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STAGE STRUCTURED PREY PREDATOR MODEL WITH MATURATION AND GESTATION DELAY FOR PREDATOR USING HOLLING TYPE 2 FUNCTIONAL RESPONSE

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Abstract. In this paper, we have proposed a prey predator model with stage structuring in predator. We have incorporated maturation and gestation delay in predator class. The positivity, boundedness, existence of equilibrium points and local stability has been discussed. We have also obtained threshold for maturation and gestation delay. If $0 < \tau_1 < \tau_1^*$ then E^* is locally asymptotically stable and when $\tau_1 > \tau_1^*$, Hopf bifurcation exist in the absence of τ. Further, for $\tau \geq 0$, E^* is locally asymptotically stable if $0 < \tau < \tau^*$ and for $\tau > \tau^*$, it looss its stability. Finally, permanence of the system is discussed with numerical example in validation to the analytical results using MATLAB software.

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1. Introduction

In our ecosystem prey predator plays a key role. For their survival each individual has to feed on the other. The rate of one species increases with the decrease in the other. Predation is a natural phenomena which is must for maintenance of the nature. If predation does not takes place then there will be an increase in prey and predators will die with time and therefore will extinct. Authors [1] proposed a model on prey predator interaction, where they discussed how does the predation takes place and what are its consequences. Such interactions are basically win-loss interaction. In addition to it, our ecosystem also consist of variety of species which goes through many stages in order to transform themselves from immature to mature. Such species are mathematically modelled by compartment modelling called as stage structured models. For example: Butterflies has to cross three stages to reach adult phase (i.e. egg, caterpillar and pupa stage). This time period is known as juvenile period. Various mathematical models with stage structure have been studied by the scholars [2, 5, 6]. Time delay also plays a vital role in interacting species models. The time taken by population to process food, digest it, before searching for new food is called gestation period. Maturation delay is also a time lag which denotes the time period required by a population to become mature. Several mathematical models incorporating different delays have been studied by authors[9, 3]. More realistic models have been developed by incorporating delay and stage structuring to study the rich dynamics of the system [4, 7, 8, 10]. The best biological example showing prey predator interaction is visible in pest-natural enemy model. For example, adult Ladybugs which feeds on aphids, scales and mites to protect the crop acts as natural enemy. Ladybugs are special predators who predate pest only in the adult stage as the larval stage ladybugs are not capable enough to feed and further only adult lady bugs are capable to reproduce. Various mathematical models involving stage structure in prey-predator population have been studied recently [12, 13, 16, 18] but in all the models, stage structuring has been considered in prey population. However, recently [17], proposed a mathematical model in which they assumed that only adult predators attack and consume the prey and have the ability to reproduce but they considered an anti-predator defense effect where prey can also attack juvenile and further adult predators may help when juvenile predators are attacked by prey. But, they didn't incorporate any delay parameters in it. Hence, motivated by the above literature, we have developed a new prey-predator mathematical model with stage structuring in predator population, i.e,juvenile and adult stage. The maturation delay and gestation delay is incorporated in the predators. Our main question to be addressed is what happens to the prey population with increase in maturation delay and gestation delay of predators. Does it stabilize or destabilize the system? The paper is organised as follows: The mathematical model is proposed in section 2. In section 3, conditions of existence of boundary and interior equilibrium points are discussed. Positivity and local stability of the equilibrium points are discussed in section 4 and 5 respectively. In section 6, permanence of the model is studied. Finally, in the last section of this paper, numerical simulation with a set of hypothetical data has been done for the validation of the analytical results.

TABLE 1. Parameters/Variable with meaning

Variable/parameter	Meaning
p	density of prey population
q_1	density of immature predator population
q_2	density of mature predator population
a_1	attack rate
h	handling time and $M = a_1 * h$
r	growth rate of prey
r_1	growth rate of immature predator
α_1	natural death rate of prey population
α_2	natural death rate of immature predator population
α_3	crowding effect
τ_1	maturation delay of predator
τ	gestation delay of predator

2. Mathematical Model

In this section, we have proposed our new model incorporating stage structuring in predator along with maturation and gestation delay to show a prey-predator interaction which gives the following set of differential equations:

$$\frac{dp}{dt} = p(t)(r - ap(t) - \frac{a_1 q_2(t)}{1 + Mp(t)}) \tag{2.1}$$

$$\frac{dq_1}{dt} = r_1 q_2(t) - \alpha_1 q_1(t) - r_1 e^{-d_1\tau_1} q_2(t - \tau_1) \tag{2.2}$$

$$\frac{dq_2}{dt} = -\alpha_2 q_2(t) + r_1 e^{-d_1\tau_1} q_2(t - \tau_1) + \frac{a_2 p(t - \tau) q_2(t - \tau)}{1 + Mp(t - \tau)} - \alpha_3 q_2^2(t) \tag{2.3}$$

where the parameters are defined as per the table above. Further, we have also considered Holling Type-II functional response to show the interaction between prey and mature predator which is given by $\frac{a_1 q_2(t)}{1+Mp(t)}$. The term $r_1 e^{-d_1\tau_1} q_2(t-\tau_1)$ represent the transformation rate of immature predator to mature predator.

3. Existence of Equilibrium points

The proposed model (2.1-2.3) has three non-negative equilibrium points:

(1) Trivial equilibrium $E_1(0,0,0)$ exists always,

(2) Boundary equilibrium point $E_2(\frac{r}{a},0,0)$,

(3) Boundary equilibrium point $E_3(0,\bar{q_1},\bar{q_2})$, exists only if (**H1**) : $r_1 e^{-d_1\tau_1} > \alpha_2$ holds, where

$$\bar{q_1} = \frac{r_1 \bar{q_2}(1-e^{-d_1\tau_1})}{\alpha_1},$$

$$\bar{q_2} = \frac{r_1 e^{-d_1\tau_1} - \alpha_2}{\alpha_3},$$

(4) Interior point $E^*(p^*,q_1^*,q_2^*)$, exists if (**H2**) : $Ma_1\alpha_2 + 2\alpha_3 Mr > Ma_1 r_1 e^{-d_1\tau_1} + \alpha_3 a + a_1 a_2$ holds, where, $\frac{r}{a} > p^*$

$$q_1^* = \frac{r_1 {q_2}^*(1-e^{-d_1\tau_1})}{\alpha_1},$$

$$q_2^* = \frac{(r-ap^*)(1+Mp^*)}{a_1},$$

and p^* is obtained from the following cubic equation:

$$\alpha_3 M^2 a(p^*)^3 + (p^*)^2(\alpha_3 rM + 2Ma) - p^*(Ma_1\alpha_2 - Ma_1 r_1 e^{-d_1\tau_1} + 2\alpha_3 Mr - \alpha_3 a - a_1 a_2)$$
$$+ a_1(r_1 e^{-d_1\tau_1} - \alpha_2) = 0$$

By Descarte's rule of sign, it has atleast one positive root provided that $Ma_1\alpha_2 + 2\alpha_3 Mr > Ma_1 r_1 e^{-d_1\tau_1} + \alpha_3 a + a_1 a_2$.

Remark 1: It is clear from (**H1**) that boundary equilibrium $E_3(0, \bar{q_1}, \bar{q_2})$ exists if $\tau_1 < \frac{1}{d_1} log \frac{r_1}{\alpha_2}$ and from **(H2)** that interior equilibrium $E^*(p^*, q_1^*, q_2^*)$ exists if $\tau_1 < \frac{1}{d_1} log \frac{Ma_1 r_1}{Ma_1\alpha_2 + 2\alpha_3 Mr - \alpha_3 a - a_1 a_2}$.

4. Positivity of the system

In this section, we will discuss the positivity of the system.

Theorem 4.1. *The solution of the given model (2.1-2.3) are always positive, and for all $t \geq 0$.*

proof. Let us first consider $q_2(t)$ for all $t \in [0, \tau]$. Using equation (2.3) of the model,

$$\frac{dq_2}{dt} = -\alpha_2 q_2(t) + r_1 e^{-d_1\tau_1} q_2(t - \tau_1) + \frac{a_2 p(t-\tau) q_2(t-\tau)}{1 + Mp(t-\tau)} - \alpha_3 q_2^2(t) \tag{4.1}$$

By Comparison theorem [14], it follows that for $t \in [0, \tau]$,

$$\frac{dq_2}{dt} \geq q_2(t-\tau) r_1 e^{-d_1\tau_1} - q_2(t)\alpha_2 - \alpha_3 q_2^2(t)$$

$$\inf q_2(t) \geq \frac{-\alpha_2 + r_1 e^{-d_1\tau_1}}{\alpha_3} = M_1 > 0.$$

where, $r_1 e^{-d_1\tau_1} > \alpha_2$

Now, from equation (2.1) of the model for $t \in [0, \tau]$ we derive that ,

$$\frac{dp(t)}{dt} = p(t)(r - ap(t) - \frac{a_1 q_2(t)}{1 + Mp(t)})$$

$$\frac{dp(t)}{dt} \geq p(t)(r - ap(t) - a_1 M_1)$$

A standard Comparison argument shows that for $t \in [0, \tau]$,

$$\inf p(t) \geq \frac{r - a_1 M_2}{a} > 0 = M_2$$

where,$r > a_1 M_2$.

Similarly, we derive from the equation (2.2) of the model that for $t \in [0, \tau]$,

$$\frac{dq_1}{dt} = r_1 q_2(t) - \alpha_1 q_1(t) - r_1 e^{-d_1 \tau_1} q_2(t - \tau_1)$$

$$\frac{dq_1}{dt} \geq r_1 M_1 (1 - e^{-d_1 \tau_1}) - \alpha_1 q_1$$

$$\inf q_1 \geq \frac{r_1 M_1 (1 - e^{-d_1 \tau_1})}{\alpha_1} = M_3 > 0$$

This completes the proof.

In the next section, we have proved the local stability of the boundary and interior equilibrium points using lemmas stated in Appendix.

5. Local Stability

The local behaviour of non negative equilibrium are given in the following theorem:

Theorem 5.1. *The local behaviour of equilibrium points of the model are as follows:*

1. The trivial equilibrium $E_1(0, 0, 0)$ *is always unstable for every* $\tau_1 \geq 0$*,*

$\tau \geq 0$.

2. The boundary equilibrium $E_2(p, 0, 0)$ *is stable if* $\tau_1 > \bar{\tau_1}$ *and* $\tau \geq 0$ *where* $\bar{\tau_1} = \frac{1}{d_1} log \frac{r_1}{\alpha_2}$.

3. The boundary equilibrium $E_3(0, \bar{q_1}, \bar{q_2})$ *is stable if* $\tau_1 > \bar{\tau_1^*}$ *where* $\bar{\tau_1^*} = \frac{1}{d_1} log(\frac{r_1}{\alpha_2 - 2\alpha_3 q_2})$ *for all* $\tau \geq 0$.

4. If $\tau_1 < \bar{\tau_1^*}$, *then the interior equilibrium* $E^*(p^*, q_1^*, q_2^*)$ *is locally asymptotically stable for every* $\tau \geq 0$.

The Jacobian corresponding to the system (2.1-2.3) is given by:

$$\begin{bmatrix} r - 2ap - \dfrac{a_1 q_2}{(1+Mp)^2} - \lambda & 0 & \dfrac{-a_1 p}{1+Mp} \\ 0 & -\alpha_1 - \lambda & r_1 - r_1 e^{-(d_1+\lambda)\tau_1} \\ e^{-\lambda\tau} \dfrac{a_2 q_2}{(1+Mp)^2} & 0 & -\alpha_2 - 2\alpha_3 q_2 + r_1 e^{-(d_1+\lambda)\tau_1} + \dfrac{a_2 p}{1+Mp} e^{-\lambda\tau} - \lambda \end{bmatrix}$$

The characteristic equation corresponding to $E_1(0, 0, 0)$ is

$$(r - \lambda)(-\alpha_1 - \lambda)(-\alpha_2 + r_1 e^{-\lambda\tau_1} e^{-d_1\tau_1} - \lambda) = 0$$

Clearly, $\lambda = r$ which shows that E_1 is always unstable for every $\tau_1 \geq 0$, $\tau \geq 0$ because it has a positive root.

The characteristic equation corresponding to $E_2(p, 0, 0)$ is

$$(-r - \lambda)(-\alpha_1 - \lambda)(-\alpha_2 + \frac{a_2 r}{a + Mr} e^{-d_1\tau} + r_1 e^{-\lambda\tau_1} e^{-d_1\tau_1} - \lambda) = 0$$

The eigen values corresponding to above characteristic equation are $\lambda = -r < 0$, $\lambda = -\alpha_1 < 0$ and $\lambda = r_1 e^{-d_1\tau_1 + \lambda_1\tau_1} - \alpha_2 + \frac{a_2 r}{a+Mr} e^{-\lambda\tau}$. $E_2(p, 0, 0)$ is stable if $r_1 e^{-d_1\tau_1} + \frac{a_2 r}{a+Mr} < \alpha_2$ which implies $\tau_1 > \bar{\tau_1}$ where

$\bar{\tau_1} = \frac{1}{d_1} log \frac{r_1}{\alpha_2}$ and unstable if $\tau_1 < \bar{\tau_1}$.

Remark 2: Boundary point E_3 cannot be stable if E_2 exists and vice-versa.

The characteristic equation corresponding to $E_3(0, \bar{q_1}, \bar{q_2})$ is:

$$(r - a_1\bar{q_2} - \lambda)(-\alpha_1 - \lambda)(-\alpha_2 + 2\alpha_3\bar{q_2} + r_1 e^{\lambda\tau_1} e^{-d_1\tau_1} - \lambda) = 0$$

$$(-\alpha_1 - \lambda)F_1(\lambda)F_2(\lambda) = 0$$

The eigen values of the above characteristic equation are $\lambda = -\alpha_1 < 0$, $\lambda = r - a_1\bar{q_2}$ and $\lambda = r_1 e^{\lambda\tau_1} e^{-d_1\tau_1} + 2\alpha_3\bar{q_2} - \alpha_2$. E_3 is locally asymptotically stable if $r < a_1\bar{q_2}$ and $r_1 e^{-d_1\tau_1} + 2\alpha_3\bar{q_2} < \alpha_2$, which implies $\tau_1 > \bar{\tau_1^*}$ where $\bar{\tau_1^*} = \frac{1}{d_1} log(\frac{r_1}{\alpha_2 - 2\alpha_3 q_2})$ and $\tau \geq 0$, otherwise unstable.

(iv) Now we examine the stability of interior equilibrium $E^*(p^*, q_1^*, q_2^*)$. Characteristic equation of E^* is:

$$(\lambda^2 + A_1\lambda + A_2) + e^{-\lambda\tau_1}(B_1\lambda + B_2) + e^{-\lambda\tau}(C_1\lambda + C_2) = 0 \tag{5.1}$$

where,

$A_1 = \alpha_2 + 2\alpha_3 q_2 - r + 2ap + \frac{a_1 q_2}{(1+Mp)^2}$

$A_2 = (r - 2ap^* - \frac{a_1 q_2^*}{(1+Mp^*)^2})(\alpha_2 + 2\alpha_3 q_2^*)$

$B_1 = -r_1 e^{-d_1\tau_1}$, $B_2 = (r - 2ap^* - \frac{a_1 q_2^*}{(1+Mp^*)^2}) r_1 e^{-d_1\tau_1}$

$C_1 = -\frac{a_2 p^*}{1+Mp^*}$, $C_2 = r - 2ap^* \frac{a_2 P^*}{1+Mp^*}$

We will discuss three cases to study the behaviour of Interior equilibrium $E^*(p^*, q_1^*, q_2^*)$.

Case 1:

When $\tau_1 = 0$, $\tau = 0$ then, from equation (5.1) we get.

$$\lambda^2 + \lambda(A_1 + B_1 + C_1) + (A_2 + B_2 + C_2) = 0 \tag{5.2}$$

It follows from the Routh-Hurwitz criterion that the necessary and sufficient conditions for all roots of (5.2) having negative real parts is given by $(A_1 + B_1 + C_1) > 0, (A_2 + B_2 + C_2) > 0$. Hence, equilibrium point $E^*(p^*, q_1^*, q_2^*)$ is locally asymptotically stable under the above condition.

We begin with the case $\tau_1 = 0$, $\tau = 0$ as it is necessary that the nontrivial equilibrium point should be locally stable for$\tau_1 = 0$, $\tau = 0$ so that we can obtain the local stability for all non negative values of delay and further can find the critical value which may destabilize the system.

Case 2:. If $\tau_1 = 0$ and $\tau > 0$, then the polynomial reduces to:

$$(\lambda^2 + A_1\lambda + A_2) + (B_1\lambda + B_2) + e^{-\tau\lambda}(C_1\lambda + C_2) = 0 \tag{5.3}$$

where, $p = (A_1 + B_1), r = (A_2 + B_2), s = C_1, q = C_2$

Here, we consider τ as the parameter to study the local stability of interior equilibrium E^*.

By lemma(A.2), we see that,

1. L1 holds if $2ap^* + \alpha_2 + 2\alpha_3 q_2^* + \frac{a_1 q_2^*}{(1+Mp^*)^2} > r + r_1 e^{-d_1\tau_1} + \frac{a_2 P^*}{1+Mp^*}$ (B1)

2. L2 holds if $r > 2ap^* + \frac{a_1 q_2^*}{(1+Mp^*)^2}$ (B2)

3. L3 holds if (B1) is true and $A_1 < \alpha_2 + 2\alpha_3 q_2^*$ and $r^2 - q^{*2} > 0$ if (B2) is true and $(r - 2ap^* - \frac{a_1 q_2^*}{(1+Mp^*)^2})(\alpha_2 + r_1 e^{-d_1\tau_1}) > (r - 2ap^*)\frac{a_2 p^*}{1+Mp^*}$

Thus, all the roots of the polynomial (5.3) has negative real parts for all $\tau > 0$ if the above conditions (1-3) holds and E^* becomes locally asymptotically stable.

Now, if any one of the above conditions gets violate say $r < 2ap^* + \frac{a_1 q_2^*}{(1+Mp^*)^2}$ i.e. $r^2 - q^2 < 0$, then the system has a pair of purely imaginary roots say, $(\lambda = \pm\iota\omega)$. Now, we obtain the condition that under what value of τ, the polynomial (5.3) has a pair of purely imaginary roots. Therefore, we substitute $\lambda = \pm\iota\omega$ in equation (5.3) and equate its real and imaginary parts which gives,

$$-\omega^2 + (A_2 + B_2) + C_2\cos\omega\tau - C_1\omega\sin\omega\tau - C_1^2\omega^2\sin\omega\tau_2 = 0 \tag{5.4}$$

and,

$$(A_1 + B_1)\omega C_2 + \omega C_1 C_2\cos\omega\tau + C_2\sin\omega\tau \tag{5.5}$$

Eliminating $\sin\omega\tau$ and $\cos\omega\tau_2$ from (5.4) and (5.5), we get,

$$\sin\omega\tau = \frac{-C_1\omega^3 + \omega C_1(A_2 + B_2) - (A_1 + B_1)\omega C_2}{C_1^2\omega^2 + C_2^2} \tag{5.6}$$

$$\cos\omega\tau = \frac{\omega^2 C_2 - (A_2 + B_2)C_2 - (A_1 + B_1)C_1\omega^2}{C_2^2\omega^2 + C_2^2} \tag{5.7}$$

Now, let ω_0 be a positive root of equation (5.4). From equation (5.3), we obtain the critical values of τ,

$$\tau_k^+ = \frac{1}{\omega_0}[\cos^{-1}(\frac{\omega_0^2 C_2 - (A_2 + B_2)C_2 - (A_1 + B_1)C_1\omega_0^2}{C_1^2 + \omega_0^2 C_1^2} + 2k\Pi]$$

for every $k = 0, 1, 2, ...$

To make sure the occurrence of the Hopf bifurcation, it is needed to check the transversality condition[15]. Without loss of generality, the time delay τ is chosen as the bifurcation parameter. The necessary condition for the existence of the Hopf bifurcation is that the critical eigenvalues cross the imaginary axis with non-zero velocity. Differentiating λ with respect to τ of (5.2) we obtain,

$$\frac{d\lambda}{d\tau} = \frac{\lambda e^{-\lambda\tau}(\lambda s + q)}{2\lambda + p + se^{-\lambda\tau} - \tau_2 e^{-\lambda\tau}(s\lambda + q)}$$

or,

$$(\frac{d\lambda}{d\tau})^{-1} = \frac{2\lambda + p + se^{-\lambda\tau}}{\lambda e^{-\lambda\tau}(s\lambda + q)} - \frac{\tau}{\lambda}$$

By simple calculations, we can see that,

$$Re[(\frac{d\lambda}{d\tau})^{-1}]_{\tau=(\tau_0)^+} > 0$$

Noting that,

$$Sign\{Re[\frac{d\lambda}{d\tau}]_{\tau=(\tau_0)^+}\} = Sign\{Re[(\frac{d\lambda}{d\tau})^{-1}]_{\tau=(\tau_0)^+}\} > 0$$

Therefore, when the delay τ is near its critical τ_0^+, then the root of equation (5.3) crosses the imaginary axis from left to right. When $\tau = \tau_0^+$, there exists a pair of purely imaginary roots for equation (5.3) and all the other roots having negative real parts. Thus, the transversality condition required for the Hopf bifurcation is proved. In the next case, we would study the effect of both the delays on the local stability of the system.

Case 3: $\tau_1 > 0$ and $\tau = 0$.

The proof is obvious as above.

Case 4: $\tau_1 > 0$ and $\tau > 0$.

We first state a result regarding the real part of the roots of the equation (5.1) to study the local stability of the E^* of system (2.1-2.3).

Proposition: If all roots of equation (5.1) have negative real parts for some $\tau > 0$, then there exists a $\tau_1^*(\tau) > 0$ such that all roots of equation (5.1) (that is, with $\tau > 0$) have negative real parts when $\tau_1 < \tau_1^*(\tau)$.

By using the above proposition and letting the following assumptions using lemma (A.2):

$(r_1 e^{-d_1\tau_1})^2 + ((r - 2ap^* - \frac{a_1 q_2^*}{(1+Mp^*)^2})(\alpha_2 + 2\alpha_3 q_2^*) + (r - 2ap^* \frac{a_2 P^*}{1+Mp^*})) <$
$(\alpha_2 + 2\alpha_3 q_2 - r + 2ap + \frac{a_1 q_2}{(1+Mp)^2} - \frac{a_2 p^*}{1+Mp^*})^2$,

$s^2 - p^2 + 2r < 0$

and

$(r - 2ap^* - \frac{a_1 q_2^*}{(1+Mp^*)^2})(\alpha_2 + 2\alpha_3 q_2^*) + r - 2ap^* \frac{a_2 P^*}{1+Mp^*} > (r - 2ap^* - \frac{a_1 q_2^*}{(1+Mp^*)^2}) r_1 e^{-d_1 \tau_1}$, then, $r^2 - q^2 > 0$. we can state the following theorem:

Theorem 5.2. *Suppose* $(B1)$, $(B2)$, *and* $r > 2ap^* + \frac{a_1 q_2^*}{(1+Mp^*)^2}$ *holds and* $\tau \in [0, \tau_0^+)$. *Then the interior equilibrium point* $E^*(p^*, q_1^*, q_2^*)$ *is locally asymptotically stable when* $\tau_1 \in [0, \bar{\tau_1^*})$.

To prove the above theorem, Again, let $\tau_1 \in [0, \bar{\tau_{11}})$ and $\tau > 0$. Suppose, the characteristic equation (5.1) has purely imaginary root $\iota\omega(\omega > 0)$, then we can obtain:

$$-\omega^2 + A_2 + B_2 \cos \omega\tau_1 - B_1 \omega \sin \omega\tau_1 + C_2 \cos \omega\tau - C_1 \omega \sin \omega\tau = 0 \tag{5.8}$$

$$\omega A_1 + B_1 \omega \cos \omega\tau_1 + B_2 \sin \omega\tau_1 + C_1 \omega \cos \omega\tau + C_2 \sin \omega\tau = 0 \tag{5.9}$$

From (5.8) and (5.9), we have,

$$\omega^4 + \bar{A}\omega^3 + \bar{B}\omega^2 + \bar{C}\omega + \bar{D} = 0 \tag{5.10}$$

where,

$\bar{A} = 2C_1 \sin \omega\tau + 2B_1 \sin \omega\tau_1$,

$\bar{B} = B_1^2 + C_1^2 - 2A_2 - 2B_2 \cos \omega\tau_1 - C_2 \cos \omega\tau + 2B_1 C_1 \sin \omega\tau_1 \sin \omega\tau_2 + 2A_1 B_1 \cos \omega\tau + 2A_1 C_1 \cos \omega\tau_2 + 2B_1 C_1 \cos \omega\tau_1 \cos \omega\tau$

$\bar{C} = 2A_2 B_1 \sin \omega\tau_1 - 2A_2 C_1 \sin \omega\tau - 2B_2 C_1 \cos \omega\tau_1 \sin \omega\tau_2 - 2B_1 C_2 \sin \omega\tau_1 \cos \omega\tau + 2A_1 B_1 \sin \omega\tau_1 + 2A_1 C_2 \sin \omega\tau + 2B_1 C_2 \cos \omega\tau_1 \sin \omega\tau_2 + 2B_2 C_1 \sin \omega\tau_1 \cos \omega\tau$

$\bar{D} = A_2^2 + B_2^2 + 2A_2B_2 \cos\omega\tau_1 + 2A_2C_2 \cos\omega\tau_2 + 2B_2C_2 \cos\omega\tau_1 \cos\omega\tau + 2B_2C_2 \sin\omega\tau_1 \sin\omega\tau$

Now, we define
$F(\omega) = \omega^4 + \bar{A}\omega^3 + \bar{B}\omega^2 + \bar{C}\omega + \bar{D}$ and assume that $Ma_1\alpha_2 + 2\alpha_3 Mr > Ma_1r_1e^{-d_1\tau_1} + \alpha_3 a + a_1a_2$ hold. It is easy to check that$F(0) < 0$ and $F(\infty) = \infty$. We can obtain that equation (5.10) has finite positive roots $\omega_1, \omega_2, ...\omega_k$. For every fixed $\omega_i, i = 1, 2, 3, ..., k$, there exists a sequence $(\tau_i^j(\tau_1)|j = 1, 2, 3.....)$, such that (5.10) holds. Let $\tau_0(\tau_1) = min(\tau_i^j(\tau_1)|i = 1, 2......., k; j = 1, 2, 3.....; \tau_1 \in [0, \bar{\tau_1^*}))$. When $\tau_1 = \tau_{10}$, then (5.1) has a pair of purely imaginary roots $\pm\iota\omega^0$. The critical value of τ is given by:

$$\tau_0(\tau_1) = \frac{1}{\omega^0} cos^{-1}(\frac{A}{B})$$

where $A = (-\omega^0)^2(-C_2 + A_1C_1 + B_1C_1 \cos\omega^0\tau_1 + C_1^2 \cos\omega^0\tau) - \omega^0((-B_1C_2 + B_2C_1) \sin\omega^0\tau_1) - (B_2C_2 \cos\omega^0\tau_1 + C_2^2 \cos\omega^0\tau)$, $B = C_2^2 + C_1^2$

If $\tau_1 = 0$, then $\tau_0 = \tau_0^+$. Now we define $\tau_1^+ = max(\tau_0(\tau_1)|\tau_1\epsilon[0, \bar{\tau_{11}}))$. We can also show that,

$$Re[\frac{d\lambda}{d\tau}]_{\tau_1 = \tau_1^+} \neq 0 \quad (5.11)$$

Thus from the above discussion, the theorem take the following form:

Theorem 5.3. *(i) The interior equilibrium E^* is locally asymptotically stable for all $\tau \geq 0$ and $\tau_1 \in [0, \tau_1^+)$.*

(ii) the interior equilibrium E^* is unstable when $\tau \geq \tau^+$ and the system undergoes a hopf bifurcation at E^* when $\tau_1 = \tau_1^+$ for all $\tau \geq 0$.

Hence, we have proved the local stability of the interior equilibrium point.

6. Permanence of the Model

In this section, we will discuss the permanence of the system.

Theorem 6.1. *The system (2.1-2.3) is permanent in the following set*

$$\Omega = \{m_1 \leq p \leq M_1, m_2 \leq q_1 \leq M_2, m_3 \leq q_2 \leq M_3\}$$

where,

$$M_1 = \frac{r}{a}, M_2 = \frac{r_1 e^{-d_1\tau_1} + \frac{a_2}{a}}{\alpha_3}, M_3 = \frac{r_1 M_2}{\alpha_1}, m_1 = \frac{M_2}{M_1},$$

$$m_2 = \frac{a_2 S - \alpha_2}{\alpha_3}, m_3 = \frac{r_1 m_2 (1 - e^{-d_1\tau_1})}{\alpha_1}$$

proof. We consider equation (2.1) of the model $i.e$

$$\frac{dp(t)}{dt} = p(t)(r - ap(t) - \frac{a_1 q_2(t)}{1 + Mp(t)})$$

$$\frac{dp(t)}{dt} \leq p(t)r - ap^2(t)$$

Then by comparison argument, it follows that,

$$\limsup_{t\to\infty} p(t) \leq \frac{r}{a}$$

Thus, for every $\epsilon > 0$ sufficiently small, $\ni T_{11} > 0$ in such a manner that if $t \geq T_{11}$, then

$$p(t) \leq \frac{r}{a} + \epsilon = M_1 \tag{6.1}$$

Now on considering equation (2.3), we get,

$$\frac{dq_2}{dt} = -\alpha_2 q_2(t) + r_1 e^{-d_1\tau_1} q_2(t-\tau_1) + \frac{a_2 p(t-\tau) q_2(t-\tau)}{1 + Mp(t-\tau)} - \alpha_3(q_2^2(t))$$

$$\frac{dq_2(t)}{dt} \leq r_1 e^{-d_1\tau_1} q_2(t) + \frac{a_2 p q_2(t)}{1+Mp} - \alpha_3(q_2(t))^2$$

Then by comparison argument, it follows that,

$$\lim sup_{t\to\infty} q_2(t) \leq \frac{r_1 e^{-d_1\tau_1} + \frac{a_2}{a} r}{\alpha_3}$$

thus, for every $\epsilon > 0$ sufficiently small, $\ni T_{12} > T_{11} + \tau$ in such a manner that if $t \geq T_{12}$,

$$q_2(t) \leq \frac{r_1 e^{-d_1\tau_1} + \frac{a_2}{a} r}{\alpha_3} + \epsilon = M_2 \tag{6.2}$$

Now on using equation (2.2) of the model, we get

$$\frac{dq_1}{dt} = r_1 q_2(t) - \alpha_1 q_1(t) - r_1 e^{-d_1\tau_1} q_2(t-\tau_1)$$

$$\frac{dq_1}{dt} \leq r_1 q_2(t) - \alpha_1 q_1(t)$$

On solving the differential equation we get,

$$\lim sup_{t\to\infty} q_1(t) \leq \frac{r_1 M_2}{\alpha_1}$$

Therefore, for every $\epsilon > 0$ sufficiently small, $\ni T_1 = T_{12} + \tau$ in such a manner that if $t \geq T_1$,

$$q_1(t) \leq \frac{r_1 M_2}{\alpha_1} + \epsilon = M_3 \tag{6.3}$$

Again from the equation (2.1) of the system we derive that,

$$\frac{dp(t)}{dt} = p(t)(r - ap(t) - \frac{a_1 q_2(t)}{1 + Mp(t)})$$

$$\frac{dp}{dt} \geq -(apM_1 + \frac{a_1 M_2}{p})$$

$$\liminf_{t\to\infty} p(t) \geq \frac{-M_2}{M_1}$$

for every $\epsilon > 0$, arbitrarily small, their exist $T_2 > T_1$ such that $t > T_2$

$$p(t) \geq \frac{-M_2}{M_1} - \epsilon = m_1 \tag{6.4}$$

From equation (2.3) it follows that,

$$\frac{dq_2(t)}{dt} = -\alpha_2 q_2(t) + r_1 e^{-d_1\tau_1} q_2(t - \tau_1) + \frac{a_2 p(t-\tau) q_2(t-\tau)}{1 + Mp(t-\tau)} - \alpha_3 q_2^2(t)$$

$$\frac{dq_2(t)}{dt} \geq -\alpha_2 q_2(t) + \frac{a_2 m_1 q_2(t-\tau)}{1 + Mm_1)} - \alpha_3 (q_2^2(t)$$

$$\frac{dq_2(t)}{dt} \geq a_2 S q_2(t-\tau) - \alpha_2 q_2(t) - \alpha_3 q_2^2(t)$$

Now, by Comparison argument we get,

$$\liminf_{t\to\infty} q_2(t) \geq \frac{a_2S - \alpha_2}{\alpha_3}$$

for every $\epsilon > 0$, arbitrarily small, their exist $T_3 > T_2 + \tau$ such that $t > T_3$,

$$q_2(t) \geq \frac{a_2S - \alpha_2}{\alpha_3} - \epsilon = m_2 \tag{6.5}$$

It follows from the equation (2.2) that,

$$\frac{dq_1}{dt} = r_1q_2(t) - \alpha_1q_1(t) - r_1e^{-d_1\tau_1}q_2(t - \tau_1)$$

$$\frac{dq_1}{dt} \geq r_1m_2 - \alpha_1q_1(t) - r_1e^{-d_1\tau_1}m_2$$

On solving the differential equation we get,

$$\liminf_{t\to\infty} q_1(t) \geq \frac{r_1m_2(1 - e^{-d_1\tau_1})}{\alpha_1} \tag{6.6}$$

for every $\epsilon > 0$, arbitrarily small, their exist $T_4 > T_3 + \tau$ such that $t > T_4$,

$$q_1(t) \geq \frac{r_1m_2(1 - e^{-d_1\tau_1})}{\alpha_1} - \epsilon = m_3 \tag{6.7}$$

Therefore, we have obtained the greatest lower bound and the least upper bounds of p, q_1, q_2. This completes the permanence of the system.

7. Numerical Simulation

In this section, we have presented numerical simulation of the system with different hypothetical values as given in the table below to validate our theoretical results. The values are

Parameter	r	a	a_1	r_1	α_1	d_1	α_2	a_2	α_3	M
Values	1	0.2	0.6	0.7	0.1	0.2	0.2	2	0.06	20

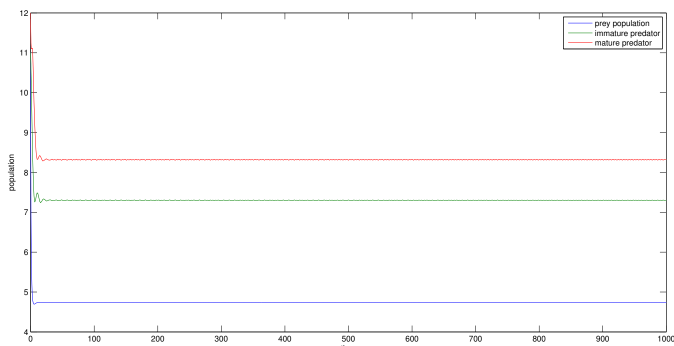

Figure 1. The interior equilibrium E^* is asymptotically stable when $\tau_1 = 3$ and $\tau = 0.018$

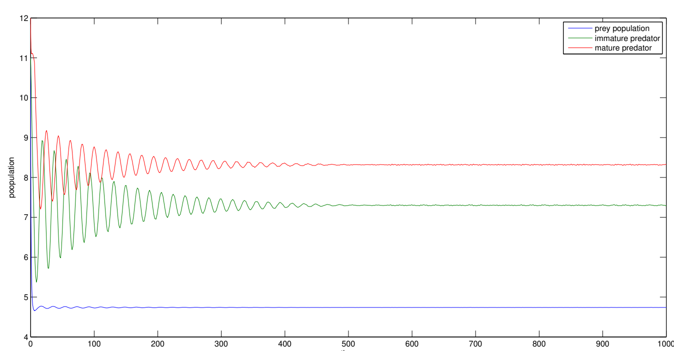

Figure 2. Behaviour of the system at $\tau_1 = 6$ and $\tau = 0.018$

Here all the parametric values are taken per week. We have observed that E^* is asymptotically stable for maturation delay $\tau_1 = 3$ and gestation delay $\tau = 0.018$(Figure 1). As we increase the maturation delay $\tau_1 = 6$, keeping the gestation delay τ fixed for the same set of initial conditions, we observe a slight periodic solution in the system but it stabilizes in long run(Figure 2). Now we further increase maturation delay $\tau_1 = 7$ and we observe the existence of Hopf bifurcation in the

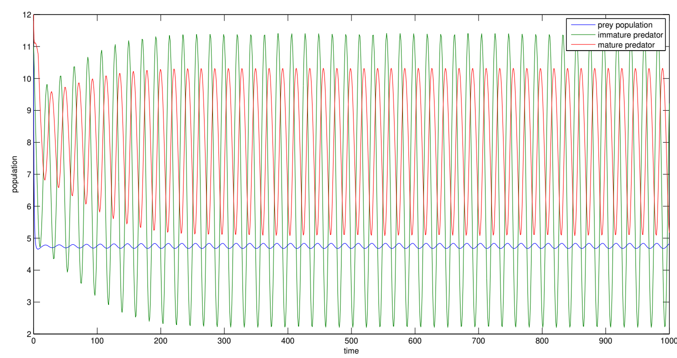

FIGURE 3. At $\tau_1 = 7$ and $\tau = 0.018$ the interior equilibrium E^* looses its stability

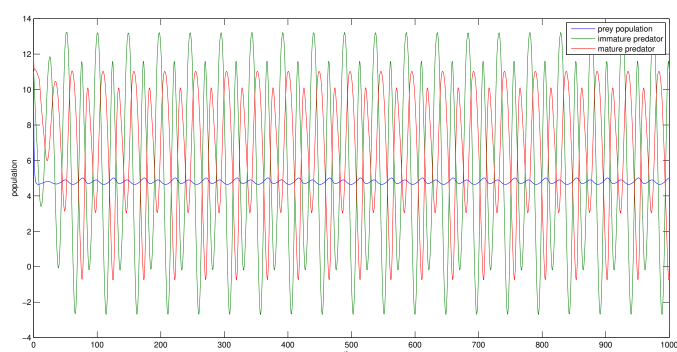

FIGURE 4. Bifurcating periodic solution occurs at $\tau_1 = 9$ and $\tau = 0.018$

system which validates our analytical results of Case 3 . Thus, it concludes that E^* is locally asymptotically stable if $0 < \tau_1 < \tau_1^* = 7$ and has Hopf bifurcation when $\tau_1 > \tau_1^* = 7$(Figure 3, 4). Now, we keep maturation delay in the stable region as per case 3 i.e $\tau_1 = 3$ and modulate gestation delay. At $\tau = 0.01$, we observe that E^* is

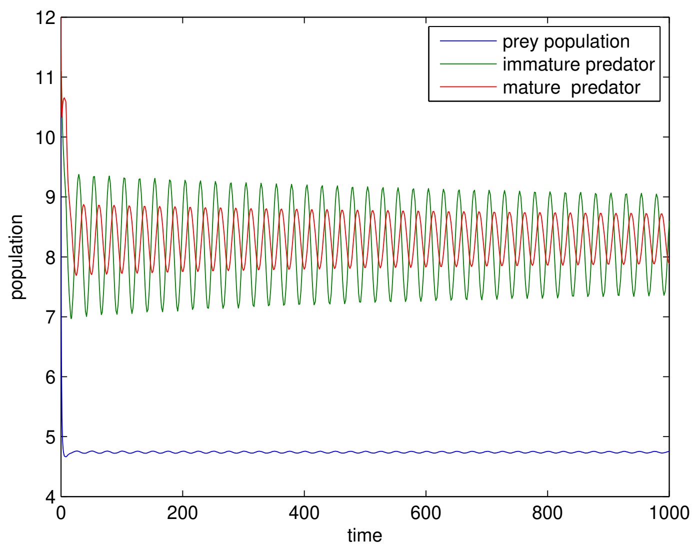

FIGURE 5. Existence of the Hopf Bifurcation for $\tau_1 = 3$ and $\tau = 0.01$

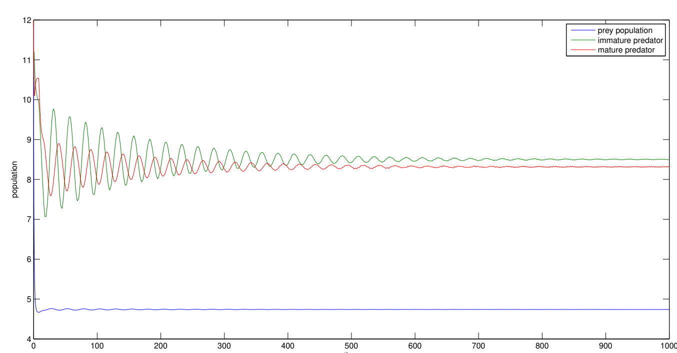

FIGURE 6. Behaviour of the system at $\tau_1 = 3$ and $\tau = 0.017$

aysmptotically stable(Figure 5,6). As we increase the gestation delay $\tau = .017$, keeping the maturation delay fixed, still we get a stabilized system but as we increase $\tau = 0.018$, we obtain periodic solution for the system. Thus, it verifies our analytical results of Case 4 that if $\tau_1 > 0$ and $\tau < \tau^* = .018$ then E^* is stable and if $\tau \geq \tau^* = .018$, Hopf bifurcation occurs(Figure 7).

8. Conclusion

To sum up, in this paper, we have studied the effect of maturation delay and gestation delay on the dynamics of prey predator model with stage structuring in predator. We have studied the existence of boundary and interior equilibrium points and the positivity of the model.Next, we have discussed the local stability of the trivial, boundary and interior equilibrium point. We have discussed three cases for the local stability analysis of E^* :

Case 1. $\tau_1 = \tau = 0$

Case 2. $\tau_1 = 0, \tau > 0$

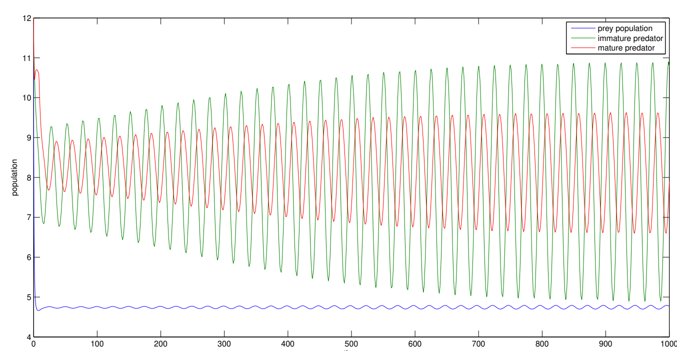

FIGURE 7. Behaviour of the system when $\tau = 0.018$

Case 3, $\tau_1 > 0, \tau = 0$

Case 4. $\tau_1 > 0, \tau > 0$

From the local stability of Case 2,3 and Case 4, we have obtained the bifurcating parameter τ_{10} and $\tau_0 = \tau$. We have shown that E^* is asymptotically stable if the maturation delay $0 < \tau_1 < \tau_1^*$ and has Hopf bifurcation when $\tau_1 > \tau_1^*$. Further, we have also proved that if $\tau_1 > 0$ and $\tau < \tau^*$ then E^* is asymptotically stable and if $\tau \geq \tau^*$ then, Hopf bifurcation occurs. The permanence of the system using comparison theorem is also obtained by evaluating the least upper bound and greatest lower bound of all the three population (p, q_1, q_2). At last, numerical example has been given to validate our theoretical results. From the above analysis, we conclude that as we increase the maturation delay , the prey population gradually decreases and may extinct in longer run and further, increase in gestation delay destabilizes the system. Hence, in the case of pest-natural enemy model, it is worth increasing the maturation delay as it would help in eradication of the pest population.

Appendix A. Known Results

Lemma A.1. *Let's consider the following equation:*

$$\frac{dP(t)}{dt} = xP(t-T) - yP(t) - zP^2(t)$$

where x, y, z *and* T *are positive constants ,* $P(t) > 0$ *for* $t \in [-T, 0]$.*,*
then we have (i) if $x > y$ *then* $\lim_{t\to\infty} P(t) = \frac{x-y}{z}$
(ii) if $x < y$ *then* $\lim_{t\to\infty} P(t) = 0$

We know, transcendental polynomial equation of second degree is

$$\lambda^2 + p\lambda + r + (s\lambda + q)e^{-\lambda\tau_1} = 0 \tag{A.1}$$

From the above equation we assume:

$(L1)p + s > 0$

$(L2)q + r > 0$

$(L3)$ either $(s^2 - p^2 - 2r)^2 < 0$ and $(r^2 - q^2) > 0$ or $(s^2 - p^2 - 2r)^2 < 4(r^2 - q^2)$

$(L4)$ either $(s^2-p^2-2r)^2 = 4(r^2-q^2)$ and $s^2-p^2-2r > 0$ or $r^2-q^2 < 0$

$(L5)$ $r^2 - q^2 > 0$, $s^2 - p^2 - 2r > 0$ and $(s^2 - p^2 - 2r)^2 > 4(r^2 -^2)$.

Lemma A.2. *For equation (A.1), we have,*

(i) If $(L1) - (L3)$ hold,then all roots of equation (A.1) have negative real parts for all $\tau \geq 0$.

(ii) If $(L1), (L2)$ and $(L4)$ hold and $\tau = \tau_j^+$, then equation (A.1) has a pair of purely imaginary roots $\pm\iota\omega_+$. When $\tau = \tau_j^+$, then all roots of (5) except $\pm\iota\omega_+$ have negative real parts.

(iii) If $(L1), (L2)$ and $(L5)$ hold and $\tau = \tau_j^+$ ($\tau = \tau_j^-$, respectively), then equation (A.1) has a pair of purely imaginary roots $\pm\iota\omega_+$ ($\pm\iota\omega_+$, respectively). Furthermore, when $\tau = \tau_j^+$ (τ_j^-,respectively), then all roots of (A.1) except $\pm\iota\omega_+$ ($\pm\iota\omega_-$, respectively) have negative real parts.

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