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A NEW ITERATIVE NATURAL TRANSFORM METHOD FOR SOLVING NONLINEAR CAPUTO TIME-FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. The main purpose of this paper is to present the solutions of a class of nonlinear Caputo time-fractional partial differential equations, in particular nonlinear Caputo time-fractional wave-like equations with variable coefficients in terms of Mittag-Leffler functions by using new technique called, new iterative natural transform method (NINTM). This method introduced an efficient tool for solving these class of equations. Numerical examples are presented to illustrate the efficiency and accuracy of the proposed method. The results obtained show that the method described by NINTM is a very simple and easy method compared to the other methods and gives the approximate solution in the form of infinite series, this series in closed form gives the corresponding exact solution of the given problem.

1. Introduction

In recent years, there is a rapid development in the concept of fractional calculus and its applications [1],[3],[15],[16]. The fractional calculus which deals with derivatives and integrals of arbitrary orders [11],[17] plays a vital role in many field of applied science and engineering. Recently, nonlinear partial differential equations with fractional order derivative are successfully applied to many mathematical models

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in mathematical biology, aerodynamics, rheology, diffusion, electrostatics, electrodynamics, control theory, fluid mechanic, analytical chemistry and so on.

In all these scientific fields, it is important to obtain exact or approximate solutions of nonlinear fractional partial differential equations (NFPDEs). But in general, there exists no method that gives an exact solution for NFPDEs anmost of the obtained solution are only approximations.

Various analytical and numerical methods have been proposed to solve NFPDEs. The most commonly used ones are: Adomian decomposition method (ADM) [20] fractional variational iteration method (FVIM) [22], fractional difference method (FDM) [17], generalized differential transform method (GDTM) [13], homotopy analysis method (HAM) [5], homotopy perturbation method (HPM) [7].

Recently, a new option has appeared, includes the combination of Laplace transform, Sumudu transform or natural transform with the previously mentioned methods to facilitate and improve the resolution speed of nonlinear fractional partial differential equations. Among wich are: Laplace homotopy analysis method [25], Laplace decomposition method [8], Laplace variational iteration method [23], homotopy perturbation Sumudu transform method [24], homotopy analysis Sumudu transform method [12], variational iteration Sumudu transform method [2], natural transform homotopy perturbation method [14], natural decomposition method [18], homotopy analysis natural transform method [19].

In this paper we will suggest a new technique to the search for solutions of nonlinear Caputo time-fractional wave-like equation with variable coefficients. This technique is a combination of two powerful methods, natural transform method and new iterative method, called new iterative natural transform method (NINTM).

Consider the following nonlinear Caputo time-fractional wave-like equations

$$
\begin{aligned}
D_t^{\alpha} v \;=\; & \sum_{i,j=1}^{n} F_{1ij}(X,t,v)\frac{\partial^{k+m}}{\partial x_i^k \partial x_j^m} F_{2ij}(v_{x_i}, v_{x_j}) \\
& + \sum_{i=1}^{n} G_{1i}(X,t,v)\frac{\partial^{p}}{\partial x_i^p} G_{2i}(v_{x_i}) + H(X,t,v) + S(X,t),
\end{aligned} \tag{1.1}
$$

with the initial conditions

$$
v(X,0) = a_0(X),\;\; v_t(X,0) = a_1(X), \tag{1.2}
$$

where D_t^{α} is the Caputo time-fractional derivative operator of order α, $1 < \alpha \leq 2$, $v = v(X,t)$ is an unknown function where $X = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$ and $t > 0$, F_{1ij}, G_{1i} $i,j \in \{1,2,...,n\}$ are nonlinear functions of X, t and v, F_{2ij}, G_{2i} $i,j \in \{1,2,...,n\}$, are nonlinear functions of derivatives of v with respect to x_i and x_j $i,j \in \{1,2,...,n\}$, respectively. Also H, S are nonlinear functions and k, m, p are integers.

In the classical case, these types of equations are of considerable significance in various fields of applied sciences, mathematical physics, nonlinear hydrodynamics, engineering physics, biophysics, human movement sciences, astrophysics and plasma physics. These equations describe the evolution of erratic motions of small particles that are immersed in fluids, fluctuations of the intensity of laser light, velocity distributions of fluid particles in turbulent flows [21].

The paper is structured as follows. In Section 2, we present necessary definitions and preliminary results about fractional calculus and natural transform. In Section 3, we present our results to solve the nonlinear Caputo time-fractional wave-like equations (1.1) with the initial conditions (1.2) by the new iterative natural transform method (NINTM). In section 4, we present three numerical examples to show the efficiency and accuracy of this method, and we present our obtained results (Graphs and Table), comparing them with their exact associated forms. These results were verified with Matlab (version R2016a). Finally, conclusions are drawn in the last section.

2. Definitions and Preliminaries

In this section, we present necessary definitions and preliminary results about fractional calculus and natural transform, which are used further in this paper.

Definition 2.1. [11] A real function $f(t), t > 0$, is considered to be in the space $C_\mu\left([0,\infty[\right), \mu \in \mathbb{R}$ if there exists a real number $p > \mu$, so that $f(t) = t^p h(t)$, where $h \in C\left([0,\infty[\right)$, and it is said to be in the space C_μ^n if $f^{(n)} \in C_\mu\left([0,\infty[\right), n \in \mathbb{N}$.

Definition 2.2. [11] The left sided Riemann-Liouville fractional integral operator of order $\alpha \geq 0$ of a function $f \in C_\mu, \mu \geq -1$ is defined as

$$I^\alpha f(t) = \begin{cases} \dfrac{1}{\Gamma(\alpha)} \int\limits_0^t (t-\xi)^{\alpha-1} f(\xi) d\xi, \alpha > 0, t > 0, \\ f(t), \qquad \alpha = 0, \end{cases} \tag{2.1}$$

where $\Gamma(.)$ is the well-known Gamma function.

Definition 2.3. [11] The left sided Caputo fractional derivative of $f(t)$ is defined as

$$D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int\limits_0^t (t-\xi)^{n-\alpha-1} f^{(n)}(\xi) d\xi, t > 0, \tag{2.2}$$

where $n-1 < \alpha \leq n$, $n \in \mathbb{N}$, $f \in C_{-1}^n$.

For the Riemann-Liouville fractional integral and Caputo fractional derivative, we have the following relation

$$I^\alpha D^\alpha f(t) = f(t) - \sum_{k=0}^{n-1} f^{(k)}(0^+) \frac{t^k}{k!}, t > 0. \tag{2.3}$$

Remark 1. In this paper, we consider the time-fractional derivative in the Caputo's sense. When $\alpha \in \mathbb{R}^+$, the Caputo time-fractional derivative is defined as

$$D_t^\alpha v(x,t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int\limits_0^t (t-\tau)^{n-\alpha-1} \frac{\partial^n v(x,\tau)}{\partial \tau^n} d\tau, n-1 < \alpha < n, \\ \frac{\partial^n v(x,t)}{\partial t^n}, \qquad \alpha = n, \end{cases} \tag{2.4}$$

where $n \in \mathbb{N}^*$.

Definition 2.4. [17] The Mittag-Leffler function is defined as follows

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha+1)}, \alpha \in \mathbb{C}, Re(\alpha) > 0. \tag{2.5}$$

A further generalization of (2.5) is given in the form

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha+\beta)}, \alpha, \beta \in \mathbb{C}, Re(\alpha) > 0, Re(\beta) > 0. \tag{2.6}$$

For $\alpha = 1$, $E_\alpha(z)$ reduces to e^z.

Definition 2.5. [4] The natural transform is defined over the set of functions is defined over the set of functions

$$A = \left\{ f(t) / \exists M, \tau_1, \tau_2 > 0, |f(t)| < M e^{\frac{|t|}{\tau_j}}, \text{ if } t \in (-1)^j \times [0, \infty), j = 1, 2, ... \right\},$$

by the following integral

$$\mathcal{N}^+ [f(t)] = R^+(s,u) = \frac{1}{u} \int_0^{+\infty} e^{-\frac{st}{u}} f(t) dt, s, u \in (0, \infty). \tag{2.7}$$

Theorem 2.1. [9],[10] *Let $n \in \mathbb{N}^*$ and $\alpha > 0$ be such that $n-1 < \alpha \leq n$ and $R^+(s,u)$ be the natural transform of the function $f(t)$, then the natural transform denoted by $R_\alpha^+(s,u)$ of the Caputo fractional derivative of the function $f(t)$ of order α, is given by*

$$\mathcal{N}^+ [D^\alpha f(t)] = R_\alpha^+(s,u) = \frac{s^\alpha}{u^\alpha} R^+(s,u) - \sum_{k=0}^{n-1} \frac{s^{\alpha-(k+1)}}{u^{\alpha-k}} \left[D^k f(t) \right]_{t=0}. \tag{2.8}$$

3. NINTM for solving nonlinear Caputo time-fractional wave-like equations

Theorem 3.1. *Consider the following nonlinear Caputo time-fractional wave-like equations (1.1) with the initial conditions (1.2).*

Then, by NINTM the solution of Eqs. (1.1) and (1.2) is given in the form of infinite series which converges rapidly to the exact solution as follows

$$v(X,t) = \sum_{i=0}^{\infty} v_i(X,t).$$

Proof. We consider the following nonlinear Caputo time-fractional wave-like equations (1.1) with the initial conditions (1.2).

First, we apply the natural transform on both sides of (1.1) subject to the initial conditions (1.2) and using the theorem 2.1, we get

$$\begin{aligned} \mathcal{N}^+\left[v(X,t)\right] &= \frac{u^\alpha}{s^\alpha}\sum_{k=0}^{n-1}\frac{s^{\alpha-(k+1)}}{u^{\alpha-k}}\left[D^k v(X,t)\right]_{t=0} + \frac{u^\alpha}{s^\alpha}\mathcal{N}^+\left[S(X,t)\right] \\ &\quad + \frac{u^\alpha}{s^\alpha}\mathcal{N}^+\left[\sum_{i,j=1}^{n} F_{1ij}(X,t,v)\frac{\partial^{k+m}}{\partial x_i^k \partial x_j^m}F_{2ij}(v_{x_i},v_{x_j})\right. \\ &\quad \left. + \sum_{i=1}^{n} G_{1i}(X,t,v)\frac{\partial^p}{\partial x_i^p}G_{2i}(v_{x_i}) + H(X,t,v)\right]. \end{aligned} \tag{3.1}$$

After that, let us take the inverse natural transform on both sides of (3.1), we have

$$\begin{aligned} v(X,t) &= \mathcal{N}^{-1}\left(\frac{1}{s}v(X,0) + \frac{u}{s^2}v_t(X,0) + \frac{u^\alpha}{s^\alpha}\mathcal{N}^+\left[S(X,t)\right]\right) \\ &\quad + \mathcal{N}^{-1}\left(\frac{u^\alpha}{s^\alpha}\mathcal{N}^+\left[\sum_{i,j=1}^{n} F_{1ij}(X,t,v)\frac{\partial^{k+m}}{\partial x_i^k \partial x_j^m}F_{2ij}(v_{x_i},v_{x_j})\right.\right. \\ &\quad \left.\left. + \sum_{i=1}^{n} G_{1i}(X,t,v)\frac{\partial^p}{\partial x_i^p}G_{2i}(v_{x_i}) + H(X,t,v)\right]\right). \end{aligned} \tag{3.2}$$

Next assume that

$$g(X,t) = \mathcal{N}^{-1}\left(\frac{1}{s}v(X,0) + \frac{u}{s^2}v_t(X,0) + \frac{u^\alpha}{s^\alpha}\mathcal{N}^+\left[S(X,t)\right]\right),$$

$$N(v(X,t)) = \mathcal{N}^{-1}\left(\frac{u^\alpha}{s^\alpha}\mathcal{N}^+\left[\sum_{i,j=1}^{n} F_{1ij}(X,t,v)\frac{\partial^{k+m}}{\partial x_i^k \partial x_j^m}F_{2ij}(v_{x_i},v_{x_j}) + \sum_{i=1}^{n} G_{1i}(X,t,v)\frac{\partial^p}{\partial x_i^p}G_{2i}(v_{x_i}) + H(X,t,v)\right]\right).$$

We can obtain

$$v(X,t) = g(X,t) + N(v(X,t)). \tag{3.3}$$

The solution of Eq. (3.3) can be written in the series form

$$v(X,t) = \sum_{i=0}^{\infty} v_i(X,t). \tag{3.4}$$

The nonlinear operator N can be decomposed as follows (see [6])

$$N\left(\sum_{i=0}^{\infty} v_i\right) = N\left(v_0\right) + \sum_{i=1}^{\infty}\left\{N\left(\sum_{j=0}^{i} v_j\right) - N\left(\sum_{j=0}^{i-1} v_j\right)\right\}. \tag{3.5}$$

From Eqs. (3.4) and (3.5), Eq. (3.3) can be represented as the following form

$$\sum_{i=0}^{\infty} v_i = g + N\left(v_0\right) + \sum_{i=1}^{\infty}\left\{N\left(\sum_{j=0}^{i} v_j\right) - N\left(\sum_{j=0}^{i-1} v_j\right)\right\}. \tag{3.6}$$

We define the recurrence relation

$$\begin{aligned} v_0 &= g, \\ v_1 &= N\left(v_0\right), \\ v_{n+1} &= N\left(\sum_{j=0}^{n} v_j\right) - N\left(\sum_{j=0}^{n-1} v_j\right), n \in \mathbb{N}. \end{aligned} \tag{3.7}$$

Then

$$v_1 + v_2 + ... + v_{n+1} = N\left(v_0 + v_1 + ... + v_n\right), n \in \mathbb{N}.$$

and

$$v = \sum_{i=0}^{\infty} v_i = g + N\left(\sum_{i=0}^{\infty} v_i\right). \tag{3.8}$$

The n-term approximate solution of (3.3) is given by

$$v = \sum_{i=0}^{n-1} v_i = v_0 + v_1 + ... + v_{n-1}. \tag{3.9}$$

4. Numerical examples and results

In this section, we apply the NINTM on three examples of nonlinear wave-like equations with Caputo time-fractional derivative and then compare our approximate solutions with the exact solutions.

We define E_n to be the absolute error between the exact solution v and the n-term approximate solution v_n, where $n = 0, 1, 2, 3, ...$, as follows

$$E_n(X,t) = \left|v(X,t) - v_n(X,t)\right|.$$

Example 4.1. *Consider the 2-dimensional nonlinear Caputo time-fractional wave-like equation with variable coefficients*

$$D_t^{\alpha} v = \frac{\partial^2}{\partial x \partial y}(v_{xx}v_{yy}) - \frac{\partial^2}{\partial x \partial y}(xyv_xv_y) - v, \tag{4.1}$$

with the initial conditions

$$v(x,y,0) = e^{xy}, v_t(x,y,0) = e^{xy}, \tag{4.2}$$

where D_t^{α} is the Caputo time-fractional derivative operator of order α, $1 < \alpha \leq 2$ and v is a function of $(x,y,t) \in \mathbb{R}^2 \times \mathbb{R}^+$.

By applying the steps involved in NINTM as presented in Section 3 to Eqs. (4.1) and (4.2), we have

$$
\begin{aligned}
g(x,y,t) &= e^{xy} + te^{xy}, \\
N(v(x,y,t)) &= \mathcal{N}^{-1}\left(\frac{u^{\alpha}}{s^{\alpha}}\mathcal{N}^{+}\left[\frac{\partial^2}{\partial x \partial y}(v_{xx}v_{yy}) - \frac{\partial^2}{\partial x \partial y}(xyv_xv_y) - v\right]\right),
\end{aligned}
$$

and

$$
\begin{aligned}
v_0(x,y,t) &= (1+t)\, e^{xy}, \\
v_1(x,y,t) &= -\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\right) e^{xy}, \\
v_2(x,y,t) &= \left(\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\right) e^{xy}, \\
v_3(x,y,t) &= -\left(\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)}\right) e^{xy}, \\
&\vdots
\end{aligned}
$$

Therefore, the solution of Eqs. (4.1) and (4.2) is given as follows

$$
\begin{aligned}
v(x,y,t) &= \left(1 + t - \frac{t^{\alpha}}{\Gamma(\alpha+1)} - \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} - \ldots\right) e^{xy} \\
&= \left(E_{\alpha}(-t^{\alpha}) + tE_{\alpha,2}(-t^{\alpha})\right) e^{xy}, \qquad (4.3)
\end{aligned}
$$

where $E_{\alpha}(-t^{\alpha})$ and $E_{\alpha,2}(-t^{\alpha})$ are the Mittag-Leffler functions defined by Eqs. (2.5) and (2.6).

Taking $\alpha = 2$ in Eq. (4.3), we obtained the following result

$$
\begin{aligned}
v(x,y,t) &= \left(1 + t - \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!} + \ldots\right) e^{xy} \\
&= (\cos t + \sin t)\, e^{xy},
\end{aligned}
$$

which is the same result as those obtained by the FNDM [9], *and NVIM* [10] *for the same test problem.*

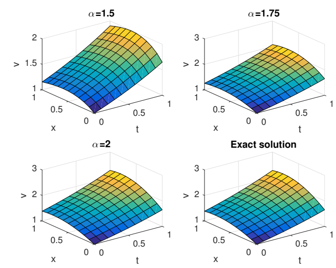

FIGURE 1. The surface graph of the 3−term approximate solution by NINTM and the exact solution for different values of α for Example 4.1 when $y = 0.5$.

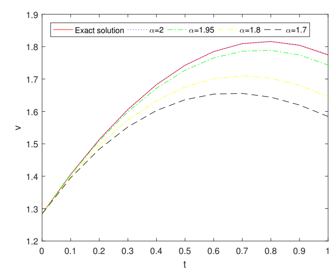

FIGURE 2. The behavior of the exact solution and the 3−term approximate solution by NINTM of v for different values of α for Example 4.1 when $x = y = 0.5$.

TABLE 1. Comparison of the absolute errors for the exact solution and 3−term approximate solution obtained by NINTM for Example 4.1, when $\alpha = 2$.

$t/x, y$	0.1	0.3	0.5	0.7
0.1	1.4226×10^{-9}	1.5411×10^{-9}	1.8085×10^{-9}	2.2991×10^{-9}
0.3	1.0648×10^{-6}	1.1535×10^{-6}	1.3536×10^{-6}	1.7208×10^{-6}
0.5	2.3382×10^{-5}	2.5330×10^{-5}	2.9725×10^{-5}	3.7787×10^{-5}
0.7	1.8000×10^{-4}	1.9499×10^{-4}	2.2882×10^{-4}	2.9089×10^{-4}
0.9	8.2963×10^{-4}	8.9872×10^{-4}	1.0547×10^{-3}	1.3407×10^{-3}

Example 4.2. *Consider the following nonlinear Caputo time-fractional wave-like equation with variable coefficients*

$$D_t^\alpha v = v^2 \frac{\partial^2}{\partial x^2}(v_x v_{xx} v_{xxx}) + v_x^2 \frac{\partial^2}{\partial x^2}(v_{xx}^3) - 18v^5 + v, \tag{4.4}$$

with the initial conditions

$$v(x,0) = e^x, v_t(x,0) = e^x, \tag{4.5}$$

where D_t^α is the Caputo time-fractional derivative operator of order α, $1 < \alpha \leq 2$, and v is a function of $(x,t) \in]0,1[\times \mathbb{R}^+$.

By applying the steps involved in NINTM as presented in Section 3 to Eqs. (4.4) and (4.5), we have

$$\begin{aligned} g(x,t) &= e^x + te^x, \\ N(v(x,t)) &= \mathcal{N}^{-1}\left(\frac{u^\alpha}{s^\alpha}\mathcal{N}^+\left[v^2 \frac{\partial^2}{\partial x^2}(v_x v_{xx} v_{xxx}) + v_x^2 \frac{\partial^2}{\partial x^2}(v_{xx}^3) - 18v^5 + v\right]\right), \end{aligned}$$

and

$$\begin{aligned} v_0(x,t) &= (1+t)\, e^x, \\ v_1(x,t) &= \left(\frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\right) e^x, \\ v_2(x,t) &= \left(\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\right) e^x, \\ &\vdots \end{aligned}$$

Therefore, the solution of Eqs. (4.4) and (4.5) is given as follows

$$\begin{aligned} v(x,t) &= \left(1 + t + \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} + ...\right) e^x \\ &= \left(E_\alpha(t^\alpha) + tE_{\alpha,2}(t^\alpha)\right) e^x, \end{aligned} \tag{4.6}$$

where $E_\alpha(t^\alpha)$ and $E_{\alpha,2}(t^\alpha)$ are the Mittag-Leffler functions, defined by Eqs. (2.5) and (2.6).

Taking $\alpha = 2$ in Eq. (4.6), we obtained the following result

$$v(x,t) = \left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!} + ...\right) e^x = e^{x+t},$$

which is the same result as those obtained by the FNDM [9], *and NVIM* [10] *for the same test problem.*

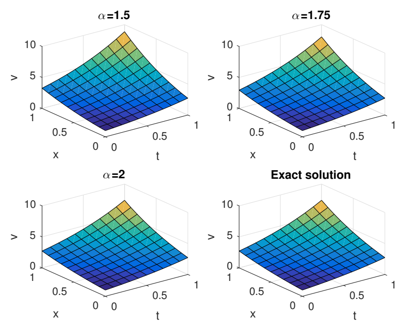

FIGURE 3. The surface graph of the 3−term approximate solution by NINTM and the exact solution for different values of α for Example 4.2.

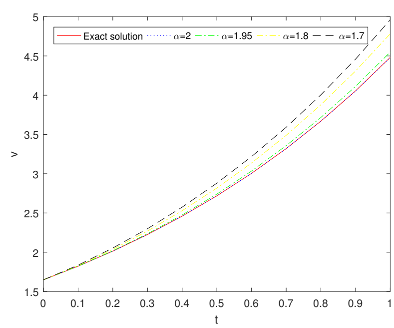

FIGURE 4. The behavior of the exact solution and the 3−term approximate solution by NINTM of v for different values of α for Example 4.2 when $x = 0.5$.

TABLE 2. Comparison of the absolute errors for the exact solution and 3−term approximate solution obtained by NINTM for Example 4.2, when $\alpha = 2$.

t/x	0.1	0.3	0.5	0.7
0.1	1.5572×10^{-9}	1.9019×10^{-9}	2.3230×10^{-9}	2.8373×10^{-9}
0.3	1.1688×10^{-6}	1.4276×10^{-6}	1.7436×10^{-6}	2.1297×10^{-6}
0.5	2.5810×10^{-5}	3.1525×10^{-5}	3.8504×10^{-5}	4.7029×10^{-5}
0.7	2.0036×10^{-4}	2.4472×10^{-4}	2.9890×10^{-4}	3.6507×10^{-4}
0.9	9.3372×10^{-4}	1.1404×10^{-3}	1.3929×10^{-3}	1.7013×10^{-3}

Example 4.3. *Consider the following one dimensional nonlinear Caputo time-fractional wave-like equation with variable coefficients*

$$D_t^\alpha v = x^2 \frac{\partial}{\partial x}(v_x v_{xx}) - x^2 (v_{xx})^2 - v, \tag{4.7}$$

with the initial conditions

$$v(x,0) = 0, v_t(x,0) = x^2, \tag{4.8}$$

where D_t^α is the Caputo time-fractional derivative operator of order α, $1 < \alpha \leq 2$, and v is a function of $(x,t) \in \,]0,1[\, \times \mathbb{R}^+$.

By applying the steps involved in NINTM as presented in Section 3 to Eqs. (4.7) and (4.8), we have

$$\begin{aligned} g(x,t) &= tx^2, \\ N(v(x,t)) &= \mathcal{N}^{-1}\left(\frac{u^\alpha}{s^\alpha}\mathcal{N}^+\left[x^2\frac{\partial}{\partial x}(v_x v_{xx}) - x^2(v_{xx})^2 - v\right]\right), \end{aligned}$$

and

$$\begin{aligned} v_0(x,t) &= tx^2, \\ v_1(x,t) &= -\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}x^2, \\ v_2(x,t) &= \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}x^2, \\ v_3(x,t) &= -\frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)}x^2, \\ &\vdots \end{aligned}$$

Therefore, the solution of Eqs. (4.7) and (4.8) is given as follows

$$\begin{aligned} v(x,t) &= \left(t - \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} - \frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)} + ...\right)x^2 \\ &= x^2\left(tE_{\alpha,2}(-t^{\alpha})\right), \end{aligned} \tag{4.9}$$

where $E_{\alpha,2}(-t^{\alpha})$ is the Mittag-Leffler function, defined by Eq. (2.6).

Taking $\alpha = 2$ in Eq. (4.9) we obtained the following result

$$v(x,t) = x^2\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + ...\right) = x^2 \sin t,$$

which is the same result as those obtained by the FNDM [9]*, and NVIM* [10] *for the same test problem.*

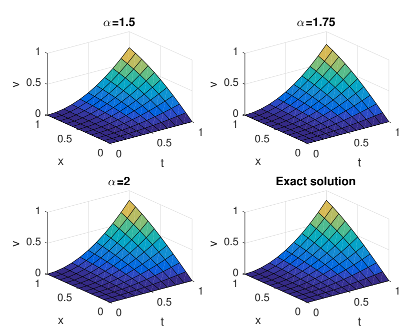

FIGURE 5. The surface graph of the 3−term approximate solution by NINTM and the exact solution for different values of α for Example 4.3.

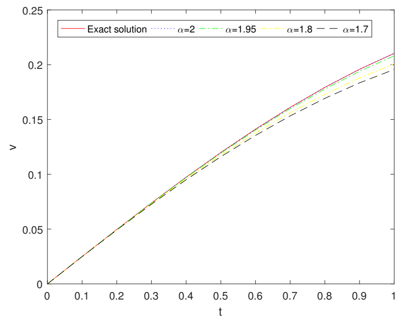

FIGURE 6. The behavior of the exact solution and the 3−term approximate solution by NINTM of v for different values of α for Example 4.3 when $x = 0.5$.

TABLE 3. Comparison of the absolute errors for the exact solution and 3−term approximate solution obtained by NINTM for Example 4.3, when $\alpha = 2$.

t/x	0.1	0.3	0.5	0.7
0.1	1.9839×10^{-13}	1.7855×10^{-12}	4.9596×10^{-12}	9.7209×10^{-12}
0.3	4.3339×10^{-10}	3.9005×10^{-9}	1.0835×10^{-8}	2.1236×10^{-8}
0.5	1.5447×10^{-8}	1.3903×10^{-7}	3.8618×10^{-7}	7.5692×10^{-7}
0.7	1.6229×10^{-7}	1.4606×10^{-6}	4.0574×10^{-6}	7.9524×10^{-6}
0.9	9.3840×10^{-7}	8.4456×10^{-6}	2.3460×10^{-5}	4.5982×10^{-5}

Remark 2. The numerical results, affirm that when α approaches 2, our results obtained by NINTM approach the exact solutions.

Remark 3. In this paper, we only apply three terms to approximate the solutions, if we apply more terms of the approximate solutions, the accuracy of the approximate solutions will be greatly improved.

In general, the results obtained show that the method described by NINTM is a very simple and easy method compared to the other methods and gives the approximate solution in the form of series, this series in closed form gives the corresponding exact solution of the given problem.

5. Conclution

In this paper, a new technique is presented for solving a class of nonlinear Catputo time- fractional partial differential equations, in particular nonlinear Caputo time-fractional wave-like equations with variable coefficients. This technique is a combination of two powerful methods, natural transform method and new iterative method, called new iterative natural transform method (NINTM). The proposed method provides the solution in a series form that converges rapidly to the exact solution if it exists. The obtained solutions are compared to the exact solutions and show the high accuracy of the proposed method. The results show that the NINTM is an appropriate and efficient method for solving nonlinear fractional partial differential equation.

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