Jordan Journal of Mathematics and Statistics (JJMS) 13(4), 2020, pp 643 - 657

BOUNDS FOR THE COEFFICIENTS OF A CLASS OF ANALYTIC FUNCTIONS ASSOCIATED WITH CONIC DOMAINS

K. AMARENDER REDDY [(1)], K. R. KARTHIKEYAN [(2)] AND G. MURUGUSUNDARAMOORTHY[(3)]

Abstract. By using the quantum differentials, we introduce presumably a new class of analytic functions associated with a conic region and obtained sufficient conditions, bounds for the coefficients of the defined function class. Furthermore, we discussed some applications for our main results.

1. Introduction

Denote by $\mathcal{A}$ the class of functions having a Taylor series expansion of the form

$$f(z) = z + \sum_{n=2}^{\infty} c_n z^n, \quad (z \in \mathcal{U} = \{z : |z| < 1\}) . \tag{1.1}$$

We let $\mathcal{S}^*$, $\mathcal{C}$ and $\mathcal{K}$ to denote the well known classes of starlike, convex and close-to-convex function respectively. Goodman[3] provides the study of various subclasses of univalent functions. In 1991, Goodman [4] introduced the geometrically defined class of uniformly convex functions (UCV) and gave the following two variable characterization

$$\mathcal{R}\left(1 + \frac{(z-\zeta)f''(z)}{f'(z)}\right) > 0, \quad (f \in \mathcal{A})$$

2000 *Mathematics Subject Classification.* 30C45.

Key words and phrases. Janowski starlike functions, conic region, sufficient conditions, coefficient inequalities, q-calculus, Sălăgean differential operator.

Received: May 10, 2019 Accepted: Jan. 2, 2020 .

for every pair (z, ζ) in the polydisk $\mathcal{U} \times \mathcal{U}$. But the two-variable characterization could not led to any sharp estimates for the class UCV. The best known bounds on the coefficients for the family UCV are $\mid a_n \mid \leq 1/n(n = 2, 3, \ldots)$. In order to find an extremal function for a family of uniformly convex functions, Ronning [11] and Ma and Minda [8] independently gave a one-variable characterizations for functions in UCV which is closely related to Goodman's characterization. There is a massive literature available on the study of uniformly convex functions.

Let $f(z)$ and $g(z)$ be analytic in $\mathcal{U}$. Then we say that the function $f(z)$ is subordinate to $g(z)$ in $\mathcal{U}$, if there exists an analytic function $w(z)$ in $\mathcal{U}$ such that $|w(z)| < |z|$ and $f(z) = g(w(z))$, denoted by $f(z) \prec g(z)$. If $g(z)$ is univalent in $\mathcal{U}$, then the subordination is equivalent to $f(0) = g(0)$ and $f(\mathcal{U}) \subset g(\mathcal{U})$. Denote by $\mathcal{P}$ the class of univalent functions with positive real part which satisfies $\mathcal{R}\{p(z)\} > 0, \quad p(0) = 1$. Janwoski [5] introduced a generalized class $P[A, B]$ of $p \in \mathcal{P}$ satisfying the following subordination condition:

$$p \in \mathcal{P} \Leftrightarrow p(z) \prec \frac{1 + Az}{1 + Bz}, \qquad -1 \leq B < A \leq 1.$$

In [6], Kanas and Wiśniowska, extended the study to conic domain as defined below:

If $P_k(z) = Q_1(k)z + Q_2(k)z^2 + \cdots$, $z \in \mathcal{U}$, then it was shown in that for, we have

$$Q_1 := Q_1(k) = \begin{cases} \frac{2A^2}{1-k^2}, & 0 \leq k < 1 \\ \frac{8}{\pi^2}, & k = 1 \\ \frac{\pi^2}{4(k^2-1)\sqrt{t}(1+t)R^2(t)}, & k > 1. \end{cases} \tag{1.2}$$

with $A = \frac{2}{\pi} \arccos t$. Motivated by the study of Janowski class with a equally popular $k - UCV$, Noor and Malik [10] introduced the following class $k - P[A, B]$.

Definition 1.1. [10] A function $p(z)$ is said to be in the class $k-P[A,\,B]$, if and only if,

$$p(z) \prec \frac{(A+1)p_k(z)-(A-1)}{(B+1)p_k(z)-(B-1)}, \qquad (k \geq 0,\, -1 \leq B < A \leq 1) \tag{1.3}$$

where $p_k(z)$ is defined by

$$p_k(z) = \begin{cases} \frac{1+z}{1-z}, & k=0 \\ 1+\frac{2}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2, & k=1 \\ 1+\frac{2}{1-k^2}\sinh^2\left[\left(\frac{2}{\pi}\arccos k\right)\arctan h\sqrt{z}\right], & 0<k<1 \\ 1+\frac{1}{k^2-1}\sin\left(\frac{\pi}{2R(t)}\int_0^{\frac{u(z)}{\sqrt{t}}}\frac{1}{\sqrt{1-x^2}\sqrt{1-(tx)^2}}dx\right)+\frac{1}{k^2-1}, & k>1. \end{cases} \tag{1.4}$$

where $u(z) = \frac{z-\sqrt{t}}{1-\sqrt{tz}}$, $t \in (0,\,1)$ and z is chosen such that $k = \cosh\left(\frac{\pi R'(t)}{4R(t)}\right)$, $R(t)$ is Legendre's complete elliptic integral of the first kind and $R'(t)$ is complementary integral of $R(t)$.

Geometrically, the function $p(z) \in k-P[A,\,B]$ takes all values from the domain $\Omega_k[A,\,B]$, $1 \leq B < A \leq 1$, $k \geq 0$ which is defined as

$$\Omega_k[A,\,B] = \left\{w : \mathcal{R}\left(\frac{(B-1)w(z)-(A-1)}{(B+1)w(z)-(A+1)}\right) > k\left|\frac{(B-1)w(z)-(A-1)}{(B+1)w(z)-(A+1)}-1\right|\right\}. \tag{1.5}$$

Further, they extended the study by defining $k-UCV[A,\,B]$ and $k-ST[A,\,B]$ which was obtained by replacing $w(z)$ in (1.5) as $\frac{zf'(z)}{f(z)}$ and $\frac{(zf'(z))'}{f'(z)}$ respectively.

Now we give a very brief introduction on q-calculus and the notations which are required for our study.

Quantum calculus popularly called as q-calculus is based on the idea of finite difference re-scaling. The difference of quantum differentials from the ordinary ones is that notion of limit is removed in q-calculus, that is q-derivative is merely a ratio which is given by

$$D_q f(z) = \frac{f(qz) - f(z)}{(q-1)z}.$$

Notice that as limit $q \to 1^-$, $D_q f(z) = f'(z)$. q-calculus has numerous applications in variety of disciplines such as theory of special functions, operator theory, quantum-mechanics, relativity etc. Notations and symbols play an very important role in the study of q-calculus. Throughout this paper, we let

$$[n]_q = \sum_{k=1}^{n} q^{k-1}, \quad [0]_q = 0, \quad (0 < q < 1)$$

and the q-shifted factorial by

$$(a;q)_n = \begin{cases} 1, & n = 0 \\ (1-a)(1-aq)\ldots(1-aq^{n-1}), & n = 1, 2, \ldots. \end{cases}$$

The q-hypergeometric series was developed by Heine as a generalization of the hypergeometric series

$$ {}_2F_1[a, b;\, c|q, z] = \sum_{n=0}^{\infty} \frac{(a;q)_n (b;q)_n}{(q;q)_n (c;q)_n} z^n. \tag{1.6}$$

Generalizing the Heine's series, we define ${}_r\phi_s$ the basic hypergeometric series by

$$ {}_r\phi_s\,(a_1,\, a_2,\, \ldots,\, a_r;\, b_1,\, b_2,\, \ldots,\, b_s; q,\, z) = \sum_{n=0}^{\infty} \frac{(a_1;q)_n (a_2;q)_n \ldots (a_r;q)_n}{(q;q)_n (b_1;q)_n \ldots (b_s;q)_n} \left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} z^n, \tag{1.7}$$

with $\binom{n}{2} = \frac{n(n-1)}{2}$, where $q \neq 0$ when $r > s+1$. In (1.6) and (1.7), it is assumed that the parameters $b_1,\, b_2,\, \ldots,\, b_s$ are such that the denominators factors in the terms of the series are never zero.

For complex parameters $a_1,\, \ldots,\, a_r$ and $b_1, \ldots,\, b_s$, $\left(\beta_j \in \mathbb{C} \setminus \mathbb{Z}_0^-;\, \mathbb{Z}_0^- = 0, -1,\, -2,\, \ldots;\right.$ $j = 1,\, \ldots,\, s)$, we define the generalized q-hypergeometric function

${}_q\Psi_s(a_1,\,\ldots,\,a_q;\,b_1,\,\ldots,\,b_s;\,q,z)$ by

$$ {}_q\Psi_s(a_1,\,a_2,\,\ldots,\,a_q;\,b_1,\,b_2,\,\ldots,\,b_s;\,q,z) = \sum_{n=0}^{\infty} \frac{(a_1;q)_n(a_2;q)_n\ldots(a_r;q)_n}{(q;q)_n(b_1;q)_n\ldots(b_s;q)_n} z^n, \tag{1.8} $$

$$ (r = s+1;\, r,\, s \in \mathcal{N}_0 = \mathcal{N}\cup\{0\};\, z \in \mathcal{U}), $$

where $\mathcal{N}$ denotes the set of positive integers. By using the ratio test, we should note that, if $|q| < 1$, the series (1.8) converges absolutely for $|z| < 1$ and $r = s+1$. For more mathematical background of these functions, one may refer to [2].

Corresponding to a function $\mathcal{G}_{r,\,s}(a_i,\,b_j;\,q,\,z)\,(i = 1,\,2,\,\ldots,\,r;\,j = 1,\,2,\,\ldots,\,s)$ defined by

$$ \mathcal{G}_{r,\,s}(a_i,\,b_j;\,q,\,z) := z\,{}_q\Psi_s(a_1,\,a_2,\,\ldots,\,a_r;\,b_1,\,b_2,\,\ldots,\,b_s;\,q,z). \tag{1.9} $$

Motivated by [9, 13], we define a q-differential operator $\mathcal{J}_\lambda^m(a_1,b_1;\,q,\,z)f\,:\mathcal{U}\longrightarrow\mathcal{U}$ as follows.

$$ \mathcal{J}_\lambda^0(a_1,b_1;\,q,\,z)f(z) = f(z)*\mathcal{G}_{r,\,s}(a_i,\,b_j;\,q,\,z). $$

(1.10)

$$ \mathcal{J}_\lambda^1(a_1,b_1;\,q,\,z)f(z) = (1-\lambda)(f(z)*\mathcal{G}_{r,\,s}(a_i,\,b_j;\,q,\,z)) + \lambda\, zD_q(f(z)*\mathcal{G}_{r,\,s}(a_i,\,b_j;\,q,\,z)). $$

$$ \mathcal{J}_\lambda^m(a_1,b_1;\,q,\,z)f(z) = \mathcal{J}_\lambda^1(\mathcal{J}_\lambda^{m-1}(a_1,b_1;\,q,\,z)f(z)). \tag{1.11} $$

If $f\in\mathcal{A}$, then from (1.10) and (1.11) we may easily deduce that

$$ \mathcal{J}_\lambda^m(a_1,b_1;\,q,\,z)f = z + \sum_{n=2}^{\infty}\left[1-\lambda+[n]_q\lambda\right]^m \gamma_n c_n z^n, \tag{1.12} $$

$$ (m\in N_0 = N\cup\{0\} \text{ and } \lambda\geq 0)\,, $$

where

$$ \gamma_n = \frac{(a_1;q)_{n-1}(a_2;q)_{n-1}\ldots(a_r;q)_{n-1}}{(q;q)_{n-1}(b_1;q)_{n-1}\ldots(b_s;q)_{n-1}}, \quad (|q|<1)\,. $$

Remark 1.1. *In this remark we list some special cases of the operator* $\mathcal{J}_\lambda^m(a_1,b_1;\,q,\,z)f$.

1. *For a choice of the parameter* $m = 0$*, the operator* $\mathcal{J}_{\lambda}^{0}(\alpha_1, \beta_1)f(z)$ *reduces to the q-analogue of Dziok- Srivastava operator* [1].
2. *For* $a_i = q^{\alpha_i}, b_j = q^{\beta_j}, \alpha_i, \beta_j \in \mathbb{C}, \beta_j \neq 0, 1, 2, \ldots, (i = 1, \ldots, r, j = 1, \ldots, s)$ *and* $q \to 1^-$*, we get the operator defined by Selvaraj and Karthikeyan* [13].
3. *For* $r = 2,\ s = 1;\ a_1 = b_1,\ a_2 = q,$ *and* $\lambda = 1$*, we get the q- analogue of the well known Sălăgean operator (see* [7, 12]*).*

Also many (well known and new) integral and differential operators can be obtained by specializing the parameters (see [1] *and* [13]*) and references cited therein.*

Definition 1.2. The function $f(z) \in \mathcal{A}$ is said to be in the class $k{-}\mathcal{ST}_q^m(a_1, b_1;\ \lambda, A, B), k \geq 0, -1 \leq B < A \leq 1$, if and only if

$$\mathcal{R}\left(\frac{(B-1)G_{m,\lambda}(a_1, b_1; q, z) - (A-1)}{(B+1)G_{m,\lambda}(a_1, b_1; q, z) - (A+1)}\right) > k\left|\left(\frac{(B-1)G_{m,\lambda}(a_1, b_1; q, z) - (A-1)}{(B+1)G_{m,\lambda}(a_1, b_1; q, z) - (A+1)}\right) - 1\right|$$

where

$$G_{m,\lambda}(a_1, b_1; q, z) = \frac{J_{\lambda}^{m+1}(a_1, b_1; q, z)f}{J_{\lambda}^{m}(a_1, b_1; q, z)f},$$

or equivalently

$$G_{m,\lambda}(a_1, b_1; q, z) \in k - P[A, B].$$

Definition 1.3. The function $f(z) \in \mathcal{A}$ is said to be in the class $k{-}\mathcal{CV}_q^m(a_1, b_1;\ \lambda, A, B), k \geq 0, -1 \leq B < A \leq 1$, if and only if

$$\mathcal{R}\left(\frac{(B-1)H_{m,\lambda}(a_1, b_1; q, z) - (A-1)}{(B+1)H_{m,\lambda}(a_1, b_1; q, z) - (A+1)}\right) > k\left|\left(\frac{(B-1)H_{m,\lambda}(a_1, b_1; q, z) - (A-1)}{(B+1)H_{m,\lambda}(a_1, b_1; q, z) - (A+1)}\right) - 1\right|$$

where

$$H_{m,\lambda}(a_1, b_1; q, z) = \frac{zD_q(J_\lambda^{m+1}(a_1, b_1; q, z)f)}{(J_\lambda^{m+1}(a_1, b_1; q, z)f)},$$

or equivalently

$$H_{m,\lambda}(a_1, b_1; q, z) \in k - P[A, B].$$

It can be easily seen that

$$f(z) \in k - \mathcal{CV}_q^m(a_1, b_1;\, \lambda, A, B) \Leftrightarrow zD_q f(z) \in k - \mathcal{ST}_q^m(a_1, b_1;\, \lambda, A, B) \tag{1.13}$$

Lemma 1.1. *Let* $h(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$ *be subordinate to* $H(z) = 1 + \sum_{n=1}^{\infty} P_n z^n$. *If* $H(z)$ *is univalent in the* $\mathcal{U}$ *and* $H(\mathcal{U})$ *is convex, then*

$$| p_n | \leq | P_1 |,\ n \geq 1.$$

Lemma 1.2. *(see* [10] *) Let* $h(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \in k - \mathcal{P}[A, B]$, *then*

$$| p_n | \leq | Q_1(k, A, B) |,\ | Q_1(k, A, B) | = \frac{A - B}{2} | Q_1(k) |,$$

where $| Q_1(k) |$ *follows from (1.2).*

2. Main Results

Theorem 2.1. *The function* $f(z) \in \mathcal{A}$ *and of the form (1.1) is in the class* $k - \mathcal{ST}_q^m(a_1, b_1;\, \lambda, A, B)$, *if it satisfies the condition*

$$\sum_{n=2}^{\infty} [2(k+1)(\lambda q[n-1]_q) + | [1 - \lambda + [n]_q \lambda](B+1) - (A+1) |] [1 - \lambda + [n]_q \lambda]^m | \gamma_n || c_n | \leq (A - B), \tag{2.1}$$

where $k \geq 0, -1 \leq B < A \leq 1$.

Proof. Assuming that (2.1) holds, then it suffices to show that

$$k \left| \left(\frac{(B-1)G_{m,\lambda}(a_1, b_1; q, z) - (A-1)}{(B+1)G_{m,\lambda}(a_1, b_1; q, z) - (A+1)} \right) - 1 \right| - \mathcal{R} \left(\frac{(B-1)G_{m,\lambda}(a_1, b_1; q, z) - (A-1)}{(B+1)G_{m,\lambda}(a_1, b_1; q, z) - (A+1)} - 1 \right) < 1.$$

We have

$$k\left|\left(\frac{(B-1)G_{m,\lambda}(a_1,b_1;q,z)-(A-1)}{(B+1)G_{m,\lambda}(a_1,b_1;q,z)-(A+1)}\right)-1\right|-\mathcal{R}\left(\frac{(B-1)G_{m,\lambda}(a_1,b_1;q,z)-(A-1)}{(B+1)G_{m,\lambda}(a_1,b_1;q,z)-(A+1)}-1\right)$$
$$\leq (k+1)\left|\frac{(B-1)G_{m,\lambda}(a_1,b_1;q,z)-(A-1)}{(B+1)G_{m,\lambda}(a_1,b_1;q,z)-(A+1)}-1\right|$$
$$=(k+1)\left|\frac{(B-1)J_\lambda^{m+1}(a_1,b_1;q,z)f-(A-1)J_\lambda^{m}(a_1,b_1;q,z)f}{(B+1)J_\lambda^{m+1}(a_1,b_1;q,z)f-(A+1)J_\lambda^{m}(a_1,b_1;q,z)f}-1\right|$$
$$=2(k+1)\left|\frac{J_\lambda^{m}(a_1,b_1;q,z)f-J_\lambda^{m+1}(a_1,b_1;q,z)f}{(B+1)J_\lambda^{m+1}(a_1,b_1;q,z)f-(A+1)J_\lambda^{m}(a_1,b_1;q,z)f}\right|$$

$$k\left|\left(\frac{(B-1)G_{m,\lambda}(a_1,b_1;q,z)-(A-1)}{(B+1)G_{m,\lambda}(a_1,b_1;q,z)-(A+1)}\right)-1\right|-\mathcal{R}\left(\frac{(B-1)G_{m,\lambda}(a_1,b_1;q,z)-(A-1)}{(B+1)G_{m,\lambda}(a_1,b_1;q,z)-(A+1)}-1\right)$$
$$\leq 2(k+1)\left|\frac{\sum_{n=2}^{\infty}[1-\lambda+[n]_q\lambda]^m\lambda[[n]_q-1]\gamma_n c_n z^n}{(B-A)z+\sum_{n=2}^{\infty}\left[(B+1)[1-\lambda+[n]_q\lambda]-(A+1)\right][1-\lambda+[n]_q\lambda]^m\gamma_n c_n z^n}\right|$$
$$\leq 2(k+1)\left[\frac{\sum_{n=2}^{\infty}[1-\lambda+[n]_q\lambda]^m\lambda[[n]_q-1]\mid\gamma_n\mid\mid c_n\mid}{(A-B)-\sum_{n=2}^{\infty}\left|(B+1)[1-\lambda+[n]_q\lambda]-(A+1)\right|[1-\lambda+[n]_q\lambda]^m\mid\gamma_n\mid\mid c_n\mid}\right]$$

The last expression is bounded above by 1 if

$$\sum_{n=2}^{\infty}[2(k+1)(\lambda q[n-1]_q)+\mid[1-\lambda+[n]_q\lambda](B+1)-(A+1)\mid] [1-\lambda+[n]_q\lambda]^m\mid\gamma_n\mid\mid c_n\mid\leq(A-B). \tag{2.2}$$

This completes the proof. □

If we let $r=2, s=1, a_1=b_1, a_2=q$ and $\lambda=1$ in Theorem 2.1, we get the following result which was obtained by see ([14]).

Corollary 2.1. *A function $f(z)\in\mathcal{A}$ is in the class $k-\mathcal{ST}_q^m(A,B)$, if it satisfies the condition*

$$\sum_{n=2}^{\infty}[2(k+1)(q[n-1]_q)+\mid[n]_q(B+1)-(A+1)\mid]\,[n]_q^m\mid c_n\mid\leq(A-B).$$

When $q \to 1^-, m = 0$, we have the following known result, proved by Noor and Malik in [10].

Corollary 2.2. *The function $f \in \mathcal{A}$ is in the class $k - \mathcal{ST}(A, B)$, if it satisfies the condition*

$$\sum_{n=2}^{\infty} [2(k+1)\lambda[n-1] + \mid [1-\lambda+n\lambda](B+1) - (A+1) \mid] \mid \gamma_n \mid\mid c_n \mid \leq (A-B). \tag{2.3}$$

Using the details of the proof (1.13) together with the result of Theorem 2.1, we have the following result. We omit the proof.

Theorem 2.2. *The function $f \in \mathcal{A}$, is in the class $k - \mathcal{CV}_q^m(a_1, b_1;\ \lambda, A, B)$, if it satisfies the condition*

$$\sum_{n=2}^{\infty} [2(k+1)(\lambda q[n-1]_q) + \mid [1-\lambda+[n]_q\lambda](B+1) - (A+1) \mid] \, [1-\lambda+[n]_q\lambda]^{m+1} \mid \gamma_n \mid\mid c_n \mid \leq (A-B), \tag{2.4}$$

where $k \geq 0$, $-1 \leq B < A \leq 1$.

Now we obtain the coefficient estimates of functions in the class $k - \mathcal{ST}_q^m(a_1, b_1;\ \lambda, A, B)$.

Theorem 2.3. *The function $f(z) \in k - \mathcal{ST}_q^m(a_1, b_1;\ \lambda, A, B)$, and is of the form (1.1), then*

$$\mid c_n \mid \leq \prod_{j=0}^{n-2} \left[\frac{\mid Q_1(k)(A-B) - 2\lambda q[j]_q \left[1-\lambda+[j+1]_q\lambda\right]^m \gamma_{j+1} B \mid}{2\lambda q[j+1]_q \left[1-\lambda+[j+2]_q\lambda\right]^m \mid \gamma_{j+2} \mid} \right], \ n \geq 2, \tag{2.5}$$

where $\mid Q_1(k) \mid$ is defined by (1.2).

Proof. By definition, for $f(z) \in k - \mathcal{ST}_q^m(a_1, b_1;\ \lambda, A, B)$, we have

$$\frac{J_\lambda^{m+1}(a_1, b_1; q, z) f}{J_\lambda^m(a_1, b_1; q, z) f} = p(z), \tag{2.6}$$

where

$$p(z) \in k - P[A, B].$$

Now from (2.6), we have

$$J_{\lambda}^{m+1}(a_1, b_1; q, z)f = (J_{\lambda}^{m}(a_1, b_1; q, z)f)\, p(z).$$

$$z+\sum_{n=2}^{\infty}[1-\lambda+[n]_q\lambda]^{m+1}\ \gamma_n c_n z^n = \left(1+\sum_{n=1}^{\infty}p_n z^n\right)\left(z+\sum_{n=2}^{\infty}[1-\lambda+[n]_q\lambda]^{m}\ \gamma_n c_n z^n\right),$$

$$z+\sum_{n=2}^{\infty}[1-\lambda+[n]_q\lambda]^{m+1}\ \gamma_n c_n z^n = \left(1+\sum_{n=1}^{\infty}p_n z^n\right)\left(\sum_{n=1}^{\infty}[1-\lambda+[n]_q\lambda]^{m}\ \gamma_n c_n z^n\right),$$

$$\begin{aligned} z+\sum_{n=2}^{\infty}[1-\lambda+[n]_q\lambda]^{m+1}\ \gamma_n c_n z^n &= \sum_{n=1}^{\infty}[1-\lambda+[n]_q\lambda]^{m}\ \gamma_n c_n z^n \\ &\quad + (\sum_{n=1}^{\infty}[1-\lambda+[n]_q\lambda]^{m}\ \gamma_n c_n z^n)(\sum_{n=1}^{\infty}p_n z^n), \end{aligned}$$

$$\sum_{n=2}^{\infty}\lambda[\,[n]_q-1]\,[1-\lambda+[n]_q\lambda]^{m}\ \gamma_n c_n z^n = \left(\sum_{n=1}^{\infty}[1-\lambda+[n]_q\lambda]^{m}\ \gamma_n c_n z^n\right)\left(\sum_{n=1}^{\infty}p_n z^n\right),$$

(2.7)
$$\sum_{n=2}^{\infty}\lambda q[n-1]_q\,[1-\lambda+[n]_q\lambda]^{m}\ \gamma_n c_n z^n = \left(\sum_{n=1}^{\infty}[1-\lambda+[n]_q\lambda]^{m}\ \gamma_n c_n z^n\right)\left(\sum_{n=1}^{\infty}p_n z^n\right).$$

By using Cauchy product formula on right hand side of (2.7), we have

(2.8)
$$\sum_{n=2}^{\infty}\lambda q[n-1]_q\,[1-\lambda+[n]_q\lambda]^{m}\ \gamma_n c_n z^n = \sum_{n=2}^{\infty}\left[\sum_{j=1}^{n-1}[1-\lambda+[j]_q\lambda]^{m}\ \gamma_j c_j p_{n-j}\right]z^n.$$

Equating coefficients of z^n on both sides of (2.8), we have

$$\lambda q[n-1]_q\left[1-\lambda+[n]_q\lambda\right]^m \gamma_n c_n=\sum_{j=1}^{n-1}\left[1-\lambda+[j]_q\lambda\right]^m \gamma_j c_j p_{n-j},$$

$$([1-\lambda+[1]_q\lambda]^m=1, c_1=1).$$

This implies that

$$| c_n |\leq \frac{1}{\lambda q[n-1]_q\left[1-\lambda+[n]_q\lambda\right]^m |\gamma_n|}\sum_{j=1}^{n-1}\left[1-\lambda+[j]_q\lambda\right]^m |\gamma_j| \mid c_j \mid\mid p_{n-j} \mid .$$

Using Lemma 1.2, we have

$$| c_n |\leq \frac{| Q_1(k) | (A-B)}{2\lambda q[n-1]_q\left[1-\lambda+[n]_q\lambda\right]^m \mid \gamma_n \mid}\sum_{j=1}^{n-1}\left[1-\lambda+[j]_q\lambda\right]^m |\gamma_j| \mid c_j \mid . \tag{2.9}$$

Now we prove that

$$\frac{| Q_1(k) | (A-B)}{2\lambda q[n-1]_q\left[1-\lambda+[n]_q\lambda\right]^m \mid \gamma_n \mid}\sum_{j=1}^{n-1}\left[1-\lambda+[j]_q\lambda\right]^m \mid \gamma_j \mid\mid c_j \mid$$
$$\leq \prod_{j=1}^{n-1}\left[\frac{| Q_1(k)(A-B)-2\lambda q[j-1]_q\left[1-\lambda+[j]_q\lambda\right]^m \mid \gamma_j \mid B \mid}{2\lambda q[j]_q\left[1-\lambda+[j+1]_q\lambda\right]^m \mid \gamma_{j+1} \mid}\right]$$
$$\leq \prod_{j=0}^{n-2}\left[\frac{| Q_1(k)(A-B)-2\lambda q[j]_q\left[1-\lambda+[j+1]_q\lambda\right]^m \mid \gamma_{j+1} \mid A \mid}{2\lambda q[j+1]_q\left[1-\lambda+[j+2]_q\lambda\right]^m \mid \gamma_{j+2} \mid}\right].$$

For this, we use the induction method. For $n=2$ from (2.9), we have

$$| c_2 |\leq \frac{| Q_1(k) | (A-B)}{2\lambda q\left[1-\lambda+[2]_q\lambda\right]^m \mid \gamma_2 \mid}.$$

From (2.5), we have

$$| c_2 |\leq \frac{| Q_1(k) | (A-B)}{2\lambda q\left[1-\lambda+[2]_q\lambda\right]^m \mid \gamma_2 \mid}.$$

For $n = 3$ from (2.9), we have

$$\begin{aligned}
\mid c_3 \mid \; &\leq \frac{\mid Q_1(k) \mid (A-B)}{2\lambda q[2]_q \left[1-\lambda+[3]_q\lambda\right]^m \mid \gamma_3 \mid}\{1+\left[1-\lambda+[2]_q\lambda\right]^m \mid \gamma_2 \mid\mid c_2 \mid\} \\
&\leq \frac{\mid Q_1(k) \mid (A-B)}{2\lambda q[2]_q \left[1-\lambda+[3]_q\lambda\right]^m \mid \gamma_3 \mid}\{1+\frac{\mid Q_1(k) \mid (A-B)}{2\lambda q}\}.
\end{aligned}$$

From (2.5), we have

$$\begin{aligned}
\mid c_3 \mid \; &\leq \frac{\mid Q_1(k) \mid (A-B)}{2\lambda q\left[1-\lambda+[3]_q\lambda\right]^m \mid \gamma_3 \mid}\left[\frac{\mid Q_1(k)(A-B)-2\lambda q\left[1-\lambda+[2]_q\lambda\right]^m \gamma_2 B \mid}{2\lambda q[2]_q\left[1-\lambda+[2]_q\lambda\right]^m \mid \gamma_2 \mid}\right] \\
&\leq \frac{\mid Q_1(k) \mid (A-B)}{2\lambda q\left[1-\lambda+[2]_q\lambda\right]^m \mid \gamma_2 \mid}\left[\frac{\mid Q_1(k) \mid (A-B)+2\lambda q\left[1-\lambda+[2]_q\lambda\right]^m \mid \gamma_2 \mid\mid B \mid}{2\lambda q[2]_q\left[1-\lambda+[3]_q\lambda\right]^m \mid \gamma_3 \mid}\right] \\
&\leq \frac{\mid Q_1(k) \mid (A-B)}{2\lambda q[2]_q\left[1-\lambda+[3]_q\lambda\right]^m \mid \gamma_3 \mid}\left[1+\frac{\mid Q_1(k) \mid (A-B)}{2\lambda q\left[1-\lambda+[2]_q\lambda\right]^m \mid \gamma_2 \mid}\right].
\end{aligned}$$

Let the hypothesis be true for $n = t$. from (2.9), we have

$$\mid c_t \mid \leq \frac{\mid Q_1(k) \mid (A-B)}{2\lambda q[t-1]_q\left[1-\lambda+[t]_q\lambda\right]^m \mid \gamma_t \mid}\sum_{j=1}^{t-1}\left[1-\lambda+[j]_q\lambda\right]^m \mid \gamma_j \mid\mid c_j \mid, \tag{2.10}$$

$$(\left[1-\lambda+[1]_q\lambda\right]^m = 1, c_1 = 1).$$

From (2.5), we have

$$\begin{aligned}
\mid c_t \mid &\leq \prod_{j=0}^{t-2}\left[\frac{\mid Q_1(k)(A-B)-2\lambda q[j]_q\left[1-\lambda+[j+1]_q\lambda\right]^m \mid \gamma_{j+1} \mid B \mid}{2\lambda q[j+1]_q\left[1-\lambda+[j+2]_q\lambda\right]^m \mid \gamma_{j+2} \mid}\right] \\
&\leq \prod_{j=0}^{t-2}\left[\frac{\mid Q_1(k) \mid (A-B)+2\lambda q[j]_q\left[1-\lambda+[j+1]_q\lambda\right]^m \gamma_{j+1} \mid B \mid}{2\lambda q[j+1]_q\left[1-\lambda+[j+2]_q\lambda\right]^m \mid \gamma_{j+2} \mid}\right].
\end{aligned}$$

By the induction hypothesis, we have

$$\begin{aligned}
&\frac{\mid Q_1(k) \mid (A-B)}{2\lambda q[t-1]_q\left[1-\lambda+[t]_q\lambda\right]^m \mid \gamma_t \mid}\sum_{j=1}^{t-1}\left[1-\lambda+[j]_q\lambda\right]^m \mid \gamma_j \mid\mid c_j \mid \\
&\leq \prod_{j=0}^{t-2}\left[\frac{\mid Q_1(k)(A-B)+2\lambda q[j]_q\left[1-\lambda+[j+1]_q\lambda\right]^m \mid \gamma_{j+1} \mid\mid}{2\lambda q[j+1]_q\left[1-\lambda+[j+2]_q\lambda\right]^m \mid \gamma_{j+2} \mid}\right].
\end{aligned}$$

Multiplying both sides by

$$\left[\frac{\mid Q_1(k)(A-B)+2\lambda q[t-1]_q\left[1-\lambda+[t]_q\lambda\right]^m\ \gamma_t\mid}{2\lambda q[t+1]_q\left[1-\lambda+[t+2]_q\lambda\right]^m\ \mid\gamma_{t+2}\mid}\right],$$

we have

$$\begin{aligned}&\left[\frac{\mid Q_1(k)(A-B)+2\lambda q[t-1]_q\left[1-\lambda+[t]_q\lambda\right]^m\ \gamma_t\mid}{2\lambda q[t+1]_q\left[1-\lambda+[t+2]_q\lambda\right]^m\ \mid\gamma_{t+2}\mid}\right]\times\\&\prod_{j=0}^{t-2}\left[\frac{\mid Q_1(k)(A-B)+2\lambda q[j]_q\left[1-\lambda+[j+1]_q\lambda\right]^m\ \gamma_{j+1}\mid}{2\lambda q[j+1]_q\left[1-\lambda+[j+2]_q\lambda\right]^m\ \mid\gamma_{j+2}\mid}\right]\\&\geq\left[\frac{\mid Q_1(k)(A-B)+2\lambda q[t-1]_q\left[1-\lambda+[t]_q\lambda\right]^m\ \gamma_t\mid}{2\lambda q[t+1]_q\left[1-\lambda+[t+2]_q\lambda\right]^m\ \mid\gamma_{t+2}\mid}\right]\times\\&\frac{\mid Q_1(k)\mid(A-B)}{2\lambda q[t-1]_q\left[1-\lambda+[t]_q\lambda\right]^m\ \mid\gamma_t\mid}\sum_{j=1}^{t-1}\left[1-\lambda+[j]_q\lambda\right]^m\mid\gamma_j\mid\mid c_j\mid.\end{aligned}$$

$$\begin{aligned}&\prod_{j=0}^{t-2}\left[\frac{\mid Q_1(k)(A-B)+2\lambda q[j]_q\left[1-\lambda+[j+1]_q\lambda\right]^m\ \mid\gamma_{j+1}\mid\mid}{2\lambda q[j+1]_q\left[1-\lambda+[j+2]_q\lambda\right]^m\ \mid\gamma_{j+2}\mid}\right]\\&\geq\frac{\mid Q_1(k)(A-B)\mid}{2\lambda q[t+1]_q\left[1-\lambda+[t+2]_q\lambda\right]^m\ \mid\gamma_{t+2}\mid}\times\\&\left[\frac{\mid Q_1(k)\mid(A-B)}{2\lambda q[t-1]_q\left[1-\lambda+[t]_q\lambda\right]^m\ \mid\gamma_t\mid}\sum_{j=1}^{t-1}\left[1-\lambda+[j]_q\lambda\right]^m\mid\gamma_j\mid\mid c_j\mid\right.\\&\qquad\left.+\sum_{j=1}^{t-1}\left[1-\lambda+[j]_q\lambda\right]^m\mid\gamma_j\mid\mid c_j\mid\right].\end{aligned}$$

$$\begin{aligned}&\prod_{j=0}^{t-2}\left[\frac{\mid Q_1(k)(A-B)+2\lambda q[j]_q\left[1-\lambda+[j+1]_q\lambda\right]^m\ \gamma_{j+1}\mid}{2\lambda q[j+1]_q\left[1-\lambda+[j+2]_q\lambda\right]^m\ \mid\gamma_{j+2}\mid}\right]\\&\geq\frac{\mid Q_1(k)(A-B)\mid}{2\lambda q[t+1]_q\left[1-\lambda+[t+2]_q\lambda\right]^m\ \mid\gamma_{t+2}\mid}\left[\mid c_t\mid+\sum_{j=1}^{t-1}\left[1-\lambda+[j]_q\lambda\right]^m\mid\gamma_j\mid\mid c_j\mid\right]\\&\geq\frac{\mid Q_1(k)(A-B)\mid}{2\lambda q[t+1]_q\left[1-\lambda+[t+2]_q\lambda\right]^m\ \mid\gamma_{t+2}\mid}\left[\sum_{j=1}^{t}\left[1-\lambda+[j]_q\lambda\right]^m\mid\gamma_j\mid\mid c_j\mid\right].\end{aligned}$$

That is

$$\frac{| Q_1(k)(A-B) |}{2\lambda q[t+1]_q \left[1-\lambda+[t+2]_q\lambda\right]^m \ | \gamma_{t+2} |} \left[\sum_{j=1}^{t} \left[1-\lambda+[j]_q\lambda\right]^m | \gamma_j || c_j |\right]$$
$$\leq \prod_{j=0}^{t-2} \left[\frac{| Q_1(k)(A-B)+2\lambda q[j]_q \left[1-\lambda+[j+1]_q\lambda\right]^m \gamma_{j+1} |}{2\lambda q[j+1]_q \left[1-\lambda+[j+2]_q\lambda\right]^m \ | \gamma_{j+2} |}\right].$$

which shows the inequality is true for $n = t+1$. Hence the required result.

□

Remark 2.1. *For appropriate choice of parameters, we get several well-known and new results obtained by various authors.*

Acknowledgement

The authors thank the referees for their valuable comments and helpful suggestions.

References

[1] M. Darus, A new look at q-hypergeometric functions, TWMS J. Appl. Eng. Math. **4** (2014), no. 1, 16–19.

[2] G. Gasper and M. Rahman, *Basic hypergeometric series*, Encyclopedia of Mathematics and its Applications, 35, Cambridge University Press, Cambridge, 1990.

[3] A. W. Goodman, *Univalent functions. Vol. II*, Mariner Publishing Co., Inc., Tampa, FL, 1983.

[4] A. W. Goodman, On uniformly convex functions, Ann. Polon. Math. **56** (1991), no. 1, 87–92.

[5] W. Janowski, Some extremal problems for certain families of analytic functions. I, Ann. Polon. Math. **28** (1973), 297–326.

[6] S. Kanas and A. Wiśniowska, Conic domains and starlike functions, Rev. Roumaine Math. Pures Appl. **45** (2000), no. 4, 647–657 (2001).

[7] K. R. Karthikeyan, M. Ibrahim and S. Srinivasan, Fractional class of analytic functions defined using q-differential operator, Aust. J. Math. Anal. Appl. **15** (2018), no. 1, Art. 9, 15 pp.

[8] W. Ma and D. Minda, Coefficient inequalities for strongly close-to-convex functions, J. Math. Anal. Appl. **205** (1997), no. 2, 537–553.

[9] G. Murugusundaramoorthy, C. Selvaraj and O. S. Babu, Subclasses of starlike functions associated with fractional q-calculus operators, J. Complex Anal. **2013**, Art. ID 572718, 8 pp.

[10] K. I. Noor and S. N. Malik, On coefficient inequalities of functions associated with conic domains, Comput. Math. Appl. **62** (2011), no. 5, 2209–2217.

[11] F. Rønning, Uniformly convex functions and a corresponding class of starlike functions, Proc. Amer. Math. Soc. **118** (1993), no. 1, 189–196.

[12] G. Ş. Sălăgean, Subclasses of univalent functions, in *Complex analysis—fifth Romanian-Finnish seminar, Part 1 (Bucharest, 1981)*, 362–372, Lecture Notes in Math., 1013, Springer, Berlin.

[13] C. Selvaraj and K. R. Karthikeyan, Differential sandwich theorems for certain subclasses of analytic functions, Math. Commun. **13** (2008), no. 2, 311–319.

[14] S. Yalçin, S. Hussain and S. Khan, Coefficient inequalities for some subclasses of analytic functions associated with conic domains involving q-calculus, Theory Appl. Math. Comput. Sci. **8** (2018), no. 1, 6–23.

(1,2) Department of Applied Mathematics and Science, College of Engineering, National University of Science and Technology, Muscat, Sultanate of Oman

Email address: (1) amarenderkommula@gmail.com

Email address: (2) kr_karthikeyan1979@yahoo.com

(3) Department of Mathematics, School of Advanced Sciences, Vellore Institute of Technology, Deemed to be University, Vellore, Tamilnadu,India

Email address, (Corresponding Author): gmsmoorthy@yahoo.com