301,0.6107)
50	30	2.1985	0.1926	(1.0235,3.3735)	0.1033	0.0086	(0.0227,0.1839)
50	40	2.2138	0.1507	(1.2129,3.2147)	0.0811	0.0054	(0.0363,0.1260)
100	40	2.2104	0.2117	(0.5996,3.8212)	0.1390	0.0074	(-0.0810,0.3591)
100	50	2.2133	0.1978	(1.1162,3.3104)	0.1176	0.0066	(0.0087,0.2264)
100	60	2.2120	0.1447	(1.3697,3.3411)	0.1018	0.0059	(0.0400,0.1637)
100	70	2.2117	0.1266	(1.4536,3.1697)	0.0899	0.0058	(0.0252,0.1446)
100	80	2.2149	0.1064	(1.5304,2.8993)	0.0806	0.0039	(0.0499,0.1112)
100	90	2.2107	0.1032	(1.5530,2.8618)	0.0729	0.0026	(0.0463,0.1061)

TABLE 3. Estimation of α and β, $\alpha = 2.5$, $\beta = 0.1$, $k = 2.1$, $p = 0.75$, $\tau = 4.5$

n	m	$\hat{\alpha}$	$RMSE(\hat{\alpha})$	$CI(\alpha)$	$\hat{\beta}$	$RMSE(\hat{\beta})$	$CI(\beta)$
20	10	1.0168	0.1276	(0.3533,3.7293)	0.0790	0.0142	(0.0281,0.2002)
20	15	1.0230	0.1219	(0.4178,3.4288)	0.0557	0.0092	(0.0212,0.1365)
30	15	1.0210	0.1230	(0.4123,3.3961)	0.0559	0.0090	(0.0215,0.1338)
30	20	1.0329	0.0952	(0.3878,2.5434)	0.0629	0.0072	(0.0244,0.1513)
50	20	1.0325	0.0975	(0.3351,3.8107)	0.0967	0.0112	(0.0339,0.3406)
50	30	1.0366	0.0702	(0.3594,2.8165)	0.0691	0.0049	(0.0261,0.1793)
50	40	1.0395	0.0599	(0.4301,2.6350)	0.0535	0.0032	(0.0218,0.1228)
100	40	1.0393	0.0598	(0.2341,3.4358)	0.0961	0.0051	(0.0367,0.2603)
100	50	1.0378	0.0534	(0.3477,3.1659)	0.0799	0.0031	(0.0300,0.2176)
100	60	1.0380	0.0490	(0.3541,2.5891)	0.0685	0.0031	(0.0254,0.1577)
100	70	1.0371	0.0449	(0.3842,2.5787)	0.0598	0.0026	(0.0229,0.1329)
100	80	1.0378	0.0418	(0.4366,2.5715)	0.0531	0.0022	(0.0223,0.1100)
100	90	1.0372	0.0393	(0.4940,2.5618)	0.0497	0.0019	(0.0222,0.1009)

TABLE 4. Average values of optimal τ and optimal GAV; with different p

n	m	$p = 0.2$		$p = 0.5$		$p = 0.75$	
		τ^*	$OptimalGAV$	τ^*	$OptimalGAV$	τ^*	$OptimalGAV$
20	10	8.7918	0.01424	10.4346	0.04187	11.0129	0.0586
20	15	4.9012	0.01116	7.2920	0.02216	9.9916	0.0262
30	15	6.1494	0.01025	7.3329	0.03008	9.9999	0.03931
30	20	4.8822	0.01003	6.6085	0.02699	7.8340	0.03124
50	20	8.8543	0.01076	8.9067	0.03072	10.0597	0.04057
50	30	5.9604	0.01021	7.0395	0.02016	9.4612	0.02630
50	40	3.0412	0.01003	3.5451	0.01411	4.1320	0.01843
100	30	6.0878	0.01548	10.5484	0.02883	12.6231	0.03980
100	40	5.2721	0.01298	8.6823	0.02558	12.0025	0.03656
100	50	4.1312	0.00991	7.4992	0.01216	11.1462	0.03105
100	60	2.9586	0.00658	6.9229	0.01173	9.8433	0.02984
100	70	2.8346	0.00630	4.9228	0.01158	6.0249	0.02922
100	80	2.3779	0.00353	4.2004	0.01013	5.5026	0.02750
100	90	2.0506	0.00134	3.3247	0.00394	4.7351	0.00531

5. CONCLUSIONS

In step-stress partially accelerated life test, the units are run at both normal and accelerated conditions. In progressive type-II censoring, first, the test units are run at normal use condition and if they do not fail for a determined time τ, then they are run at accelerated condition unit number of failures (m) is reached. During the experiment with the occurrence of each failure, numbers of test units are removed randomly. In this article, MLEs of the model parameters of the Lomax distribution for step-stress partially accelerated life test were studied under progressive type-II censored data with binomial removals. The expectation maximization (EM) algorithm was used in estimating the scale parameter and acceleration factor because the normal equations were non-linear. In addition, the asymptotic confidence intervals of them are obtained. From the simulation results, it is clear that the performance of the MLE is good in terms of the $RMSE$ and CI.

The optimum test plans were developed under the presumption of Lomax lifetimes of test units and progressive type-II censoring with binomial removals. The minimization of the GAV of the MLE of model parameters was selected as an optimality criterion. Finally, the decrease of the stress change time optimal with the increase of n and $\frac{m}{n}$ is confirmed.

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References

[1] K. S. Lomax, *Business failures: Another example of the analysis of failure data*, Am. Stat. Assoc., **49**(1954), 847-852.

[2] N. Balakrishnan, R. Aggarwala, *Progressive censoring: theory, methods, and applications*, Springer Science & Business Media, 2000.

[3] N. Balakrishnan, *Progressive censoring methodology: an appraisal*, Springer , **16**(2007), 211-296.

[4] H. K. Yuen and S. K. Tse, *Parameters estimation for weibull distribution lifetimes under progressive censoring with random removals*, J. Stat. Comput. Simul, **55**(1996), 57-71.

[5] S.Dey and T. Dey, *Statistical inference for the Rayleigh distribution under progressively Type-II censoring with binomial removal*, Applied Mathematical Modelling, **38**(2014), 974-982.

[6] S. K. Tse, C. Yang, and HK. Yuen, *Statistical analysis of Weibull distributed lifetime data under Type II progressive censoring with binomial removals*. Journal of Applied Statistics, **27**(2000), 1033-1043.

[7] S. J. Wu and C. Chang, *Parameter estimations based on exponential progressive type II censored data with binomial removals*, International journal of information and management sciences, **13 (4)**(2002), 37-46.

[8] S. J. Wu, Y. J. Chen, and C. Chang, *Statistical inference based on progressively censored samples with random removals from the Burr type XII distribution*, Journal of Statistical Computation and Simulation, **77(1)**(2007), 19-27.

[9] M. Mubarak, *Parameter estimation based on the frechet progressive type ii censored data with binomial removals*, International Journal of Quality, Statistics, and Reliability, **2012**(2011), 1-5.

[10] H. Chernoff, *Optimal accelerated life designs for estimation*, Technomenics, **4**(1962), 381-408.

[11] S. Bessler, H. Chernoff and A. W. Marshall, *An optimal sequential accelerated life test*, Technomenics, **4**(1962), 367-379.

[12] W. Nelson, *Accelerated Testing: Statistical Models, Test Plans, and Data Analysis*, John Wiley & Sons, New York 1990

[13] A. A. Ismail, *Optimum failure-censored step-stress life test plans for the Lomax distribution*, Strength Mater. , **48(3)**(2016), 437-443.

[14] A. A. Ismail, *planning step-stress life test for the generalized Rayleigh distribution under progressive type-II censoring with binomial removals*, Strength Mater. , **49(2)**(2017), 292-306.

[15] N. Chandra and M. A. Khan, *Maximum likelihood estimation for step-stress partially accelerated life test based on censored data*, Mathematical Journal of Interdisciplinary Sciences , **3(1)**(2014), 37-54.

[16] f. K. Wang, Y. F. Cheng and W. L. lu, *Partially accelerated life tests for the Weibull distribution under Multiply censored data*, Communications in Statistics-Simulation and Computation , **41**(2012), 1667-1678.

[17] A. Asgharzadeh, and R. Valiollahi, *Estimation of the scale parameter of the Lomax distribution under progressive censoring*, International journal of statistics and Economics. Series B (methodological), **6**6(2011), 37-48.

[18] A. Helu, H. Samawi. and M. Z. Raqab, *Estimation on Lomax progressive censoring using the EM algorithm*, Journal of Statistical Computation and Simulation, **85(5)**(2015), 1035-1052.

[19] GJ. McLachlan , T. Krishnan , *The EM Algorithm and Extensions*, New York: Wiley, 1997.

[20] A. P. Dempster, and N. M. Laird, and D. B. Rubin, *Maximum likelihood from incomplete data via the EM algorithm*, Journal of the royal statistical society. Series B (methodological), **39(1)**(1977), 1-38.

[21] J. F. Lawless, *Statistical models and methods for lifetime data*, John Wiley & Sons, **362**, 2011.

[22] D.S. Bai, J. G. Kim, and Y. R.Chun, *Design of failure-censored accelerated life test sampling plans for lognormal and Weibull distribution* , Eng.Optimiz., **21**,(1993), 197-212.

[23] R.H. Wang, H. L. Fei, *Uniqueness of the maximum likelihood estimate of the Weibull distribution tampered failure rate model.* , Comm. Statist. Theory and Methods, **32**,(2003), 2321-2338.

(1) Department of Statistics, Faculty of Mathematical Sciences, Payame Noor University (PNU), 19395-4697 Tehran, I.R. Iran

Email address: roshanak.zaman@gmail.com

(2)(Corresponding Author) Department of Statistics, Faculty of Mathematical Sciences, University of Payame Noor, 19395-4697 Tehran, I.R. Iran

Email address: pnasiri45@yahoo.com

(3) Department of Statistics, Faculty of Mathematical Sciences, University of Payame Noor, 19395-4697 Tehran, I.R. Iran

Email address: ali_shadrokh@yahoo.com