Jordan Journal of Mathematics and Statistics (JJMS) 13(4), 2020, pp 487 - 503

ON CERTAIN CHROMATIC TOPOLOGICAL INDICES OF SOME MYCIELSKI GRAPHS

SMITHA ROSE (1) AND SUDEV NADUVATH (2)

Abstract. As a coloring analogue of different Zagreb indices, in the recent literature, the notion of chromatic Zagreb indices has been introduced and studied for some basic graph classes in trees. In this paper, we study the chromatic Zagreb indices and chromatic irregularity indices of some special classes of graphs called Mycielski graphs of paths and cycles.

1. Introduction

For the terms and definitions, which are not introduced in this paper, we refer to [7, 3, 4, 11]. Throughout our study, we consider $G = (V, E)$ as a finite, nontrivial, undirected, simple and connected graph.

A *topological index* of a graph G is a real number which is preserved under isomorphism (see [10]), which makes them vital in various fields of mathematical chemistry. As the earliest irregularity measurement introduced in the literature, the Zagreb indices, in the manipulation of the vertex degrees, are well studied for years. Investigating the effect of interchanging vertex degrees with minimal coloring, obeying additional coloring conditions, a whole new area for research is opened fresh. As a coloring analogue of different Zagreb indices, chromatic Zagreb indices have been introduced in [8].

2010 *Mathematics Subject Classification.* 05C15,05C38.

Key words and phrases. Mycielski graphs, chromatic Zagreb indices, irregularity indices.
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Received: March 31, 2019 Accepted: Jan. 2, 2020 .

If $\mathcal{C} = \{c_1, c_2, c_3, \dots, c_\ell\}$ is a set of colors (or labels or weights), then a *proper vertex coloring* of a graph G is an assignment $\varphi : V(G) \to \mathcal{C}$ of the vertices of G with colors in $\mathcal{C}$ such that adjacent vertices of G have different colors. The cardinality of the minimum set of colors allowing proper coloring of G is called the *chromatic number* of G and is denoted $\chi(G)$. The set of vertices of G which have the color c_i is called the *color class* of that color c_i in G. The cardinality of the color class of a color c_i is said to be the *strength* of that color in G and is denoted by $\theta(c_i)$. We can also define a function $\zeta : V(G) \to \{1, 2, 3, \dots, \ell\}$ such that $\zeta(v_i) = s \iff \varphi(v_i) = c_s, c_s \in \mathcal{C}$. Also, we denote the number of edges with end points having colors c_t and c_s by η_{ts}, where $t < s, 1 \leq t, s \leq \chi(G)$.

A vertex coloring consisting of the colors having minimum subscripts may be called a *minimum parameter coloring*. Unless stated otherwise, the colorings we consider in this paper are minimum parameter colorings. For any minimum parameter set of colors $\mathcal{C}$ with cardinality $|\mathcal{C}| = \ell$, a graph G has $\ell!$ minimum parameter colorings. We denote these colorings by $\varphi_t(G),\ 1 \leq t \leq \ell!$.

If we color the vertices of G in such a way that c_1 is assigned to the maximum possible number of vertices, then c_2 is assigned to the maximum possible number of remaining uncolored vertices and proceed in this manner until all vertices are colored, then such a coloring is called a φ^--*coloring* of G. In a similar manner, if c_ℓ is assigned to the maximum possible number of vertices, then $c_{\ell-1}$ is assigned to the maximum possible number of remaining uncolored vertices and proceed in this manner until all vertices are colored, then such a coloring is called a φ^+-*coloring* of G.

Analogous to the definitions of Zagreb and irregularity indices of graphs (see [1, 6, 12, 13]), the notions of different chromatic Zagreb indices and chromatic irregularity indices have been introduced in [8] as follows:

Definition 1.0.1. [8] Let G be a graph and let $\mathcal{C} = \{c_1, c_2, c_3, \dots, c_\ell\}$ be a proper coloring of G such that $\varphi(v_i) = c_s; 1 \leq i \leq n, 1 \leq s \leq \ell$. Then for $1 \leq t \leq \ell\,!$,

(i) The *first chromatic Zagreb index* of G, denoted by $M_1^{\varphi_t}(G)$, is defined as $M_1^{\varphi_t}(G) = \sum\limits_{i=1}^{n}(\zeta(v_i))^2 = \sum\limits_{j=1}^{\ell}\theta(c_j)\cdot j^2$.

(ii) The *second chromatic Zagreb index* of G, denoted by $M_2^{\varphi_t}(G)$, is defined as $M_2^{\varphi_t}(G) = \sum\limits_{i=1}^{n-1}\sum\limits_{j=2}^{n}(\zeta(v_i)\cdot\zeta(v_j)),\ v_iv_j \in E(G)$.

(iii) The *chromatic irregularity index* of G, denoted by $M_3^{\varphi_t}(G)$, is defined as $M_3^{\varphi_t}(G) = \sum\limits_{i=1}^{n-1}\sum\limits_{j=2}^{n}|\zeta(v_i)-\zeta(v_j)|,\ v_iv_j \in E(G)$.

In view of the above notions, the minimum and maximum chromatic Zagreb indices and the corresponding irregularity indices are defined in [8] as follows.

$$
\begin{aligned}
M_1^{\varphi^-}(G) &= \min\{M_1^{\varphi_t}(G) : 1 \leq t \leq \ell!\},\\
M_1^{\varphi^+}(G) &= \max\{M_1^{\varphi_t}(G) : 1 \leq t \leq \ell!\},\\
M_2^{\varphi^-}(G) &= \min\{M_2^{\varphi_t}(G) : 1 \leq t \leq \ell!\},\\
M_2^{\varphi^+}(G) &= \max\{M_2^{\varphi_t}(G) : 1 \leq t \leq \ell!\},\\
M_3^{\varphi^-}(G) &= \min\{M_3^{\varphi_t}(G) : 1 \leq t \leq \ell!\},\\
M_3^{\varphi^+}(G) &= \max\{M_3^{\varphi_t}(G) : 1 \leq t \leq \ell!\}.
\end{aligned}
$$

In a similar way, we define the chromatic *total irregularity* indices as follows:

Definition 1.0.2. The *chromatic total irregularity index* of a graph G corresponding to a proper coloring $\varphi : V(G) \to \mathcal{C} = \{c_1, c_2, \dots c_\ell\}$ is defined as

$$M_4^{\varphi_t}(G) = \frac{1}{2}\sum_{i=1}^{n-1}\sum_{j=2}^{n}|\zeta(v_i)-\zeta(v_j)|,\ v_i, v_j \in V(G).$$

Then, the minimal and maximal chromatic total irregularities are defined as

$$M_4^{\varphi^+}(G) = \min\{M_4^{\varphi_t}(G) : 1 \leq t \leq \ell!\}$$
$$M_4^{\varphi^-}(G) = \max\{M_4^{\varphi_t}(G) : 1 \leq t \leq \ell!\}.$$

This definition is the chromatic analogue of the total irregularity of graphs defined in [1].

2. Chromatic Zagreb Index of Mycielskian of a graph

Motivated by the studies mentioned above, we study the chromatic Zagreb indices and chromatic irregularity indices of Mycielskian of certain fundamental graph classes in the following discussion.

Definition 2.0.1. [9] Let G be a graph with the vertex set $V(G) = \{v_1, \ldots, v_n\}$. The *Mycielski graph* or the *Mycielskian* of a graph G, denoted by $\mu(G)$, is the graph with vertex set $V(\mu(G)) = \{v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n, w\}$ such that

(i) $v_i v_j \in E(\mu(G))$ if and only if$v_i v_j \in E(G)$,

(ii) $v_i u_j \in E(\mu(G))$ if and only if $v_i v_j \in E(G)$, and

(iii) $u_i w \in E(\mu(G))$ for all $i = 1, \ldots, n$.

An illustration to Mycielskian of a graph is provided in Figure 1.

Note that the set of vertices $\{u_1, u_2, \ldots, u_n\}$ is an independent set in $\mu(G)$ and can be denoted by U. The vertices u_i and v_i may be called the *twin vertices* and the vertex w may be called the *root vertex* of $\mu(G)$. For the ease of the notation, we represent the Mycielski graph of a graph G by $\breve{G}$.

First we find out the chromatic topological indices defined above for the Mycielskian of paths. Let $\breve{P}_n$ denote the Mycielskian of a path on n vertices. Then, $\breve{P}_n$ has $2n+1$ vertices and $4n-3$ edges. We introduce here the notation η_{ts} to denote the number of edges with end points t and s respectively, where $t < s, 1 \leq t, s \leq \chi(\breve{P}_n)$.

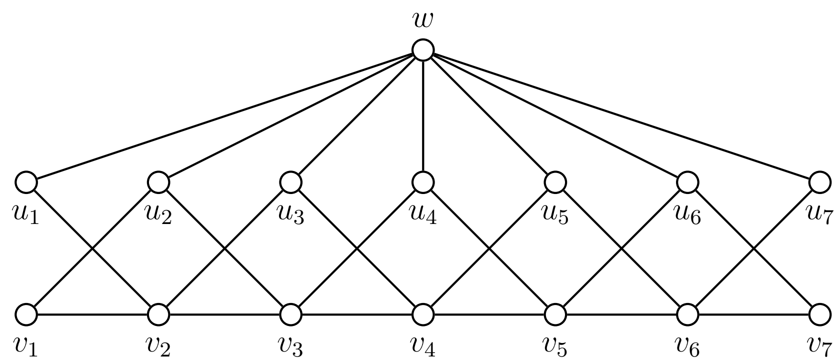

FIGURE 1. The Mycielski graph $\mu(P_7)$

Theorem 2.1. *For the Mycielskian of a path P_n, we have*

(i) $M_1^{\varphi^-}(\breve{P}_n) = \begin{cases} \frac{15n+3}{2}; & \text{if } n \text{ is odd} \\ \frac{15n+8}{2}; & \text{if } n \text{ is even;} \end{cases}$

(ii) $M_2^{\varphi^-}(\breve{P}_n) = 13n - 11;$

(iii) $M_3^{\varphi^-}(\breve{P}_n) = 5n - 4;$

(iv) $M_4^{\varphi^-}(\breve{P}_n) = \begin{cases} \frac{7n^2+4n-3}{8}; & \text{if } n \text{ is odd} \\ \frac{7n^2+6n}{8}; & \text{if } n \text{ is even.} \end{cases}$

Proof. Note that $\chi(\breve{P}_n) = 3$ and the chromatic topological indices can be calculated for all 3! minimum parameter colorings. To find $M_1^{\varphi^-}, M_2^{\varphi^-}$ and $M_3^{\varphi^-}$, we follow the coloring pattern as described below:

Since the maximum independent set of $\breve{P}_n$ is the set U, assign the color c_1 to all its vertices. Since the path P_n is bipartite, we can assign the color c_2 to $\lceil \frac{n}{2} \rceil$ vertices in V and the color c_3 to the remaining $\lfloor \frac{n}{2} \rfloor$ vertices in V. Since the root vertex w is adjacent to all vertices of U and w is not adjacent to any vertex in V, w can have the color c_2. For the Mycielskian of the path P_n, $t, s = 1, 2, 3$.

Part (i): In order to find $M_1^{\varphi^-}$ of $\breve{P}_n$, we first color the vertices as mentioned above and then proceed to consider the following cases.

Case-1: Let n be odd. Then, we have $\theta(c_1) = n$, $\theta(c_2) = \frac{n+3}{2}$ and $\theta(c_3) = \frac{n-1}{2}$. Now, by the definition of first chromatic Zagreb index, we have

$$M_1^{\varphi^-}(\breve{P}_n) = \sum_{i=1}^{3}(\theta(c_i))i^2 = \frac{15n+3}{2}.$$

Case-2: Let n be even. Then, we have $\theta(c_1) = n$, $\theta(c_2) = 1 + \frac{n}{2} = \frac{n+2}{2}$ and $\theta(c_3) = \frac{n}{2}$. Then, from the definition of first chromatic Zagreb index we have

$$M_1^{\varphi^-}(\breve{P}_n) = \sum_{i=1}^{3}(\theta(c_i))i^2 = \frac{15n+8}{2}.$$

Part (ii): We first color the vertices as per the instructions in the introductory part of this proof. Here, we observe that $\eta_{12} = 2n-1, \eta_{23} = \eta_{13} = n-1$. Now, the result follows from substitution and simplification as

$$M_2^{\varphi^-}(\breve{P}_n) = \sum_{1\leq t,s\leq \chi(\breve{P}_n)}^{t<s} ts\eta_{ts} = 13n - 11$$

Part (iii): To find the minimum irregularity measurement, we color the vertices using minimum parameter coloring. Now, $\eta_{12} = 2n-1$ edges and $\eta_{23} = n-1$ edges contributes the distance 1 to the total summation while $\eta_{13} = n-1$ contributes the distance 2. Then, we have

$$M_3^{\varphi^-}(\breve{P}_n) = \sum_{i=1}^{n-1}\sum_{j=2}^{n}|\zeta(v_i) - \zeta(v_j)| = 2n - 1 + 3(n-1) = 5n - 4.$$

Part (iv): After minimum parameter coloring, to find the total irregularity of $\breve{P}_n$, we consider all the possible vertex pairs from $\breve{P}_n$ and find their possible color distances. Half of their total sum will give us the total irregularity of $\breve{P}_n$. Since the vertex pairs with same colors do not contribute to the color distance, we omit such cases. The possibility of the vertex pairs which contribute to the color distance can be classified into three, namely, $\{1,3\}, \{1,2\}, \{2,3\}$ combinations. The $\{1,3\}$ combination has the color distance 2 while the other two combinations have the color distance 1. Each vertex of color c_1 pairs with $\theta(c_3)$ vertices of color c_3. So the total color distance for the $\{1,3\}$ combination is $2\theta(c_1)\theta(c_3)$. Similarly, the total color distance for the $\{1,2\}$ combination is $\theta(c_1)\theta(c_2)$ and for the $\{2,3\}$ combination is $\theta(c_2)\theta(c_3)$. Now we consider the following two cases:

Case-1: Let n be odd. Then, we have $\theta(c_1)=n$, $\theta(c_2)=\frac{n+3}{2}$ and $\theta(c_3)=\frac{n-1}{2}$. Now, by the definition of fourth chromatic Zagreb index, we have

$$\begin{aligned} M_4^{\varphi^-}(\breve{P}_n) &= \frac{1}{2}\sum_{u,v\in V(\breve{P}_n)} |\varphi(u)-\varphi(v)| \\ &= 2\theta(c_1)\theta(c_3)+\theta(c_1)\theta(c_2)+\theta(c_2)\theta(c_3) \\ &= \frac{7n^2+4n-3}{8}. \end{aligned}$$

Case-2: Let n be even. Then, we have $\theta(c_1)=n$, $\theta(c_2)=1+\frac{n}{2}=\frac{n+2}{2}$ and $\theta(c_3)=\frac{n}{2}$. The total summation will follow from the definition of fourth chromatic Zagreb index.

$$\begin{aligned} M_4^{\varphi^+}(\breve{P}_n) &= \frac{1}{2}\sum_{u,v\in V(\breve{P}_n)} |\varphi(u)-\varphi(v)| \\ &= 2\theta(c_1)\theta(c_3)+\theta(c_1)\theta(c_2)+\theta(c_2)\theta(c_3) \\ &= \frac{7n^2+6n}{8}. \end{aligned}$$

□

Theorem 2.2. *For the Mycielskian of a path P_n, we have*

(i) $M_1^{\varphi^+}(\breve{P}_n)=\begin{cases}\frac{23n+11}{2}; & \text{if } n \text{ is odd}\\ \frac{23n+8}{2}; & \text{if } n \text{ is even}\end{cases}$

(ii) $M_2^{\varphi^+}(\breve{P}_n)=17n-11$

(iii) $M_3^{\varphi^+}(\breve{P}_n)=6n-4$

(iv) $M_4^{\varphi^+}(\breve{P}_n)=\begin{cases}\frac{7n^2+12n-3}{8}; & \text{if } n \text{ is odd}\\ \frac{7n^2+10n}{8}; & \text{if } n \text{ is even}\end{cases}$

Proof. The chromatic Zagreb indices with any minimum parameter set of colors can be calculated in 3! ways, since the chromatic number of $\breve{P}_n$ is 3. All through this proof, for all the four parts, we first apply the minimum parameter coloring as follows, if not mentioned otherwise. We color the vertices of U with c_3, c_2 to $\lceil \frac{n}{2} \rceil$ vertices in V and the color c_1 to $\lfloor \frac{n}{2} \rfloor$ vertices of V. The root vertex is colored with c_2. As in Theorem 2.1, let $\theta(c_i)$ denotes the cardinality of the color class of c_i and η_{ts} denotes the number of edges with end points t, s where $t < s, 1 \leq t, s \leq \chi(\breve{P}_n)$. For $t, s = 1, 2, 3$, we have

Part (i): In the case of the maximum value of chromatic Zagreb index, we assign a maximum color index for the maximum independence set, which adds the maximum value to the sum of the squares. Here, we consider the following two cases:

Case-1: Let n be odd. Then, we have $\theta(c_1) = \frac{n-1}{2}$, $\theta(c_2) = \frac{n+3}{2}$ and $\theta(c_3) = n$. Now, by the definition of first chromatic Zagreb index, we have

$$M_1^{\varphi^+}(\breve{P}_n) = \sum_{i=1}^{3}(\theta(c_i))i^2 = \frac{23n + 11}{2}.$$

Case-2: Let n be even. Then, we have $\theta(c_1) = \frac{n}{2}$, $\theta(c_2) = 1 + \frac{n}{2} = \frac{n+2}{2}$ and $\theta(c_3) = n$. By the definition of first chromatic Zagreb index, we have

$$M_1^{\varphi^+}(\breve{P}_n) = \sum_{i=1}^{3}(\theta(c_i))i^2 = \frac{23n + 8}{2}.$$

Part (ii): We follow the coloring pattern as stated, but in this part the root vertex w is colored with c_2. Here we have $\eta_{12} = 2n - 1, \eta_{23} = \eta_{13} = n - 1$ and through total summation, the result follows.

Part (iii): After coloring the vertices as mentioned, we note that $\eta_{12} = 2n - 1$ edges and $\eta_{23} = n - 1$ edges contribute the distance 1 to the total summation while $\eta_{13} = n - 1$ contributes the distance 2. Thus, we have

$$M_3^{\varphi^+}(\breve{P}_n) = \sum_{i=1}^{n-1}\sum_{j=2}^{n} |\zeta(v_i) - \zeta(v_j)| = 2(n - 1) + 2(2n - 1) = 6n - 4.$$

Part (iv): To evaluate the maximum value of the fourth chromatic Zagreb index, we follow the coloring stated, except that the root vertex is colored with c_1. With the same logic described in part(iv) of the Theorem 2.1, we proceed to the following two cases:

Case-1: Let n be odd. Then, we have $\theta(c_1) = \frac{n+3}{2}$, $\theta(c_2) = \frac{n-1}{2}$ and $\theta(c_3) = n$. Now, by the definition of fourth chromatic Zagreb index, we have

$$\begin{aligned} M_4^{\varphi^-}(\breve{P}_n) &= \frac{1}{2}\sum_{u,v\in V(\breve{P}_n)} |\varphi(u)-\varphi(v)| \\ &= 2\theta(c_1)\theta(c_3)+\theta(c_1)\theta(c_2)+\theta(c_2)\theta(c_3) \\ &= \frac{7n^2+12n-3}{8}. \end{aligned}$$

Case-2: Let n be even. Then, we have $\theta(c_1) = \frac{n+2}{2}$, $\theta(c_2) = \frac{n}{2}$ = and $\theta(c_3) = n$. Then, the result follows from the definition of the chromatic total irregularity index.

$$\begin{aligned} M_4^{\varphi^+}(\breve{P}_n) &= \frac{1}{2}\sum_{u,v\in V(\breve{P}_n)} |\varphi(u)-\varphi(v)| \\ &= 2\theta(c_1)\theta(c_3)+\theta(c_1)\theta(c_2)+\theta(c_2)\theta(c_3) \\ &= \frac{7n^2+10n}{8}. \end{aligned}$$

□

3. Chromatic Topological Indices of Mycielskian of Cycles

Next we discuss the chromatic Zagreb indices of Mycielskian of a cycle in the following theorems.

Theorem 3.1. *For the Mycielskian of a cycle C_n, we have*

(i) $M_1^{\varphi^-}(\breve{C}_n) = \begin{cases} \frac{15n+27}{2}; & \textit{if } n \textit{ is odd} \\ \frac{15n+8}{2}; & \textit{if } n \textit{ is even} \end{cases}$

(ii) $M_2^{\varphi^-}(\breve{C}_n) = \begin{cases} 13n+11; & \textit{if } n \textit{ is odd} \\ 13n; & \textit{if } n \textit{ is even} \end{cases}$

(iii) $M_3^{\varphi^-}(\breve{C}_n) = \begin{cases} 5n+4; & \textit{if } n \textit{ is odd} \\ 5n; & \textit{if } n \textit{ is even} \end{cases}$

(iv) $M_4^{\varphi^-}(\breve{C}_n) = \begin{cases} \frac{7n^2+14n-1}{8}; & \textit{if } n \textit{ is odd} \\ \frac{7n^2+6n}{8}; & \textit{if } n \textit{ is even} \end{cases}$

Proof. Part (i): As explained in Theorem 2.1, the set U is the largest independence set in $\breve{C}_n$ also and we assign color c_1 to all vertices in U. As stated earlier, every vertex of V is adjacent to at least one vertex in U and hence no vertices in V can have the color c_1. To proceed further, we have to consider the following cases.

Case-1: Let n be odd. Then, C_n is 3- colorable and we can color the vertices in V using three colors, say c_2, c_3, c_4 such that $\lfloor \frac{n}{2} \rfloor$ vertices have colors c_2 and c_3, while one vertex has color c_4. The root vertex w can be colored using the color c_2. Then we have, $\theta(c_1) = n$, $\theta(c_2) = \frac{n+1}{2}$, $\theta(c_3) = \frac{n-1}{2}$ and $\theta(c_4) = 1$. Hence the total summation can be given as

$$M_1^{\varphi^-}(\breve{C}_n) = \sum_{i=1}^{4} (\theta(c_i))i^2 = \frac{15n+27}{2}.$$

Case-2: Let n be even. Then, C_n can be colored using two colors, say c_2 and c_3, the corresponding color classes contain $\frac{n}{2}$ vertices each. Since w is adjacent to all vertices in U and w is not adjacent to any vertex in V, we can assign the color c_2 to w. If $\theta(c_i)$ denotes the cardinality of the color class of c_i, then we have $\theta(c_1) = n$,

$\theta(c_2) = 1 + \frac{n}{2} = \frac{n+2}{2}$ and $\theta(c_3) = \frac{n}{2}$. The total summation will follow from the definition of first chromatic Zagreb index.

$$M_1^{\varphi^-}(\breve{C}_n) = \sum_{i=1}^{3}(\theta(c_i))i^2 = \frac{15n+8}{2}.$$

Part (ii): We color the vertices as per the instructions in part (i) for even and odd cases of n. Now consider the following cases:

Case- 1: Let n be odd. Here we see that $\eta_{12} = 2n-1, \eta_{13} = n-1, \eta_{14} = 2, \eta_{23} = n-2, \eta_{24} = \eta_{34} = 1$. Hence, we have the sum

$$M_2^{\varphi^-}(\breve{C}_n) = \sum_{1\leq t,s\leq \chi(\breve{C}_n)}^{t<s} ts\eta_{ts} = 13n+11.$$

Case- 2: Let n be even. Here we see that $\eta_{12} = 2n, \eta_{23} = \eta_{13} = n$. The definition of second chromatic Zagreb index, gives the sum

$$M_2^{\varphi^-}(\breve{C}_n) = \sum_{1\leq t,s\leq \chi(\breve{C}_n)}^{t<s} ts\eta_{ts} = 13n.$$

Part (iii): To find the minimum irregularity measurement, we proceed to the following two cases, after coloring the vertices using minimal parameter coloring.

Case- 1: Let n be odd. Here we see that $\eta_{12} = 2n-1, \eta_{13} = n-1, \eta_{14} = 2, \eta_{23} = n-2, \eta_{24} = \eta_{34} = 1$. Now $\eta_{14} = 2$ edges contributes the distance 3, $\eta_{13} = n-1$ edges and $\eta_{24} = 1$edge contributes the distance 2 to the total summation while all other edges contributes the distance 1. Then the result follows from the following calculations:

$$M_3^{\varphi^-}(\breve{C}_n) = \sum_{i=1}^{n-1}\sum_{j=2}^{n}|\zeta(v_i)-\zeta(v_j)| = 2n-1+2(n-1)+n-2+9 = 5n+4.$$

Case- 2: Let n be even. Now $\eta_{12} = 2n$ edges and $\eta_{23} = n$ edges contribute the distance 1 to the total summation while $\eta_{13} = n$ contributes the distance 2. The result follows from the following calculations:

$$M_3^{\varphi^-}(\breve{C}_n) = \sum_{i=1}^{n-1}\sum_{j=2}^{n} |\zeta(v_i) - \zeta(v_j)| = 2n + n + 2n = 5n.$$

Part (iv): First color the vertices according to minimal parameter coloring described in the first part of the proof. In order to calculate the total irregularity of $\breve{C}_n$, all the possible vertex pairs from $\breve{C}_n$ have to be considered and their possible color distances are determined. We observe that vertex pairs with same colors contribute nothing to the color distance and we discard such cases. The possibility of the vertex pairs which contribute to the color distance can be classified according to the following two cases.

Case- 1: Let n be odd. Here the possible combinations which contributes to the color distances are $\{1,2\}, \{2,3\}, \{3,4\}$ contributing 1, $\{1,3\}, \{2,4\}$ contributing 2 and $\{1,4\}$ contributing 3. We have $\theta(c_1) = n$, $\theta(c_2) = \frac{n+1}{2}$, $\theta(c_3) = \frac{n-1}{2}$ and $\theta(c_4) = 1$. Now for the combination $\{t, s\}$, each vertex of color c_t pairs with $\theta(c_s)$ vertices of color c_s. So the total color distance for the $\{t, s\}$ combination is the color distance times $2\theta(c_t)\theta(c_s)$. Then, from the definition, we have

$$\begin{aligned} M_4^{\varphi^-}(\breve{C}_n) &= \frac{1}{2}\sum_{u,v\in V(\breve{C}_n)} |\varphi(u) - \varphi(v)| \\ &= \frac{7n^2 + 14n - 1}{8}. \end{aligned}$$

Case- 2: Let n be even. The combinations possible are charted as $\{1,2\}, \{2,3\}$ contributing 1 and $\{1,3\}$ contributing 2. Here, we observe that $\theta(c_1) = n$, $\theta(c_2) =$

$1+\frac{n}{2}=\frac{n+2}{2}$ and $\theta(c_3)=\frac{n}{2}$. With the same logic as in case (1), we have

$$\begin{aligned} M_4^{\varphi^-}(\breve{C}_n) &= \frac{1}{2}\sum_{u,v\in V(\breve{C}_n)} |\varphi(u)-\varphi(v)| \\ &= \frac{1}{2}\{2\theta(c_1)\theta(c_3)+\theta(c_1)\theta(c_2)+\theta(c_2)\theta(c_3)\} \\ &= \frac{7n^2+6n}{8}. \end{aligned}$$

□

Theorem 3.2. *For the Mycielskian of a cycle C_n, we have*

(i) $M_1^{\varphi^+}(\breve{C}_n) = \begin{cases} \frac{45n+7}{2}; & \text{if } n \text{ is odd} \\ \frac{23n+8}{2}; & \text{if } n \text{ is even} \end{cases}$

(ii) $M_2^{\varphi^+}(\breve{C}_n) = \begin{cases} 38n-19; & \text{if } n \text{ is odd} \\ 17n; & \text{if } n \text{ is even} \end{cases}$

(iii) $M_3^{\varphi^+}(\breve{C}_n) = \begin{cases} 5n+4; & \text{if } n \text{ is odd} \\ 5n; & \text{if } n \text{ is even} \end{cases}$

(iv) $M_4^{\varphi^+}(\breve{C}_n) = \begin{cases} \frac{7n^2+16n+1}{8}; & \text{if } n \text{ is odd} \\ \frac{7n^2+6n}{8}; & \text{if } n \text{ is even} \end{cases}$

Proof. The Mycielskian of a cycle $\breve{C}_n$, has chromatic number 4 when n is odd and 3 when n is even. We first color $\breve{C}_n$ by minimal parameter coloring in such a way that we get the maximum values for the chromatic Zagreb indices and irregularity measurements.

Now we suppose n is odd. Since U comprises the largest independence set in $\breve{C}_n$, we assign color c_4 to all vertices in U. Every vertex of V is adjacent to at least one vertex in U. Hence we color $\lfloor\frac{n}{2}\rfloor$ vertices and the root vertex w with c_3 and the other $\lfloor\frac{n}{2}\rfloor$ vertices with color c_2 . The remaining one vertex is colored with the color c_1. Hence, we observe the following values when n is odd. $\theta(c_1)=1, \theta(c_2)=\frac{n-1}{2}, \theta(c_3)=\frac{n+1}{2}$ and $\theta(c_4)=n$. Also we have $\eta_{34}=2n-1, \eta_{24}=n-1, \eta_{14}=2, \eta_{23}=n-2, \eta_{12}=\eta_{13}=1$.

Let n be even. We color vertices of U with c_3 and root vertex w with c_2. Then, C_n can be colored using two colors, say c_2 and c_1, the corresponding color classes contain $\frac{n}{2}$ vertices each. Thus we note the following values when n is even. $\theta(c_1)=\frac{n}{2}$, $\theta(c_2)=\frac{n+2}{2}$ and $\theta(c_3)=n$. Also we have $\eta_{23}=2n$ and $\eta_{12}=\eta_{13}=n$.

Now we proceed for the four parts of the theorem.

Part (i): To find the first chromatic Zagreb index, we consider the following cases:

Case-1: Let n be odd. Then we have the total summation as

$$M_1^{\varphi^+}(\breve{C}_n)=\sum_{i=1}^{4}(\theta(c_i))i^2=\frac{45n+7}{2}.$$

Case-2: Let n be even. The total summation will follow from the definition as

$$M_1^{\varphi^+}(\breve{C}_n)=\sum_{i=1}^{3}(\theta(c_i))i^2=\frac{23n+8}{2}.$$

Part (ii): We color the vertices as per the instructions in introduction for even and odd cases of n. Now consider the following cases:

Case- 1: Let n be odd. Then, we have the sum

$$M_2^{\varphi^+}(\breve{C}_n)=\sum_{1\le t,s\le \chi(\breve{C}_n)}^{t<s} ts\eta_{ts}=38n-19.$$

Case- 2: Let n be even. Then, the total summation is the direct consequence of the definition of second chromatic Zagreb index

$$M_2^{\varphi^+}(\breve{C}_n)=\sum_{1\le t,s\le \chi(\breve{C}_n)}^{t<s} ts\eta_{ts}=17n.$$

Part (iii): To find the minimum irregularity measurement, consider the following cases:

Case- 1: Let n be odd. Here we see that, $\eta_{12}+\eta_{23}+\eta_{34}$ edges contributes 1 to the color distance, $\eta_{13}+\eta_{24}$ edges contributes 2, while η_{14} edges contributes 3. Then the result follows from the following calculations:

$$M_3^{\varphi^+}(\breve{C}_n) = \sum_{i=1}^{n-1}\sum_{j=2}^{n} |\zeta(v_i) - \zeta(v_j)| = 2n - 1 + 2(n-1) + n - 2 + 9 = 5n + 4.$$

Case- 2: Let n be even. Here $\eta_{12}+\eta_{23} = 3n$ edges contributes the distance 1 to the total summation while $\eta_{13} = n$ contributes the distance 2. The result follows from the following calculations:

$$M_3^{\varphi^+}(\breve{C}_n) = \sum_{i=1}^{n-1}\sum_{j=2}^{n} |\zeta(v_i) - \zeta(v_j)| = 2n + n + 2n = 5n.$$

Part (iv): To calculate the total irregularity of $\breve{C}_n$, all the possible vertex pairs from $\breve{C}_n$ have to be considered and their possible color distances are determined. The possibility of the vertex pairs which contribute to the color distance can be classified according to the following two cases.

Case- 1: Let n be odd. Here the possible combinations which contributes to the color distances are $\{1,2\},\{2,3\},\{3,4\}$ contributing 1, $\{1,3\},\{2,4\}$ contributing 2 and $\{1,4\}$ contributing 3. Just as we proceeded in *part*(4) of the previous theorem, we calculate the total irregularity as given below:

$$\begin{aligned} M_4^{\varphi^+}(\breve{C}_n) &= \frac{1}{2}\sum_{u,v\in V(\breve{C}_n)} |\varphi(u) - \varphi(v)| \\ &= \frac{7n^2 + 14n - 1}{8} \end{aligned}$$

Case- 2: Let n be even. The combinations possible are charted as $\{1,2\},\{2,3\}$ contributing 1 and $\{1,3\}$ contributing 2. Observe that $\theta(c_1) = n$, $\theta(c_2) = 1 + \frac{n}{2} = \frac{n+2}{2}$ and $\theta(c_3) = \frac{n}{2}$. With the same logic as in case (1), we have

$$\begin{aligned} M_4^{\varphi^+}(\breve{C}_n) &= \frac{1}{2}\sum_{u,v\in V(\breve{C}_n)} |\varphi(u) - \varphi(v)| \\ &= \frac{7n^2 + 6n}{8}. \end{aligned}$$

□

4. Conclusion

Chromatic topological indices can find a variety of applications in mathematical chemistry, optimization techniques, distribution theory and even in sociology. An overview of chromatic Zagreb indices and irregularity indices of Mycielskian of paths and cycles is provided in this paper. More research areas will be opened if Mycielskian of different graph classes like bipartite graphs, complete graphs are considered. Also comparative study on chromatic Zagreb indices and irregularity indices of graph classes and their operations will be interesting. One can also work on chromatic Zagreb indices and irregularity indices of some associated graphs such as line graphs, subdivision of graphs, total graphs, etc. Even the chromatic version of other topological indices gives fresh areas of research with tremendous applications.

Acknowledgement

The first author gratefully acknowledge the academic helps rendered by Center for Studies in Discrete Mathematics, Vidya Academy of Science and Technology, Thrissur, Kerala, India. The authors would like to thank the editor and the referees for their critical and insightful comments, which improved the content and presentation style of the article significantly.

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(1) Department of Mathematics, St. Mary's College, Thrissur, Kerala, India.

Email address: `jesualcmc@gmail.com`

(2) Department of Mathematics, CHRIST (Deemed to be University), Bangalore, Karnataka, India.

Email address: `sudev.nk@christuniversity.in`