THE VERTEX DETOUR MONOPHONIC NUMBER OF A GRAPH

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Abstract. In this paper we determine bounds for $x$-detour monophonic number and characterize graphs which realize these bounds. A connected graph of order $p$ with vertex detour monophonic numbers either $p-1$ or $p-2$ for every vertex is characterized. It is shown that for each triple $a, b$ and $p$ of integers with $1 \leq a \leq b \leq p - 4$, there is a connected graph $G$ of order $p$ such that $x$-monophonic number is $a$ and $x$-detour monophonic number is $b$ for some vertex $x$ in $G$. Also, for integers $a, b$ and $p$ with $1 \leq a \leq p - b$ and $b \geq 2$, there is a connected graph $G$ of order $p$ such that $x$-detour monophonic number is $a$ and monophonic eccentricity of $x$ is $b$ for some vertex $x$ in $G$.

1. Introduction

By a graph $G = (V, E)$ we mean a finite undirected connected graph without loops or multiple edges. The order and size of $G$ are denoted by $p$ and $q$ respectively. For basic graph theoretic terminology we refer to Harary [4]. For vertices $x$ and $y$ in a connected graph $G$, the distance

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$d(x, y)$ is the length of a shortest $x$-$y$ path in $G$. An $x$-$y$ path of length $d(x, y)$ is called an $x$-$y$ geodesic. The neighborhood of a vertex $v$ is the set $N(v)$ consisting of all vertices $u$ which are adjacent with $v$. The closed neighborhood of a vertex $v$ is the set $N[v] = N(u) \cup \{v\}$. A vertex $v$ is a simplicial vertex of $G$ if the subgraph induced by its neighbors is complete. A nonseparable graph is connected, nontrivial, and has no cut vertices. A block of a graph is a maximal nonseparable subgraph. A caterpillar is a tree for which the removal of all the end vertices gives a path. The closed interval $I[x, y]$ consists of all vertices lying on some $x$-$y$ geodesic of $G$, while for $S \subseteq V$, $I[S] = \bigcup_{x,y \in S} I[x, y]$. A set $S$ of vertices is a geodetic set if $I[S] = V$, and the minimum cardinality of a geodetic set is the geodetic number $g(G)$. A geodetic set of cardinality $g(G)$ is called a $g$-set of $G$. The geodetic number of a graph was introduced in [1, 5] and further studied in [2, 3]. The concept of vertex geodomination number was introduced in [6] and further studied in [7]. Let $x$ be a vertex of a connected graph $G$. A set $S$ of vertices of $G$ is an $x$-geodominating set of $G$ if each vertex $v$ of $G$ lies on an $x$-$y$ geodesic in $G$ for some element $y$ in $S$. The minimum cardinality of an $x$-geodominating set of $G$ is defined as the $x$-geodomination number of $G$ and is denoted by $g_x(G)$. An $x$-geodominating set of cardinality $g_x(G)$ is called a $g_x$-set of $G$.

A chord of a path $P$ is an edge joining two non-adjacent vertices of $P$. A path $P$ is called monophonic if it is a chordless path. A longest $x$-$y$ monophonic path $P$ is called an $x$-$y$ detour monophonic path. The closed interval $I_m[x, y]$ consists of all vertices lying on some $x$-$y$ monophonic path of $G$. For any two vertices $u$ and $v$ in a connected graph $G$, the monophonic distance $d_m(u, v)$ from $u$ to $v$ is defined as the length of a longest $u$-$v$ monophonic path in $G$. The monophonic
eccentricity $e_m(v)$ of a vertex $v$ in $G$ is $e_m(v) = \max\{d_m(v,u) : u \in V(G)\}$. The monophonic radius, $\text{rad}_m G$ of $G$ is $\text{rad}_m G = \min\{e_m(v) : v \in V(G)\}$ and the monophonic diameter, $\text{diam}_m G$ of $G$ is $\text{diam}_m G = \max\{e_m(v) : v \in V(G)\}$. The monophonic distance was introduced and studied in [8].

The concept of vertex monophonic number was introduced and studied in [9]. Let $x$ be a vertex of a connected graph $G$. A set $S$ of vertices of $G$ is an $x$-monophonic set of $G$ if each vertex $v$ of $G$ lies on an $x$-$y$ monophonic path in $G$ for some element $y$ in $S$. The minimum cardinality of an $x$-monophonic set of $G$ is defined as the $x$-monophonic number of $G$ and is denoted by $m_x(G)$. An $x$-monophonic set of cardinality $m_x(G)$ is called a $m_x$-set of $G$. The following theorems will be used in the sequel.

**Theorem 1.1.** [4] Let $v$ be a vertex of a connected graph $G$. The following statements are equivalent:

1. $v$ is a cut vertex of $G$.
2. There exist $u$ and $w$ distinct from $v$ such that $v$ is on every $u$-$w$ path.
3. There exists a partition of the set of vertices $V - \{v\}$ into subsets $U$ and $W$ such that for any vertices $u \in U$ and $w \in W$, the vertex $v$ is on every $u$-$w$ path.

**Theorem 1.2.** [4] Every non-trivial connected graph has at least two vertices which are not cut vertices.

**Theorem 1.3.** [4] Let $G$ be a connected graph with at least three vertices. The following statements are equivalent:

1. $G$ is a block.
2. Every two vertices of $G$ lie on a common cycle.
Theorem 1.4. [9] Let $x$ be any vertex of a connected graph $G$. Then every simplicial vertex of $G$ other than the vertex $x$ (whether $x$ is simplicial vertex or not) belongs to every $m_x$-set.

Throughout this paper $G$ denotes a connected graph with at least two vertices.

2. Vertex detour monophonic number

Definition 2.1. Let $x$ be a vertex of a connected graph $G$. A set $S$ of vertices of $G$ is an $x$-detour monophonic set if each vertex $u$ of $G$ lies on an $x$-$y$ detour monoponic path in $G$ for some $y$ in $S$. The minimum cardinality of an $x$-detour monophonic set of $G$ is defined as the $x$-detour monophonic number of $G$ and is denoted by $dm_x(G)$. An $x$-detour monophonic set of cardinality $dm_x(G)$ is called a $dm_x$-set of $G$.

We observe that for any vertex $x$ in $G$, $x$ does not belong to any $dm_x$-set of $G$.

Example 2.2. For the graph $G$ given in Figure 2.1, the minimum vertex monophonic sets, the vertex monophonic numbers, the minimum vertex detour monophonic sets and the vertex detour monophonic numbers are given in Table 2.1.
Table 2.1

<table>
<thead>
<tr>
<th>vertex</th>
<th>minimum vertex monophonic sets</th>
<th>vertex monophonic number</th>
<th>minimum vertex detour monophonic sets</th>
<th>vertex detour monophonic number</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t$</td>
<td>${z, w}$</td>
<td>2</td>
<td>${z, w}$</td>
<td>2</td>
</tr>
<tr>
<td>$y$</td>
<td>${z, w}$</td>
<td>2</td>
<td>${w, z, t}$, ${w, z, u}$</td>
<td>3</td>
</tr>
<tr>
<td>$z$</td>
<td>${w}$</td>
<td>1</td>
<td>${u, w}$, ${w, y}$</td>
<td>2</td>
</tr>
<tr>
<td>$u$</td>
<td>${z, w, y}$</td>
<td>3</td>
<td>${w, z, y}$</td>
<td>3</td>
</tr>
<tr>
<td>$v$</td>
<td>${z, w}$</td>
<td>2</td>
<td>${w, t, z}$, ${w, u, z}$</td>
<td>3</td>
</tr>
<tr>
<td>$w$</td>
<td>${z}$</td>
<td>1</td>
<td>${t, z}$, ${z, u}$</td>
<td>2</td>
</tr>
</tbody>
</table>

**Theorem 2.3.** Let $x$ be a vertex of a connected graph $G$.

(1) Every simplicial vertex of $G$ other than the vertex $x$ (whether $x$ is simplicial vertex or not) belongs to every $dm_x$-set of $G$.

(2) No cut vertex of $G$ belongs to any $dm_x$-set of $G$.

**Proof.** (1) Let $x$ be a vertex of $G$. Then $x$ does not belong to any $dm_x$-set of $G$. Let $u \neq x$ be a simplicial vertex and $S_x$ a $dm_x$-set of $G$. Suppose that $u \notin S_x$. Then $u$ is an internal vertex of an $x$-$y$ detour monophonic path, say $P$, for some $y \in S_x$. Let $v$ and $w$ be the neighbors of $u$ on $P$. Then $v$ and $w$ are not adjacent and so $u$ is not a simplicial vertex, which is a contradiction.

(2) Let $y$ be a cut vertex of $G$. Then by Theorem 1.1, there exists a partition of the set of vertices $V - \{y\}$ into two subsets $U$ and $W$ such that for any pair of vertices $u \in U$ and $w \in W$, the vertex $y$ is on every $u$-$w$ path. Hence, if $x \in U$, then for any vertex $w$ in $W$, $y$ lies on every $x$-$w$ path so that $y$ is an internal vertex of an $x$-$w$ detour monophonic path. Let $S_x$ be any $dm_x$-set of $G$. Suppose that $S_x \cap W = \emptyset$. Then for any $w_1 \in W$, there exists an element $z$ in $S_x$ such that $w_1$ lies in some $x$-$z$ detour monophonic path $P : x = z_0, z_1, \ldots, w_1, \ldots, z_n = z$ in $G$. 


Now, the \( x-w_1 \) subpath of \( P \) and \( w_1-z \) subpath of \( P \) both contain \( y \) so that \( P \) is not a path in \( G \), which is a contradiction. Hence \( S_x \cap W \neq \emptyset \).

Let \( w_2 \in S_x \cap W \). Then \( y \) is an internal vertex of an \( x-w_2 \) detour monophonic path. If \( y \in S_x \), let \( S = S_x - \{y\} \). It is clear that every vertex that lies on an \( x-y \) detour monophonic path also lies on an \( x-w_2 \) detour monophonic path. Hence it follows that \( S \) is an \( x \)-detour monophonic set of \( G \), which contradicts the fact that \( S_x \) is a minimum \( x \)-detour monophonic set of \( G \). Thus \( y \) does not belong to any \( dm_x \)-set. Similarly, if \( x \in W \), \( y \) does not belong to any \( dm_x \)-set. If \( x = y \), then obviously \( y \) does not belong to any \( dm_x \)-set. \( \Box \)

**Note 2.4.** Even if \( x \) is a simplicial vertex of \( G \), \( x \) does not belong to any \( dm_x \)-set.

**Theorem 2.5.** For any non-trivial tree \( T \) with \( k \) end vertices, \( dm_x(T) = k - 1 \) or \( k \) according as \( x \) is an end-vertex or not. In fact, if \( W \) is the set of all end-vertices of \( T \), then \( W - \{x\} \) is the unique \( dm_x \)-set of \( T \).

*Proof.* Let \( W \) be the set of all end-vertices of \( T \). It follows from Theorem 2.3 and Note 2.4 that \( W - \{x\} \) is the unique \( dm_x \)-set of \( T \) for any end-vertex \( x \) in \( T \) and \( W \) is the unique \( dm_x \)-set of \( T \) for any cut vertex \( x \) in \( T \). Thus \( W - \{x\} \) is the unique \( dm_x \)-set of \( T \). \( \Box \)

**Theorem 2.6.** For any vertex \( x \) in a connected graph \( G \) of order \( p \), 
\[ 1 \leq dm_x(G) \leq p - 1. \]

*Proof.* It is clear from the definition of \( dm_x \)-set that \( dm_x(G) \geq 1 \). Also, since the vertex \( x \) does not belong to any \( dm_x \)-set, it follows that \( dm_x(G) \leq p - 1. \) \( \Box \)
Remark 2.7. The bounds for $dm_x(G)$ in Theorem 2.6 are sharp, for example $dm_x(C_{2n}) = 1$ for any vertex $x$ in $C_{2n}$, and $dm_x(K_p) = p - 1$ for any vertex $x$ in $K_p$.

Now we proceed to characterize graphs for which the bounds in Theorem 2.6 are attained.

**Definition 2.8.** Let $x$ be any vertex in a connected graph $G$. A vertex $y$ in $G$ is said to be an $x$-detour monophonic superior vertex if for any vertex $z$ with $d_m(x, y) < d_m(x, z)$, $z$ lies on an $x-y$ detour monophonic path.

**Table 2.2**

<table>
<thead>
<tr>
<th>vertex detour monophonic superior vertices</th>
<th>$t$</th>
<th>$y$</th>
<th>$z$</th>
<th>$u$</th>
<th>$v$</th>
<th>$w$</th>
</tr>
</thead>
<tbody>
<tr>
<td>vertex detour monophonic superior vertices</td>
<td>${w}$</td>
<td>${z}$</td>
<td>${w}$</td>
<td>${y, w}$</td>
<td>${z}$</td>
<td>${z}$</td>
</tr>
</tbody>
</table>

**Example 2.9.** For the graph $G$ given in Figure 2.1, the vertex detour monophonic superior vertices are given in Table 2.2.

We give below a property related with monophonic eccentric vertex of $x$ and $x$-detour monophonic superior vertex in a graph $G$.

**Theorem 2.10.** Let $x$ be any vertex in $G$. Then every monophonic eccentric vertex of $x$ is an $x$-detour monophonic superior vertex.

**Proof.** Let $y$ be a monophonic eccentric vertex of $x$ so that $e_m(x) = d_m(x, y)$. If $y$ is not an $x$-detour monophonic superior vertex, then there exists a vertex $z$ in $G$ such that $d_m(x, y) < d_m(x, z)$ and $z$ does not lie on any $x-y$ detour monophonic path and hence $e_m(x) < d_m(x, z)$, which is a contradiction. □
Note 2.11. The converse of Theorem 2.10 is not true. For the even cycle $C_{2n}(n \geq 3)$, the eccentric vertex of $x$ is an $x$-detour monophonic superior vertex but it is not a monophonic eccentric vertex of $x$.

**Theorem 2.12.** Let $G$ be a connected graph. For a vertex $x$ in $G$, $dm_x(G) = 1$ if and only if there exists an $x$-detour monophonic superior vertex $y$ in $G$ such that every vertex of $G$ is on an $x - y$ detour monophonic path.

*Proof.* Let $dm_x(G) = 1$ and $S = \{y\}$ be a $dm_x$-set of $G$. If $y$ is not an $x$-detour monophonic superior vertex, then there is a vertex $z$ in $G$ with $d_m(x, y) < d_m(x, z)$ and $z$ does not lie on any $x - y$ detour monophonic path. Thus $S$ is not a $dm_x$-set of $G$, which is a contradiction. The converse is clear from the definition. \qed

**Theorem 2.13.** For any vertex $x$ in a connected graph $G$ of order $p$, $dm_x(G) = p - 1$ if and only if $\deg x = p - 1$.

*Proof.* Let $x$ be any vertex in a connected graph $G$ of order $p$. Let $dm_x(G) = p - 1$. Suppose that $\deg x < p - 1$. Then there exists a vertex $u$ in $G$ which is not adjacent to $x$. Since $G$ is connected, there is a detour monophonic path from $x$ to $u$, say $P$, with length greater than or equal to 2. It is clear that $(V(G) - V(P)) \cup \{u\}$ is an $x$-detour monophonic set of $G$ and hence $dm_x(G) \leq p - 2$, which is a contradiction. Conversely, if $\deg x = p - 1$, then all other vertices of $G$ are adjacent to $x$ and hence all these vertices form the $dm_x$-set. Thus $dm_x(G) = p - 1$. \qed

**Corollary 2.14.** A graph $G$ is complete if and only if $dm_x(G) = p - 1$ for every vertex $x$ in $G$. 

Theorem 2.15. Let $G$ be a connected graph. Then $G = K_1 + \cup m_j K_j$ if and only if $dm_x(G) = p - 1$ or $p - 2$ for any vertex $x$ in $G$.

Proof. Let $G = K_1 + \cup m_j K_j$. Then $G$ has at most one cut vertex. If $G$ has no cut vertex, then $G = K_p$ and so by Corollary 2.14, $dm_x(G) = p - 1$ for every vertex $x$ in $G$. Suppose that $G$ has exactly one cut vertex. Then all the remaining vertices are simplicial and hence by Theorem 2.3, $dm_x(G) = p - 1$ or $p - 2$ for any vertex $x$ in $G$.

Conversely, suppose that $dm_x(G) = p - 1$ or $p - 2$ for any vertex $x$ in $G$. If $p = 2$, then $G = K_2 = K_1 + K_1$. If $p \geq 3$, then by Theorem 1.2, there exists a vertex $x$, which is not a cut vertex of $G$. If $G$ has two or more cut vertices, then by Theorem 2.3, $dm_x(G) \leq p - 3$, which is a contradiction. Thus, the number of cut vertices $k$ of $G$ is at most one.

Case 1: $k = 0$. Then the graph $G$ is a block. If $p = 3$, $G = K_3 = K_1 + K_2$. For $p \geq 4$, we claim that $G$ is complete. If $G$ is not complete, then there exist two vertices $x$ and $y$ in $G$ such that $d(x, y) \geq 2$. By Theorem 1.3, $x$ and $y$ lie on a common cycle and hence $x$ and $y$ lie on a smallest cycle $C : x, x_1, \ldots, y, \ldots, x_n, x$ of length at least 4. If $d_m(x, y) = 2$, then $V(G) - \{x, x_1, x_n\}$ is an $x$-detour monophonic set of $G$ and so $dm_x(G) \leq p - 3$, which is a contradiction to the assumption. If $d_m(x, y) > 2$, then let $P$ be an $x - y$ detour monophonic path of order at least 4. Clearly $(V(G) - V(P)) \cup \{y\}$ is an $x$-detour monophonic set of $G$ and so $dm_x(G) \leq p - 3$, which is a contradiction. Hence $G$ is the complete graph $K_p$ and so $G = K_1 + K_{p-1}$.

Case 2: $k = 1$. Let $x$ be the cut vertex of $G$. If $p = 3$, then $G = P_3 = K_1 + m_j K_1$, where $\sum m_j = 2$. If $p \geq 4$, we claim that $G = K_1 + \cup m_j K_j$, where $\sum m_j \geq 2$. It is enough to prove that every block of $G$ is complete. Suppose there exists a block $B$, which is not
complete. Let \( u \) and \( v \) be two vertices in \( B \) such that \( d(u, v) \geq 2 \). Then by Theorem 1.3, both \( u \) and \( v \) lie on a common cycle so that \( u \) and \( v \) lie on a smallest cycle of length at least 4. Then as in Case 1, \( dm_u(G) \leq p - 3 \), which is a contradiction. Thus every block of \( G \) is complete so that \( G = K_1 + \cup m_jK_j \), where \( K_1 \) is the vertex \( x \) and \( \sum m_j \geq 2 \).

\[ \square \]

**Theorem 2.16.** Let \( G \) be a connected graph of order \( p \geq 3 \) with exactly one cut vertex. Then \( G = K_1 + \cup m_jK_j \), where \( \sum m_j \geq 2 \) if and only if \( dm_x(G) = p - 1 \) or \( p - 2 \) for any vertex \( x \) in \( G \).

*Proof.* The proof is contained in Theorem 2.15.

Now, Corollary 2.14 and Theorem 2.15 lead to the natural question whether there exists a graph \( G \) for which \( dm_x(G) = p - 2 \) for every vertex \( x \) in \( G \). This is answered in the next theorem.

**Theorem 2.17.** There is no graph \( G \) of order \( p \) with \( dm_x(G) = p - 2 \) for every vertex \( x \) in \( G \).

*Proof.* If \( dm_x(G) = p - 2 \) for every vertex \( x \) in \( G \), then use Theorem 2.15 to get \( G = K_1 + \cup m_jK_j \). If \( x \) is \( K_1 \), then use Theorem 2.13 to get \( dm_x(G) = p - 1 \). But this contradicts the assumption. Thus there is no graph \( G \) with \( dm_x(G) = p - 2 \) for every vertex \( x \) in \( G \).

\[ \square \]

**Theorem 2.18.** For any non-trivial tree \( T \) with monophonic diameter \( d_m \),

\( dm_x(T) = p - d_m \) or \( p - d_m + 1 \) for any vertex \( x \) in \( T \) if and only if \( T \) is a caterpiller.

*Proof.* Let \( T \) be any non-trivial tree. Let \( P : v_0, v_1, \ldots, v_{d_m} \) be a monophonic path of length \( d_m \). Let \( k \) be the number of end vertices of \( T \).
and \( l \) be the number of internal vertices of \( T \) other than \( v_1, \ldots, v_{d_m-1} \). Then \( d_m - 1 + l + k = p \). By Theorem 2.5, \( dm_x(T) = k \) or \( k - 1 \) for any vertex \( x \) in \( T \). Hence \( dm_x(T) = p - d_m - l + 1 \) or \( p - d_m - l \) for any vertex \( x \) in \( T \). Hence \( dm_x(T) = p - d_m + 1 \) or \( p - d_m \) for any vertex \( x \) in \( T \) if and only if \( l = 0 \), if and only if all the internal vertices of \( T \) lie on the monophonic diametral path \( P \), if and only if \( T \) is a caterpillar.

\[ \square \]

**Theorem 2.19.** For any vertex \( x \) in the cycle \( C_n \) \((n \geq 3)\), \( dm_x(C_n) = 1 \) or 2 according as \( n \) is even or odd.

*Proof.* Let \( C_n : u_1, u_2, \ldots, u_n, u_1 \) be the cycle of order \( n \). Let \( x \) be any vertex in \( C_n \), say \( x = u_1 \). If \( n \) is even, then \( S_x = \{u_{\frac{n}{2}+1}\} \) is an \( x \)-detour monophonic set and so \( dm_x(C_n) = 1 \). If \( n \) is odd, then \( S_x = \{u_2, u_3\} \) is a minimum \( x \)-detour monophonic set and so \( dm_x(C_n) = 2 \). \( \square \)

**Theorem 2.20.** Let \( W_n = K_1 + C_{n-1} \) \((n \geq 5)\) be the wheel.

1. If \( n = 5 \), then \( dm_x(W_n) = n - 1 \) or 1 according as \( x \) is \( K_1 \) or \( x \) is in \( C_{n-1} \).
2. If \( n \) is odd and \( n \geq 7 \), then \( dm_x(W_n) = n - 1 \) or 2 according as \( x \) is \( K_1 \) or \( x \) is in \( C_{n-1} \).
3. If \( n \) is even, then \( dm_x(W_n) = n - 1 \) or 3 according as \( x \) is \( K_1 \) or \( x \) is in \( C_{n-1} \).

*Proof.* Let \( C_{n-1} : u_1, u_2, \ldots, u_{n-1}, u_1 \) be a cycle of order \( n-1 \geq 4 \) and \( u \) be the vertex of \( K_1 \). If \( x = u \), then by Theorem 2.13, \( dm_x(W_n) = n - 1 \).

Let \( x \) be any vertex in \( C_{n-1} \), say \( x = u_1 \). If \( n = 5 \), then every vertex of \( W_n \) lies on an \( x-u_3 \) detour monophonic path and so \( \{u_3\} \) is an \( x \)-detour monophonic set of \( W_n \). Hence it follows that \( dm_x(W_n) = 1 \). If
$n \geq 7$ and $n$ is odd, then no 1-element subset of $V(W_n)$ is an $x$-detour monophonic set of $W_n$. Since $\{u_{n+1}, u\}$ is an $x$-detour monophonic set of $W_n$, it follows that $dm_x(W_n) = 2$. If $n$ is even, then neither 1-element nor 2-element subset of $V(W_n)$ will form an $x$-detour monophonic set of $G$. It is clear that $\{u, u_2, u_3\}$ is an $x$-detour monophonic set of $G$ and so $dm_x(G) = 3$. \hfill \square

**Theorem 2.21.** Let $K_{n_1,n_2,\ldots,n_k}$ ($n_i \geq 2$) be a complete $k$-partite graph with partition $(V_1, V_2, \ldots, V_k)$. Then $dm_x(K_{n_1,n_2,\ldots,n_k})$ is $n_i - 1$ according as $x \in V_i$. Moreover, if $n_1 = n_2 = \ldots = n_k = r + 1$, then $dm_x(K_{n_1,n_2,\ldots,n_k}) = r$ for every vertex $x$ in $K_{n_1,n_2,\ldots,n_k}$.

**Proof.** Let $x \in V_i$. Then it is clear that $V_i - \{x\}$ is a minimum $x$-detour monophonic set of $G$ and so $dm_x(K_{n_1,n_2,\ldots,n_k}) = n_i - 1$. \hfill \square

For a vertex $v$ in a graph $G$, the *link* $L(v)$ of $v$ is the subgraph induced by the neighbors of $v$.

**Theorem 2.22.** For every integer $k \geq 1$ and every $k$ graphs $G_1, G_2, \ldots, G_k$, there exists a connected graph $G$ with a unique $dm_x$-set $\{v_1, v_2, \ldots, v_k\}$ for some $x$ in $G$ such that $L(v_i) = G_i$ for $1 \leq i \leq k$.

**Proof.** We construct a graph $G$ with the desired property. For each integer $i$ ($1 \leq i \leq k$), let $F_i = K_2 + G_i$, where $V(K_2) = \{u_i, v_i\}$. Then the graph $G$ is constructed from the graph $F_i$ by adding a new vertex $x$ and the $k$ edges $xu_i$ ($1 \leq i \leq k$). Thus in $G$, $L(v_i) = G_i$ for $1 \leq i \leq k$.

If $k = 1$, then every vertex of $G$ lies on an $x - v_1$ detour monophonic path and hence $S = \{v_1\}$ is the unique minimum $x$-detour monophonic set of $G$. If $k \geq 2$, then $x$ is a cut-vertex of $G$. Now, we show that every $x$-detour monophonic set of $G$ contains an element of every component of $G - \{x\}$. Suppose that there is an $x$-detour monophonic set $S$ of
Let $v \in V(B)$. Since $S$ is an $x$-detour monophonic set, there exists an element $y \in S$ such that $v$ lies in some $x-y$ detour monophonic path $P : x, y_1, \ldots, v, \ldots, y_n = y$ in $G$. Now the $x-v$ subpath of $P$ and $v-y$ subpath of $P$ both contain $x$ and it follows that $P$ is not a path, which is a contradiction. Thus every $x$-detour monophonic set of $G$ contains an element of every component of $G - \{x\}$ and so $dm_x(G) \geq k$.

Let $S = \{v_1, v_2, \ldots, v_k\}$. Clearly $S$ is an $x$-detour monophonic set of $G$ and so $dm_x(G) = k$.

Next, we show that $S$ is a unique $dm_x$-set of $G$. Assume, to the contrary, that $S'$ is a $dm_x$-set of $G$ distinct from $S$. Clearly $S'$ must contain exactly one vertex from each subgraph $F_i$ $(1 \leq i \leq k)$. Since $S \neq S'$, we may assume that $v_1 \notin S'$. Since $u_1$ is a cut vertex of $G$, it follows from Theorem 2.3 that $y \in S$ for some vertex $y$ in $G_1$. Since $d_m(z, u_1) = 1$ for any $z$ in $G_1$, $v_1$ is not an internal vertex of any detour monophonic path from $x$, which is a contradiction.

The graph $G$ constructed in the proof of Theorem 2.22 has a cut-vertex and so is not 2-connected. However, we can extend Theorem 2.22 by modifying the structure of the graph $G$ in the proof of Theorem 2.22 to construct a 2-connected graph with the properties described in Theorem 2.22.

**Theorem 2.23.** For every integer $k \geq 1$ and every $k$ graphs $G_1, G_2, \ldots, G_k$ of order at least two, there exists a 2-connected graph $G$ with a unique $dm_x$-set $\{v_1, v_2, \ldots, v_k\}$ for some $x$ in $G$ such that $L(v_i) = G_i$ for $1 \leq i \leq k$.

**Proof.** For each integer $i$ $(1 \leq i \leq k)$, let $F_i = K_3 + G_i$, where $V(K_3) = \{u_i, v_i, w_i\}$. Then a 2-connected graph $G$ is constructed from the graph
by adding a new vertex $x$ and the $4k$ new edges $xu_i, xw_i, u_iw_i$ and $u_iw_{i+1}$ for $1 \leq i \leq k$, where the subscripts are expressed modulo $k$. For $k = 3$, the graph $G$ is shown in Figure 2.2. Thus in $G$, $L(v_i) = G_i$ for $1 \leq i \leq k$. Now claim that every $x$-detour monophonic set of $G$ contains an element of every $F_i(1 \leq i \leq k)$. Suppose that there is an $x$-detour monophonic set $S$ of $G$ such that $S$ contains no vertex of some $F_i$, say $F_1$. Let $y \in V(G_1)$. Since $S$ is an $x$-detour monophonic set, there exists an element $w \in S$ such that $y$ lies in some $x - w$ detour monophonic path $P$ in $G$. Since $\{u_1, w_1\}$ is a cut set of $G$ and it separate $F_1$ from $G$, the path $P$ contains both the vertices $u_1$ and $w_1$ and so $P$ is not a detour monophonic path, which is a contradiction. Thus every $x$-detour monophonic set of $G$ contains an element of every $F_i(1 \leq i \leq k)$ and hence $dm_x(G) \geq k$. Let $S = \{v_1, v_2, \ldots, v_k\}$. Since every vertex of $F_i$ lies on an $x - v_i$ detour monophonic path, $S$ is an $x$-detour monophonic set of $G$ and so $dm_x(G) = k$.

Next, we show that $S$ is the unique $dm_x$-set of $G$. Assume, to the contrary, that $S'$ is a $dm_x$-set of $G$ distinct from $S$. Clearly $S'$ must contain exactly one vertex from each subgraph $F_i(1 \leq i \leq k)$. Since $S \neq S'$, we may assume that $v_1 \notin S'$. Since $d_m(x,y) \leq 2$ for any $y \in V(F_1) - \{v_1\}$ and $d_m(x,v_1) = 3$, we have $v_1$ is not an internal vertex of any detour monophonic path from $x$, which is a contradiction.
Next we present a theorem, which gives the relation between \( m_x(G) \) and \( dm_x(G) \) of a graph \( G \).

**Theorem 2.24.** Let \( x \) be any vertex in a connected graph \( G \). Then
\[
1 \leq m_x(G) \leq dm_x(G) \leq p - 1.
\]

*Proof.* It is clear from the definition of \( m_x \)-set that \( m_x(G) \geq 1 \). Since every \( x \)-detour monophonic set is an \( x \)-monophonic set, we have \( m_x(G) \leq dm_x(G) \). Also, since the vertex set \( x \) does not belong to any \( dm_x \)-set, it follows that \( dm_x(G) \leq p - 1 \). \( \square \)

The bounds of Theorem 2.24 are sharp. The cycle \( C_{2n}(n \geq 2) \) has \( m_x(C_{2n}) = dm_x(C_{2n}) = 1 \) for every vertex \( x \) in \( C_{2n} \). Also, the non-trivial path \( P_n \) has \( m_x(P_n) = dm_x(P_n) = 1 \) for an end-vertex \( x \) in \( P_n \). For any vertex \( x \) in the complete graph \( K_p(p \geq 2) \), \( m_x(K_p) = dm_x(K_p) = p - 1 \). Also, all the inequalities in Theorem 2.24 are strict. For the graph \( G \) given in Figure 2.1, \( m_y(G) = 2, \ dm_y(G) = 3 \) and \( p = 6 \) so that \( 1 < m_x(G) < dm_x(G) < p \).

**Corollary 2.25.** Let \( x \) be any vertex in a connected graph \( G \). If \( dm_x(G) = 1 \), then \( m_x(G) = 1 \).

*Proof.* This follows from Theorem 2.24. \( \square \)

**Theorem 2.26.** Let \( x \) be any vertex in a connected graph \( G \) of order \( p \). Then \( dm_x(G) \leq p - e_m(x) \).

*Proof.* Let \( x \) be any vertex in \( G \) and \( y \) a monophonic eccentric vertex of \( x \). Then \( d_m(x, y) = e_m(x) \). Let \( P : x = x_0, x_1, x_2, \ldots, x_n = y \) be an \( x-y \) detour monophonic path in \( G \). Let \( S = V(G) - \{x_0, x_1, \ldots, x_{n-1}\} \).

Since each \( x_i (0 \leq i \leq n - 1) \) lies on an \( x-y \) detour monophonic path, \( S \) is an \( x \)-detour monophonic set of \( G \) so that \( dm_x(G) \leq p - e_m(x) \). \( \square \)
Remark 2.27. The bound in Theorem 2.26 is sharp. For any vertex $x$ in the odd cycle $C_{2n+1}$, $e_m(x) = 2n - 1$ and $dm_x(C_{2n+1}) = 2$. Thus $dm_x(C_{2n+1}) = p - e_m(x)$.

Theorem 2.28. For each triple $a, b$ and $p$ of integers with $1 \leq a \leq b \leq p - 4$, there is a connected graph $G$ of order $p$ such that $m_x(G) = a$ and $dm_x(G) = b$ for some vertex $x$ in $G$.

Proof. Case 1. $1 \leq a = b \leq p - 4$. Let $G$ be a tree of order $p$ with $a + 1$ end-vertices. Let $x$ be an end-vertex of $G$. Then $G$ has the desired properties.

Case 2. $1 \leq a < b \leq p - 4$. Let $G$ be a graph obtained from the cycle $C_{p-b+1} : u_1, u_2, \ldots, u_{p-b+1}, u_1$ of order $p - b + 1$ by (i) adding $a$ new vertices $v_1, v_2, \ldots, v_a$ and joining each vertex $v_i (1 \leq i \leq a)$ to $u_{p-b}$; and (ii) adding $b - a - 1$ new vertices $w_1, w_2, \ldots, w_{b-a-1}$ and joining each $w_i (1 \leq i \leq b - a - 1)$ to every vertex $y \in \{u_1, u_2, \ldots, u_{p-b}\}$. The graph $G$ has order $p$ and is shown in Figure 2.3. Let $S = \{v_1, v_2, \ldots, v_a\}$ be the set of all simplicial vertices of $G$. Then by Theorems 1.4 and 2.3, every $x$-monophonic set and every $x$-detour monophonic set of $G$ contains $S$ for the vertex $x = u_1$. It is clear that every vertex of $G$ lies on an $x$-$v_i$ $(1 \leq i \leq a)$ monophonic path so that $m_x(G) = a$. It is clear that $S$ is not an $x$-detour monophonic set of $G$. Also, it is easily seen that $w_i (1 \leq i \leq b - a - 1)$ is not an internal vertex of any detour monophonic path starting from $x$. Thus every $x$-detour monophonic set of $G$ contains $S_1 = S \cup \{w_1, w_2, \ldots, w_{b-a-1}\}$. Since every vertex $u_i (1 \leq i \leq p - b)$ lies on an $x$-$v_1$ detour monophonic path and the vertex $u_{p-b+1}$ lies on an $x$-$u_3$ detour monophonic path, $S_1 \cup \{u_3\}$ is an $x$-detour monophonic set of $G$. Hence $dm_x(G) = b$. \qed
Theorem 2.29. For integers $a$, $b$ and $p$ with $1 \leq a \leq p - b$ and $b \geq 2$, there is a connected graph $G$ of order $p$ such that $dm_x(G) = a$ and $e_m(x) = b$ for some vertex $x$ in $G$.

Proof. Let $P_{b+1} : u_1, u_2, \ldots, u_{b+1}$ be a path of order $b + 1$. Let $G$ be the graph obtained from $P_{b+1}$ by adding $p - b - 1$ new vertices $v_1, v_2, \ldots, v_{a-1}, w_1, w_2, \ldots, w_{p-b-a}$ and joining each $v_i (1 \leq i \leq a-1)$ to $u_b$; also joining each $w_i (1 \leq i \leq p - b - a)$ to both $u_1$ and $u_3$. The graph $G$ has order $p$ and is shown in Figure 2.4. Let $x = u_1$ be a vertex in $G$. Then, clearly $e_m(x) = b$. If $b \geq 3$, then $S = \{v_1, v_2, \ldots, v_{a-1}, u_{b+1}\}$ is the set of all end vertices of $G$. Since every vertex of $G$ lies on an $x$-$y$ detour monophonic path for some $y \in S$ and by Theorem 2.3, $S$ is the unique minimum $x$-detour monophonic set of $G$. Hence $dm_x(G) = |S| = a$.

If $b = 2$, then $S' = S - \{u_3\}$ is the set of all end-vertices of $G$. By Theorem 2.3, every $x$-detour monophonic set of $G$ contains $S'$. Also, it is easily seen that $S'$ is not an $x$-detour monophonic set of $G$ and so
$dm_x(G) > a - 1$. Then $S' \cup \{u_3\}$ is an $x$-detour monophonic set of $G$ and so $dm_x(G) = a$. \hfill \Box

References


