Jordan Journal of Mathematics and Statistics (JJMS) 12(4), 2020, pp 601 - 623

ON JENSEN'S ADDITIVE INEQUALITY FOR POSITIVE CONVEX FUNCTIONS OF SELFADJOINT OPERATORS IN HILBERT SPACES

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ABSTRACT. In this paper we obtain some additive refinements and reverses of Jensen's inequality for positive convex/concave functions of selfadjoint operators in Hilbert spaces. Natural applications for power and exponential functions are provided.

1. Introduction

The famous *Young inequality* for scalars says that if a, $b>0$ and $\nu\in[0,1]$, then

$$a^{1-\nu}b^{\nu}\leq(1-\nu)\,a+\nu b \tag{1.1}$$

with equality if and only if $a=b$. The inequality (1.2) is also called *ν-weighted arithmetic-geometric mean inequality.*

We have the following inequality that provides a refinement and a reverse for the celebrated Young's inequality

$$\frac{1}{2}\nu\,(1-\nu)\,\frac{(b-a)^{2}}{\max\{a,b\}}\leq(1-\nu)\,a+\nu b-a^{1-\nu}b^{\nu}\leq\frac{1}{2}\nu\,(1-\nu)\,\frac{(b-a)^{2}}{\min\{a,b\}} \tag{1.2}$$

for any a, $b>0$ and $\nu\in[0,1]$.

2000 *Mathematics Subject Classification.* 47A63, 47A30, 15A60, 26D15, 26D10.
Key words and phrases. Young's Inequality, Convex functions, Jensen's inequality, Selfadjoint operator, Functions of selfadjoint operators.

Received: May 1, 2019 Accepted: Oct. 14, 2019 .

This result was obtained in 1978 by Cartwright and Field [2] who established a more general result for n variables and gave an application for a probability measure supported on a finite interval.

We observe that, if $a,\ b \in [\gamma, \Delta] \subset (0, \infty)$, then from (1.2) we have

$$\frac{1}{2\Delta}\nu(1-\nu)(b-a)^2 \le (1-\nu)a + \nu b - a^{1-\nu}b^{\nu} \le \frac{1}{2\gamma}\nu(1-\nu)(b-a)^2 \tag{1.3}$$

for $\nu \in [0,1]$.

Moreover, since

$$\frac{(b-a)^2}{\max\{a,b\}} = \min\{a,b\}\frac{(b-a)^2}{ab} = \min\{a,b\}\left(\frac{b}{a}+\frac{a}{b}-2\right)$$

and

$$\frac{(b-a)^2}{\min\{a,b\}} = \max\{a,b\}\frac{(b-a)^2}{ab} = \max\{a,b\}\left(\frac{b}{a}+\frac{a}{b}-2\right),$$

then from (1.2) we have the following inequality as well

$$\begin{aligned}\frac{1}{2}\nu(1-\nu)\gamma\left(\frac{b}{a}+\frac{a}{b}-2\right) &\le (1-\nu)a + \nu b - a^{1-\nu}b^{\nu} \\ &\le \frac{1}{2}\nu(1-\nu)\Delta\left(\frac{b}{a}+\frac{a}{b}-2\right)\end{aligned} \tag{1.4}$$

for any $a,\ b \in [\gamma, \Delta] \subset (0, \infty)$ and $\nu \in [0,1]$.

We recall that *Specht's ratio* is defined by [16]

$$S(h) := \begin{cases} \dfrac{h^{\frac{1}{h-1}}}{e\ln\left(h^{\frac{1}{h-1}}\right)} \text{ if } h \in (0,1)\cup(1,\infty), \\ \\ 1 \text{ if } h = 1. \end{cases} \tag{1.5}$$

It is well known that $\lim_{h\to 1} S(h) = 1$, $S(h) = S\left(\frac{1}{h}\right) > 1$ for $h > 0$, $h \ne 1$. The function is decreasing on $(0,1)$ and increasing on $(1,\infty)$.

In [17], Tominaga also proved the following additive reverse of Young's inequality

$$(1-\nu)a + \nu b - a^{1-\nu}b^{\nu} \le S\left(\frac{a}{b}\right)L(a,b) \tag{1.6}$$

where $a,\, b>0,\ \nu\in[0,1]$ and $L(a,b)$ is the logarithmic mean, namely

$$L(a,b):=\begin{cases}\frac{b-a}{\ln b-\ln a} & \text{if}\quad a\neq b\\ \\ b & \text{if}\quad a=b.\end{cases}$$

If for positive numbers $a,\, b$ we have $a,b\in[\gamma,\Delta]\subset(0,\infty)$ and $\nu\in[0,1]$, then by (1.6) we get [17]

$$(1-\nu)\,a+\nu b-a^{1-\nu}b^{\nu}\leq S\left(\frac{\Delta}{\gamma}\right)L\left(1,\frac{\Delta}{\gamma}\right) \tag{1.7}$$

Kittaneh and Manasrah [10], [11] provided a refinement and an additive reverse for Young inequality as follows:

$$r\left(\sqrt{a}-\sqrt{b}\right)^{2}\leq(1-\nu)\,a+\nu b-a^{1-\nu}b^{\nu}\leq R\left(\sqrt{a}-\sqrt{b}\right)^{2} \tag{1.8}$$

where $a,\, b>0,\ \nu\in[0,1]$, $r=\min\{1-\nu,\nu\}$ and $R=\max\{1-\nu,\nu\}$. The case $\nu=\frac{1}{2}$ reduces (1.8) to an identity.

If $a,\, b\in[\gamma,\Delta]\subset(0,\infty)$, then $\left|\sqrt{a}-\sqrt{b}\right|\leq\sqrt{\Delta}-\sqrt{\gamma}$ and by (1.8) we get

$$(1-\nu)\,a+\nu b-a^{1-\nu}b^{\nu}\leq R\left(\sqrt{\Delta}-\sqrt{\gamma}\right)^{2}. \tag{1.9}$$

In the recent paper [5] we obtained the following reverses of Young's inequality as well:

$$(1-\nu)\,a+\nu b-a^{1-\nu}b^{\nu}\leq\nu\,(1-\nu)\,(a-b)\,(\ln a-\ln b) \tag{1.10}$$

where $a,\, b>0,\ \nu\in[0,1]$.

Observe that for $a,\, b\in[\gamma,\Delta]\subset(0,\infty)$ we have

$$0\leq(a-b)\,(\ln a-\ln b)=|a-b|\,|\ln a-\ln b|\leq(\Delta-\gamma)\,(\ln\Delta-\ln\gamma)$$

and by (1.10) we get

$$(1-\nu)\,a+\nu b-a^{1-\nu}b^{\nu}\leq\nu\,(1-\nu)\,(\Delta-\gamma)\,(\ln\Delta-\ln\gamma)\,. \tag{1.11}$$

For any $a,\ b>0$ and $\nu\in[0,1]$ we have [6]

$$\frac{1}{2}\nu(1-\nu)(\ln a-\ln b)^{2}\min\{a,b\}\leq(1-\nu)a+\nu b-a^{1-\nu}b^{\nu}\leq\frac{1}{2}\nu(1-\nu)(\ln a-\ln b)^{2}\max\{a,b\} \tag{1.12}$$

This inequality was obtained in the case $a<b$ in [1] as well.

If $a,\ b\in[\gamma,\Delta]\subset(0,\infty)$, then by (1.12) we get

$$\frac{1}{2}\nu(1-\nu)\gamma(\ln a-\ln b)^{2}\leq(1-\nu)a+\nu b-a^{1-\nu}b^{\nu}\leq\frac{1}{2}\nu(1-\nu)\Delta(\ln a-\ln b)^{2} \tag{1.13}$$

for any $\nu\in[0,1]$.

If $a,\ b\in[\gamma,\Delta]\subset(0,\infty)$ and $\nu\in[0,1]$, then we have [7]

$$(1-\nu)a+\nu b-a^{1-\nu}b^{\nu}\leq\max\{\iota_{\gamma,\Delta}(\nu),\iota_{\gamma,\Delta}(1-\nu)\} \tag{1.14}$$

where

$$\iota_{\gamma,\Delta}(\nu):=(1-\nu)\gamma+\nu\Delta-\gamma^{1-\nu}\Delta^{\nu}. \tag{1.15}$$

We consider the function $f_{\nu}:[0,\infty)\to[0,\infty)$ defined for $\nu\in(0,1)$ by

$$f_{\nu}(x)=1-\nu+\nu x-x^{\nu}.$$

For $[k,K]\subset[0,\infty)$ define

$$\Delta_{\nu}(k,K):=\begin{cases} f_{\nu}(k) \text{ if } K<1, \\ \max\{f_{\nu}(k),f_{\nu}(K)\} \text{ if } k\leq1\leq K, \\ f_{\nu}(K) \text{ if } 1<k \end{cases} \tag{1.16}$$

and

$$\delta_{\nu}(k,K) := \begin{cases} f_{\nu}(K) \text{ if } K<1, \\ \\ 0 \text{ if } k\le 1\le K, \\ \\ f_{\nu}(k) \text{ if } 1<k. \end{cases} \tag{1.17}$$

In the recent paper [8] we obtained the following refinement and reverse for the *additive Young's inequality:*

$$\delta_{\nu}(k,K)\,a \le (1-\nu)\,a + \nu b - a^{1-\nu}b^{\nu} \le \Delta_{\nu}(k,K)\,a, \tag{1.18}$$

for positive numbers a, b with $\frac{b}{a}\in[k,K]\subset(0,\infty)$ and $\nu\in[0,1]$ where $\Delta_{\nu}(k,K)$ and $\delta_{\nu}(k,K)$ are defined by (1.16) and (1.17) respectively.

Now, if $a,\ b\in[\gamma,\Delta]\subset(0,\infty)$ and $\nu\in[0,1]$, then $\frac{b}{a}\in\left[\frac{\gamma}{\Delta},\frac{\Delta}{\gamma}\right]$ and by (1.18) we have

$$(1-\nu)\,a+\nu b - a^{1-\nu}b^{\nu} \le \max\left\{f_{\nu}\left(\frac{\gamma}{\Delta}\right), f_{\nu}\left(\frac{\Delta}{\gamma}\right)\right\}a, \tag{1.19}$$

and since

$$f_{\nu}\left(\frac{\gamma}{\Delta}\right) = \frac{(1-\nu)\,\Delta+\nu\gamma-\gamma^{\nu}\Delta^{1-\nu}}{\Delta}$$

and

$$f_{\nu}\left(\frac{\Delta}{\gamma}\right) = \frac{(1-\nu)\,\gamma+\nu\Delta-\Delta^{\nu}\gamma^{1-\nu}}{\gamma}$$

then by (1.19) we get

$$\begin{aligned} &(1-\nu)\,a+\nu b-a^{1-\nu}b^{\nu} \\ &\le \max\left\{\frac{(1-\nu)\,\Delta+\nu\gamma-\gamma^{\nu}\Delta^{1-\nu}}{\Delta}, \frac{(1-\nu)\,\gamma+\nu\Delta-\Delta^{\nu}\gamma^{1-\nu}}{\gamma}\right\}a, \end{aligned} \tag{1.20}$$

for any $a,\ b\in[\gamma,\Delta]\subset(0,\infty)$ and $\nu\in[0,1]$.

Let $A : H \to H$ a bounded linear operator on the Hilbert space H. The spectrum of A denoted in the following by $Sp(A)$ is defined as

$$Sp(A) := \{\lambda \in \mathbb{C} | \lambda 1_H - A \text{ is not invertible}\},$$

where 1_H is the identity operator on H.

Let A be a selfadjoint linear operator on a complex Hilbert space $(H; \langle ., . \rangle)$. The *Gelfand map* establishes a $*$-isometrically isomorphism Φ between the set $C(Sp(A))$ of all *continuous functions* defined on the *spectrum* of A, denoted $Sp(A)$, an the C^*-algebra $C^*(A)$ generated by A and the identity operator 1_H on H as follows (see for instance [14, p. 3]):

For any $f, g \in C(Sp(A))$ and any $\alpha, \beta \in \mathbb{C}$ we have

(i) $\Phi(\alpha f + \beta g) = \alpha \Phi(f) + \beta \Phi(g)$;

(ii) $\Phi(fg) = \Phi(f)\Phi(g)$ and $\Phi(\bar{f}) = \Phi(f)^*$;

(iii) $\|\Phi(f)\| = \|f\| := \sup_{t \in Sp(A)} |f(t)|$;

(iv) $\Phi(f_0) = 1_H$ and $\Phi(f_1) = A$, where $f_0(t) = 1$ and $f_1(t) = t$, for $t \in Sp(A)$.

With this notation we define

$$f(A) := \Phi(f) \quad \text{for all } f \in C(Sp(A))$$

and we call it the *continuous functional calculus* for a selfadjoint operator A.

If A is a selfadjoint operator and f is a real valued continuous function on $Sp(A)$, then $f(t) \geq 0$ for any $t \in Sp(A)$ implies that $f(A) \geq 0$, i.e. $f(A)$ is a *positive operator* on H. Moreover, if both f and g are real valued functions on $Sp(A)$ then the following important property holds:

$$\text{(P)} \qquad f(t) \geq g(t) \quad \text{for any } t \in Sp(A) \quad \text{implies that} \quad f(A) \geq g(A)$$

in the operator order of $B(H)$.

The following result that provides a vector operator version for the Jensen inequality is well known, see for instance [13] or [14, p. 5]:

Theorem 1.1. *Let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A) \subseteq [m, M]$ for some scalars m, M with $m < M$. If f is a convex function on $[m, M]$, then*

$$f(\langle Ax, x\rangle) \leq \langle f(A)x, x\rangle \tag{1.21}$$

for each $x \in H$ with $\|x\| = 1$.

As a special case of Theorem 1.1 we have the *Hölder-McCarthy inequality* [12]: Let A be a selfadjoint positive operator on a Hilbert space H, then

(i) $\langle A^r x, x\rangle \geq \langle Ax, x\rangle^r$ for all $r > 1$ and $x \in H$ with $\|x\| = 1$;

(ii) $\langle A^r x, x\rangle \leq \langle Ax, x\rangle^r$ for all $0 < r < 1$ and $x \in H$ with $\|x\| = 1$;

(iii) If A is invertible, then $\langle A^r x, x\rangle \geq \langle Ax, x\rangle^r$ for all $r < 0$ and $x \in H$ with $\|x\| = 1$.

In [3] (see also [4, p. 16]) we obtained the following additive reverse of (1.21):

Theorem 1.2. *Let I be an interval and $f : I \to \mathbb{R}$ be a convex and differentiable function on $\mathring{I}$ (the interior of I) whose derivative f' is continuous on $\mathring{I}$. If A is a selfadjoint operators on the Hilbert space H with $Sp(A) \subset \mathring{I}$, then*

$$(0 \leq) \langle f(A)x, x\rangle - f(\langle Ax, x\rangle) \leq \langle f'(A)Ax, x\rangle - \langle Ax, x\rangle \langle f'(A)x, x\rangle \tag{1.22}$$

for any $x \in H$ with $\|x\| = 1$.

In the recent paper [9] we established the following multiplicative inequalities:

Theorem 1.3. *Let $f : [m, M] \to [0, \infty)$ be a continuous function and assume that*

$$0 < \gamma = \min_{t\in[m,M]} f(t) < \max_{t\in[m,M]} f(t) = \Delta < \infty. \tag{1.23}$$

Then for any A, a selfadjoint operator with

$$m1_H \leq A \leq M1_H, \tag{1.24}$$

we have the inequality

$$\frac{(1-\nu) f(\langle Ax, x\rangle) + \nu \langle f(A) x, x\rangle}{\langle f^{\nu}(A) x, x\rangle f^{1-\nu}(\langle Ax, x\rangle)} \le \exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{\Delta}{\gamma}-1\right)^2\right] \tag{1.25}$$

for any $x \in H$ *with* $\|x\| = 1,$ *where* $\nu \in [0,1]$.

Moreover, if f *is convex on* $[m, M]$, *then for any* $\nu \in [0,1]$,

$$\frac{f^{\nu}(\langle Ax, x\rangle)}{\langle f^{\nu}(A) x, x\rangle} \le \exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{\Delta}{\gamma}-1\right)^2\right] \tag{1.26}$$

while, if f *is concave on* $[m, M]$, *then*

$$\frac{\langle f(A) x, x\rangle}{f(\langle Ax, x\rangle)} \le \exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{\Delta}{\gamma}-1\right)^2\right] \frac{\langle f^{\nu}(A) x, x\rangle}{f^{\nu}(\langle Ax, x\rangle)}. \tag{1.27}$$

For some meaningful examples of functions satisfying the above inequalities (1.26)-(1.27), see [9]. For related results, see [15].

Motivated by the above results, in this paper we obtain several additive refinements and reverses of Jensen's inequality for positive convex/concave functions of selfadjoint operators in Hilbert spaces. Natural applications for power functions are provided.

2. Upper Bounds

By using the definition of $\iota_{\gamma,\Delta}(\nu)$ from (1.15), we define for γ, $\Delta \in (0,\infty)$ and $\nu \in [0,1]$, the function

$$\begin{aligned} \varphi(\gamma, \Delta, \nu) & \\ &:= \max\{\iota_{\gamma,\Delta}(\nu), \iota_{\gamma,\Delta}(1-\nu)\} \\ &= \max\left\{(1-\nu)\gamma + \nu\Delta - \gamma^{1-\nu}\Delta^{\nu}, \nu\gamma + (1-\nu)\Delta - \gamma^{\nu}\Delta^{1-\nu}\right\}. \end{aligned} \tag{2.1}$$

We observe that

$$\varphi(\gamma, \Delta, \nu) = \varphi(\Delta, \gamma, \nu) = \varphi(\gamma, \Delta, 1-\nu)$$

for γ, $\Delta \in (0,\infty)$ and $\nu \in [0,1]$.

Using the inequality (1.3) for $(a,b)=(\gamma,\Delta)$ and $(a,b)=(\Delta,\gamma)$ we get

$$\varphi(\gamma,\Delta,\nu)\le\frac{1}{2\gamma}\nu(1-\nu)(\Delta-\gamma)^2 \tag{2.2}$$

for any $0<\gamma<\Delta$ and $\nu\in[0,1]$.

From Tominaga's inequality (1.6) we have

$$\varphi(\gamma,\Delta,\nu)\le S\left(\frac{\Delta}{\gamma}\right)L(\gamma,\Delta)$$

while from (1.7) we have

$$\varphi(\gamma,\Delta,\nu)\le S\left(\frac{\Delta}{\gamma}\right)L\left(1,\frac{\Delta}{\gamma}\right)=\frac{1}{\gamma}S\left(\frac{\Delta}{\gamma}\right)L(\gamma,\Delta)$$

giving that

$$\varphi(\gamma,\Delta,\nu)\le S\left(\frac{\Delta}{\gamma}\right)L(\gamma,\Delta)\begin{cases}1 & \text{if } \gamma\le 1,\\ \\ \frac{1}{\gamma} & \text{if } \gamma>1,\end{cases} \tag{2.3}$$

for any $0<\gamma<\Delta$ and $\nu\in[0,1]$.

From (1.9) we have

$$\varphi(\gamma,\Delta,\nu)\le\max\{\nu,1-\nu\}\left(\sqrt{\Delta}-\sqrt{\gamma}\right)^2 \tag{2.4}$$

for any $0<\gamma<\Delta$ and $\nu\in[0,1]$, while from (1.10) we have

$$\varphi(\gamma,\Delta,\nu)\le\nu(1-\nu)(\Delta-\gamma)(\ln\Delta-\ln\gamma) \tag{2.5}$$

for any $0<\gamma<\Delta$ and $\nu\in[0,1]$.

From (1.13) we also have

$$\varphi(\gamma,\Delta,\nu)\le\frac{1}{2}\nu(1-\nu)\Delta(\ln\Delta-\ln\gamma)^2 \tag{2.6}$$

for any $0<\gamma<\Delta$ and $\nu\in[0,1]$.

Therefore, by the inequality (1.14), we can state that for any $a, b \in [\gamma, \Delta] \subset (0, \infty)$ (with $\gamma < \Delta$) we have the following reverse of Young's inequality

$$(1 - \nu) a + \nu b - a^{1-\nu} b^{\nu} \leq \varphi(\gamma, \Delta, \nu) \leq \Phi(\gamma, \Delta, \nu) \tag{2.7}$$

for any $\nu \in [0, 1]$, where, as pointed out above, the upper bound $\Phi(\gamma, \Delta, \nu)$ can be one of the right hand side of the inequalities (2.2)-(2.6), namely

$$\Phi(\gamma, \Delta, \nu) := \tag{2.8}$$

$$\frac{1}{2\gamma}\nu(1-\nu)(\Delta-\gamma)^2, \; S\left(\frac{\Delta}{\gamma}\right) L(\gamma, \Delta) \begin{cases} 1 & \text{if } \gamma \leq 1, \\ \\ \frac{1}{\gamma} & \text{if } \gamma > 1, \end{cases}$$

$$\max\{\nu, 1-\nu\}\left(\sqrt{\Delta} - \sqrt{\gamma}\right)^2, \; \nu(1-\nu)(\Delta-\gamma)(\ln\Delta - \ln\gamma),$$

$$\frac{1}{2}\nu(1-\nu)\Delta(\ln\Delta - \ln\gamma)^2,$$

for any $0 < \gamma < \Delta$ and $\nu \in [0, 1]$.

Theorem 2.1. *Let $f : [m, M] \to [0, \infty)$ be a continuous function and assume that it satisfies the condition (1.23). Then for any A, a selfadjoint operator with the property (1.24), we have the inequality*

$$\begin{aligned} 0 &\leq (1-\nu) f(\langle Ax, x\rangle) + \nu \langle f(A)x, x\rangle - f^{1-\nu}(\langle Ax, x\rangle)\langle f^{\nu}(A)x, x\rangle \\ &\leq \varphi(\gamma, \Delta, \nu) \leq \Phi(\gamma, \Delta, \nu) \end{aligned} \tag{2.9}$$

for any $x \in H$ with $\|x\| = 1$.

Moreover, if f is convex on $[m, M]$, then

$$f^{1-\nu}(\langle Ax, x\rangle)\left[f^{\nu}(\langle Ax, x\rangle) - \langle f^{\nu}(A)x, x\rangle\right] \leq \varphi(\gamma, \Delta, \nu) \leq \Phi(\gamma, \Delta, \nu) \tag{2.10}$$

for any $\nu \in [0, 1]$ and for any $x \in H$ with $\|x\| = 1$.

Proof. Let $t\in[m,M]$ and $x\in H$ with $\|x\|=1$. Then $f(t)\,,\ f(\langle Ax,x\rangle)\in[\gamma,\Delta]$ and by (2.7) we have

$$(1-\nu)f(\langle Ax,x\rangle)+\nu f(t)-f^{1-\nu}(\langle Ax,x\rangle)f^{\nu}(t)\le\varphi(\gamma,\Delta,\nu)\le\Phi(\gamma,\Delta,\nu) \tag{2.11}$$

for any $t\in[m,M]\,,\ \nu\in[0,1]$ and $x\in H$ with $\|x\|=1$.

If we use the functional calculus for the operator A with $m1_H\le A\le M1_H$, then by (2.11) we get

$$\begin{aligned}&(1-\nu)f(\langle Ax,x\rangle)1_H+\nu f(A)-f^{1-\nu}(\langle Ax,x\rangle)f^{\nu}(A)\\&\le\varphi(\gamma,\Delta,\nu)1_H\le\Phi(\gamma,\Delta,\nu)1_H\end{aligned} \tag{2.12}$$

for any $\nu\in[0,1]$ and $x\in H$ with $\|x\|=1$.

If we take in (2.12) the inner product over $y\in H$ with $\|y\|=1$, then we get

$$\begin{aligned}&(1-\nu)f(\langle Ax,x\rangle)+\nu\langle f(A)y,y\rangle-f^{1-\nu}(\langle Ax,x\rangle)\langle f^{\nu}(A)y,y\rangle\\&\le\varphi(\gamma,\Delta,\nu)\le\Phi(\gamma,\Delta,\nu)\end{aligned} \tag{2.13}$$

for any $\nu\in[0,1]$, which by taking $y=x$, implies the desired inequality (2.9).

If f is convex on $[m,M]$, then by Jensen's inequality we have $f(\langle Ax,x\rangle)\le\langle f(A)x,x\rangle$, for $x\in H$ with $\|x\|=1$, then

$$\begin{aligned}&f(\langle Ax,x\rangle)-f^{1-\nu}(\langle Ax,x\rangle)\langle f^{\nu}(A)x,x\rangle\\&\le(1-\nu)f(\langle Ax,x\rangle)+\nu\langle f(A)x,x\rangle-f^{1-\nu}(\langle Ax,x\rangle)\langle f^{\nu}(A)x,x\rangle\\&\le\varphi(\gamma,\Delta,\nu)\le\Phi(\gamma,\Delta,\nu),\end{aligned}$$

which implies the desired result (2.10). □

Remark 2.1. *If for some $\nu\in(0,1)$ the function f^{ν} is concave, then $\langle f^{\nu}(A)x,x\rangle\le f^{\nu}(\langle Ax,x\rangle)$ for any $x\in H$ with $\|x\|=1$. Therefore by (2.10) we have the meaningful* *inequality*

$$0 \le f^{1-\nu}(\langle Ax,x\rangle)\left[f^{\nu}(\langle Ax,x\rangle)-\langle f^{\nu}(A)x,x\rangle\right] \le \varphi(\gamma,\Delta,\nu) \le \Phi(\gamma,\Delta,\nu) \tag{2.14}$$

for any $x \in H$ with $\|x\| = 1$.

If we consider the convex function $f(t) = t^r$, $r \ge 1$ and take $\nu \in (0,1)$ with $r\nu < 1$, then the function f^{ν} is concave and by (2.14) we get for any $x \in H$ with $\|x\| = 1$ that

$$\begin{aligned} 0 &\le \langle Ax,x\rangle^{(1-\nu)r}\left[\langle Ax,x\rangle^{\nu r} - \langle A^{\nu r}x,x\rangle\right] \le \varphi(m^r,M^r,\nu) \\ &\le \Phi(m^r,M^r,\nu), \end{aligned} \tag{2.15}$$

where $m1_H \le A \le M1_H$ with $0 < m < M$. Since $\langle Ax,x\rangle^{(1-\nu)r} \ge m^{(1-\nu)r}$ for any $x \in H$ with $\|x\| = 1$, then by (2.15) we get the following additive reverse of the Hölder-McCarthy inequality

$$\begin{aligned} (0 \le)\, \langle Ax,x\rangle^{\nu r} - \langle A^{\nu r}x,x\rangle &\le \frac{1}{m^{(1-\nu)r}}\varphi(m^r,M^r,\nu) \\ &\le \frac{1}{m^{(1-\nu)r}}\Phi(m^r,M^r,\nu) \end{aligned} \tag{2.16}$$

for any $x \in H$ with $\|x\| = 1$.

3. Some Inequalities Via Cartwright-Field Result

By making use of the Cartwright-Field celebrated inequalities we can state the following result:

Theorem 3.1. *Let $f : [m,M] \to [0,\infty)$ be a continuous function and assume that it satisfies the condition (1.23). Then for any A, a selfadjoint operator with the property*

(1.24), we have the inequality

$$
\begin{aligned}
(3.1)\qquad &(0 \leq)\, \frac{1}{2\Delta}\nu\,(1-\nu)\,(\langle f\,(A)\,x,x\rangle - f\,(\langle Ax,x\rangle))^2 \\
&\leq \frac{1}{2\Delta}\nu\,(1-\nu)\,\big(\big\langle f^2\,(A)\,x,x\big\rangle - 2f\,(\langle Ax,x\rangle)\,\langle f\,(A)\,x,x\rangle + f^2\,(\langle Ax,x\rangle)\big) \\
&\leq (1-\nu)\,f\,(\langle Ax,x\rangle) + \nu\,\langle f\,(A)\,x,x\rangle - f^{1-\nu}\,(\langle Ax,x\rangle)\,\langle f^{\nu}\,(A)\,x,x\rangle \\
&\leq \frac{1}{2\gamma}\nu\,(1-\nu)\,\big(\big\langle f^2\,(A)\,x,x\big\rangle - 2f\,(\langle Ax,x\rangle)\,\langle f\,(A)\,x,x\rangle + f^2\,(\langle Ax,x\rangle)\big)
\end{aligned}
$$

for any $x \in H$ *with* $\|x\| = 1$.

Moreover, if f *is convex on* $[m, M]$, *then*

$$
\begin{aligned}
(3.2)\qquad &(0 \leq)\, \frac{1}{2\Delta}\nu\,(1-\nu)\,(\langle f\,(A)\,x,x\rangle - f\,(\langle Ax,x\rangle))^2 \\
&\leq \langle f\,(A)\,x,x\rangle - f^{1-\nu}\,(\langle Ax,x\rangle)\,\langle f^{\nu}\,(A)\,x,x\rangle
\end{aligned}
$$

and

$$
\begin{aligned}
(3.3)\qquad &f^{1-\nu}\,(\langle Ax,x\rangle)\,[f^{\nu}\,(\langle Ax,x\rangle) - \langle f^{\nu}\,(A)\,x,x\rangle] \\
&\leq \frac{1}{2\gamma}\nu\,(1-\nu)\,\big(\big\langle f^2\,(A)\,x,x\big\rangle - 2f\,(\langle Ax,x\rangle)\,\langle f\,(A)\,x,x\rangle + f^2\,(\langle Ax,x\rangle)\big)
\end{aligned}
$$

for any $\nu \in [0, 1]$ *and for any* $x \in H$ *with* $\|x\| = 1$.

If f *is concave on* $[m, M]$, *then*

$$
\begin{aligned}
(3.4)\qquad &(0 \leq)\, \frac{1}{2\Delta}\nu\,(1-\nu)\,(f\,(\langle Ax,x\rangle) - \langle f\,(A)\,x,x\rangle)^2 \\
&\leq f^{1-\nu}\,(\langle Ax,x\rangle)\,[f^{\nu}\,(\langle Ax,x\rangle) - \langle f^{\nu}\,(A)\,x,x\rangle]
\end{aligned}
$$

for any $\nu \in [0, 1]$ *and for any* $x \in H$ *with* $\|x\| = 1$.

Proof. Let $t\in[m,M]$ and $x\in H$ with $\|x\|=1$. Then $f(t), f(\langle Ax,x\rangle)\in[\gamma,\Delta]$ and by (1.3) we have

$$\begin{aligned}
0 &\leq \frac{1}{2\Delta}\nu(1-\nu)\left(f^2(t)-2f(\langle Ax,x\rangle)f(t)+f^2(\langle Ax,x\rangle)\right) \\
&\leq (1-\nu)f(\langle Ax,x\rangle)+\nu f(t)-f^{1-\nu}(\langle Ax,x\rangle)f^{\nu}(t) \\
&\leq \frac{1}{2\gamma}\nu(1-\nu)\left(f^2(t)-2f(\langle Ax,x\rangle)f(t)+f^2(\langle Ax,x\rangle)\right)
\end{aligned} \tag{3.5}$$

for any $t\in[m,M]$ and $x\in H$ with $\|x\|=1$.

If we use the functional calculus for the operator A with $m1_H\leq A\leq M1_H$, then by (3.5) we get

$$\begin{aligned}
0 &\leq \frac{1}{2\Delta}\nu(1-\nu)\left(f^2(A)-2f(\langle Ax,x\rangle)f(A)+f^2(\langle Ax,x\rangle)1_H\right) \\
&\leq (1-\nu)f(\langle Ax,x\rangle)1_H+\nu f(A)-f^{1-\nu}(\langle Ax,x\rangle)f^{\nu}(A) \\
&\leq \frac{1}{2\gamma}\nu(1-\nu)\left(f^2(A)-2f(\langle Ax,x\rangle)f(A)+f^2(\langle Ax,x\rangle)1_H\right)
\end{aligned} \tag{3.6}$$

for any $x\in H$ with $\|x\|=1$.

If we take the inner product in (3.6) over $y\in H$ with $\|y\|=1$, then we get

$$\begin{aligned}
0 &\leq \frac{1}{2\Delta}\nu(1-\nu)\left(\left\langle f^2(A)y,y\right\rangle-2f(\langle Ax,x\rangle)\langle f(A)y,y\rangle+f^2(\langle Ax,x\rangle)\right) \\
&\leq (1-\nu)f(\langle Ax,x\rangle)+\nu\langle f(A)y,y\rangle-f^{1-\nu}(\langle Ax,x\rangle)\langle f^{\nu}(A)y,y\rangle \\
&\leq \frac{1}{2\gamma}\nu(1-\nu)\left(\left\langle f^2(A)y,y\right\rangle-2f(\langle Ax,x\rangle)\langle f(A)y,y\rangle+f^2(\langle Ax,x\rangle)\right),
\end{aligned} \tag{3.7}$$

which by taking $y=x$, produces the second, third and fourth inequalities in (3.1).

By Hölder-McCarthy inequality we have

$$\left\langle f^2(A)x,x\right\rangle\geq\langle f(A)x,x\rangle^2,$$

for any $x \in H$ with $\|x\| = 1$, which implies that

$$\begin{aligned}
&\left\langle f^2(A)x, x\right\rangle - 2f(\langle Ax, x\rangle)\langle f(A)x, x\rangle + f^2(\langle Ax, x\rangle) \\
&\geq \langle f(A)x, x\rangle^2 - 2f(\langle Ax, x\rangle)\langle f(A)x, x\rangle + f^2(\langle Ax, x\rangle) \\
&= (\langle f(A)x, x\rangle - f(\langle Ax, x\rangle))^2
\end{aligned}$$

proving the first inequality in (3.1).

If f is convex, then by Jensen's inequality we have $\langle f(A)x, x\rangle \geq f(\langle Ax, x\rangle)$ for any $x \in H$ with $\|x\| = 1$. Using the first two inequalities in (3.1) we get

$$\begin{aligned}
(3.8)\qquad &(0 \leq)\, \frac{1}{2\Delta}\nu(1-\nu)(\langle f(A)x, x\rangle - f(\langle Ax, x\rangle))^2 \\
&\leq \frac{1}{2\Delta}\nu(1-\nu)\left(\left\langle f^2(A)x, x\right\rangle - 2f(\langle Ax, x\rangle)\langle f(A)x, x\rangle + f^2(\langle Ax, x\rangle)\right) \\
&\leq (1-\nu)f(\langle Ax, x\rangle) + \nu\langle f(A)x, x\rangle - f^{1-\nu}(\langle Ax, x\rangle)\langle f^{\nu}(A)x, x\rangle \\
&\leq \langle f(A)x, x\rangle - f^{1-\nu}(\langle Ax, x\rangle)\langle f^{\nu}(A)x, x\rangle
\end{aligned}$$

proving the inequality (3.2).

From the fourth inequality in (3.1) we have

$$\begin{aligned}
(3.9)\qquad &f^{1-\nu}(\langle Ax, x\rangle)(f^{\nu}(\langle Ax, x\rangle) - \langle f^{\nu}(A)x, x\rangle) \\
&= f(\langle Ax, x\rangle) - f^{1-\nu}(\langle Ax, x\rangle)\langle f^{\nu}(A)x, x\rangle \\
&\leq (1-\nu)f(\langle Ax, x\rangle) + \nu\langle f(A)x, x\rangle - f^{1-\nu}(\langle Ax, x\rangle)\langle f^{\nu}(A)x, x\rangle \\
&\leq \frac{1}{2\gamma}\nu(1-\nu)\left(\left\langle f^2(A)x, x\right\rangle - 2f(\langle Ax, x\rangle)\langle f(A)x, x\rangle + f^2(\langle Ax, x\rangle)\right)
\end{aligned}$$

for any $x \in H$ with $\|x\| = 1$, proving the inequality (3.3).

If f is concave on $[m, M]$, then by the first two inequalities in (3.1) we get

$$(3.10)\qquad (0 \leq) \frac{1}{2\Delta}\nu(1-\nu)(\langle f(A)x,x\rangle - f(\langle Ax,x\rangle))^2$$
$$\leq \frac{1}{2\Delta}\nu(1-\nu)\left(\left\langle f^2(A)x,x\right\rangle - 2f(\langle Ax,x\rangle)\langle f(A)x,x\rangle + f^2(\langle Ax,x\rangle)\right)$$
$$\leq (1-\nu)f(\langle Ax,x\rangle) + \nu\langle f(A)x,x\rangle - f^{1-\nu}(\langle Ax,x\rangle)\langle f^{\nu}(A)x,x\rangle$$
$$\leq f(\langle Ax,x\rangle) - f^{1-\nu}(\langle Ax,x\rangle)\langle f^{\nu}(A)x,x\rangle$$
$$= f^{1-\nu}(\langle Ax,x\rangle)[f^{\nu}(\langle Ax,x\rangle) - \langle f^{\nu}(A)x,x\rangle]$$

for any $x \in H$ with $\|x\| = 1$, proving the inequality (3.4). □

Remark 3.1. *The function $f(t) = t^r$, $r \geq 1$ is convex on $\mathbb{R}_+$. Then for any $\nu \in [0,1]$ and a selfadjoint operator $m1_H \leq A \leq M1_H$ with $0 < m < M$ we have from (3.2) that*

$$(3.11)\qquad (0 \leq) \frac{1}{2M^r}\nu(1-\nu)(\langle A^r x,x\rangle - \langle Ax,x\rangle^r)^2$$
$$\leq \langle A^r x,x\rangle - \langle Ax,x\rangle^{(1-\nu)r}\langle A^{\nu r}x,x\rangle$$

for any $x \in H$ with $\|x\| = 1$.

If we take in (3.11) $\nu = \frac{1}{2}$, then we get the inequality

$$(3.12)\qquad (0 \leq) \frac{1}{8M^r}(\langle A^r x,x\rangle - \langle Ax,x\rangle^r)^2 \leq \langle A^r x,x\rangle - \langle Ax,x\rangle^{r/2}\left\langle A^{r/2}x,x\right\rangle$$

for any $x \in H$ with $\|x\| = 1$.

If we take $r = 2$ in (3.11), then we get

$$(3.13)\qquad (0 \leq) \frac{1}{2M^2}\nu(1-\nu)\left(\left\langle A^2 x,x\right\rangle - \langle Ax,x\rangle^2\right)^2$$
$$\leq \left\langle A^2 x,x\right\rangle - \langle Ax,x\rangle^{2(1-\nu)}\left\langle A^{2\nu}x,x\right\rangle$$

for any $x \in H$ with $\|x\| = 1$.

The function $f(t) = t^q$, $q \in (0,1)$ is concave on $\mathbb{R}_+$. Then for any $\nu \in [0,1]$ and a selfadjoint operator $m1_H \leq A \leq M1_H$ with $0 < m < M$ we have from (3.4)

$$
\begin{aligned}
(3.14)\qquad & (0 \leq) \frac{1}{2M^q}\nu(1-\nu)\left(\langle Ax,x\rangle^q - \langle A^q x,x\rangle\right)^2 \\
& \leq \langle Ax,x\rangle^{(1-\nu)q}\left[\langle Ax,x\rangle^{\nu q} - \langle A^{\nu q}x,x\rangle\right]
\end{aligned}
$$

for any $x \in H$ with $\|x\| = 1$.

Since $\langle Ax,x\rangle^{(1-\nu)q} \leq M^{(1-\nu)q}$ for any $x \in H$ with $\|x\| = 1$, then by (3.14) we have

$$
(3.15)\qquad (0 \leq) \frac{1}{2M^{(2-\nu)q}}\nu(1-\nu)\left(\langle Ax,x\rangle^q - \langle A^q x,x\rangle\right)^2 \leq \langle Ax,x\rangle^{\nu q} - \langle A^{\nu q}x,x\rangle
$$

for any $x \in H$ with $\|x\| = 1$.

If we use the second Cartwright-Field inequality that holds for any a, $b \in [\gamma, \Delta]$ and $\nu \in [0,1]$, namely

$$
\begin{aligned}
\frac{1}{2}\nu(1-\nu)\gamma\left(\frac{b}{a}+\frac{a}{b}-2\right) &\leq (1-\nu)a + \nu b - a^{1-\nu}b^{\nu} \\
&\leq \frac{1}{2}\nu(1-\nu)\Delta\left(\frac{b}{a}+\frac{a}{b}-2\right)
\end{aligned}
$$

we can state the following result as well:

Theorem 3.2. *With the assumptions of Theorem 3.1 we have*

$$
\begin{aligned}
(3.16)\qquad & \frac{1}{2}\gamma\nu(1-\nu) \\
& \times\left(\left\langle f^{1/2}(A)x,x\right\rangle f^{-1/2}(\langle Ax,x\rangle) - \left\langle f^{-1/2}(A)x,x\right\rangle f^{1/2}(\langle Ax,x\rangle)\right)^2 \\
& \leq \frac{1}{2}\gamma\nu(1-\nu)\left(\langle f(A)x,x\rangle f^{-1}(\langle Ax,x\rangle) + f(\langle Ax,x\rangle)\left\langle f^{-1}(A)x,x\right\rangle - 2\right) \\
& \leq (1-\nu)f(\langle Ax,x\rangle) + \nu\langle f(A)x,x\rangle - f^{1-\nu}(\langle Ax,x\rangle)\langle f^{\nu}(A)x,x\rangle \\
& \leq \frac{1}{2}\Delta\nu(1-\nu)\left(\langle f(A)x,x\rangle f^{-1}(\langle Ax,x\rangle) + f(\langle Ax,x\rangle)\left\langle f^{-1}(A)x,x\right\rangle - 2\right)
\end{aligned}
$$

for any $\nu \in [0,1]$ and for any $x \in H$ with $\|x\| = 1$.

Moreover, if f is convex on $[m,M]$, then

$$(3.17)\qquad \begin{aligned}&(0\leq)\,\frac{1}{2}\gamma\nu\left(1-\nu\right)\\&\times\left(\left\langle f^{1/2}\left(A\right)x,x\right\rangle f^{-1/2}\left(\left\langle Ax,x\right\rangle\right)-\left\langle f^{-1/2}\left(A\right)x,x\right\rangle f^{1/2}\left(\left\langle Ax,x\right\rangle\right)\right)^{2}\\&\leq\left\langle f\left(A\right)x,x\right\rangle-f^{1-\nu}\left(\left\langle Ax,x\right\rangle\right)\left\langle f^{\nu}\left(A\right)x,x\right\rangle\end{aligned}$$

and

$$(3.18)\qquad \begin{aligned}&f^{1-\nu}\left(\left\langle Ax,x\right\rangle\right)\left[f^{\nu}\left(\left\langle Ax,x\right\rangle\right)-\left\langle f^{\nu}\left(A\right)x,x\right\rangle\right]\\&\leq\frac{1}{2}\Delta\nu\left(1-\nu\right)\left(\left\langle f\left(A\right)x,x\right\rangle f^{-1}\left(\left\langle Ax,x\right\rangle\right)+f\left(\left\langle Ax,x\right\rangle\right)\left\langle f^{-1}\left(A\right)x,x\right\rangle-2\right)\end{aligned}$$

for any $\nu\in[0,1]$ and for any $x\in H$ with $\|x\|=1$.

If f is concave on $[m,M]$, then

$$(3.19)\qquad \begin{aligned}&(0\leq)\,\frac{1}{2}\gamma\nu\left(1-\nu\right)\\&\times\left(\left\langle f^{1/2}\left(A\right)x,x\right\rangle f^{-1/2}\left(\left\langle Ax,x\right\rangle\right)-\left\langle f^{-1/2}\left(A\right)x,x\right\rangle f^{1/2}\left(\left\langle Ax,x\right\rangle\right)\right)^{2}\\&\leq f^{1-\nu}\left(\left\langle Ax,x\right\rangle\right)\left[f^{\nu}\left(\left\langle Ax,x\right\rangle\right)-\left\langle f^{\nu}\left(A\right)x,x\right\rangle\right]\end{aligned}$$

for any $\nu\in[0,1]$ and for any $x\in H$ with $\|x\|=1$.

The proof follows along the lines of the proof in Theorem 3.1 and we omit the details.

4. Related Results

By the use of the Kittaneh-Manasrah inequality (1.8) we have:

Theorem 4.1. *With the assumptions of Theorem 3.1 we have*

$$
\begin{aligned}
(4.1)\qquad & r\left(\left\langle f^{1/2}(A)x,x\right\rangle - f^{1/2}(\langle Ax,x\rangle)\right)^2 \\
& \le r\left(\langle f(A)x,x\rangle + f(\langle Ax,x\rangle) - 2\left\langle f^{1/2}(A)x,x\right\rangle f^{1/2}(\langle Ax,x\rangle)\right) \\
& \le (1-\nu) f(\langle Ax,x\rangle) + \nu \langle f(A)x,x\rangle - f^{1-\nu}(\langle Ax,x\rangle)\langle f^{\nu}(A)x,x\rangle \\
& \le R\left(\langle f(A)x,x\rangle + f(\langle Ax,x\rangle) - 2\left\langle f^{1/2}(A)x,x\right\rangle f^{1/2}(\langle Ax,x\rangle)\right)
\end{aligned}
$$

for any $\nu \in [0,1]$ *and for any* $x \in H$ *with* $\|x\| = 1$*, where* $r = \min\{1-\nu,\nu\}$ *and* $R = \max\{1-\nu,\nu\}$.

Moreover, if f *is convex on* $[m,M]$*, then*

$$
\begin{aligned}
(4.2)\qquad & r\left(\left\langle f^{1/2}(A)x,x\right\rangle - f^{1/2}(\langle Ax,x\rangle)\right)^2 \\
& \le \langle f(A)x,x\rangle - f^{1-\nu}(\langle Ax,x\rangle)\langle f^{\nu}(A)x,x\rangle
\end{aligned}
$$

and

$$
\begin{aligned}
(4.3)\qquad & f^{1-\nu}(\langle Ax,x\rangle)\left[f^{\nu}(\langle Ax,x\rangle) - \langle f^{\nu}(A)x,x\rangle\right] \\
& \le R\left(\langle f(A)x,x\rangle + f(\langle Ax,x\rangle) - 2\left\langle f^{1/2}(A)x,x\right\rangle f^{1/2}(\langle Ax,x\rangle)\right)
\end{aligned}
$$

for any $\nu \in [0,1]$ *and for any* $x \in H$ *with* $\|x\| = 1$.

If f *is concave on* $[m,M]$*, then*

$$
\begin{aligned}
(4.4)\qquad & r\left(\left\langle f^{1/2}(A)x,x\right\rangle - f^{1/2}(\langle Ax,x\rangle)\right)^2 \\
& \le f^{1-\nu}(\langle Ax,x\rangle)\left[f^{\nu}(\langle Ax,x\rangle) - \langle f^{\nu}(A)x,x\rangle\right]
\end{aligned}
$$

for any $\nu \in [0,1]$ *and for any* $x \in H$ *with* $\|x\| = 1$.

The function $f(t) = t^p$, $p \ge 1$ is convex on $\mathbb{R}_+$. Then for any $\nu \in [0,1]$ and a positive selfadjoint operator A we have from (4.2) that

$$
(4.5)\qquad r\left(\left\langle A^{p/2}x,x\right\rangle - \langle Ax,x\rangle^{p/2}\right)^2 \le \langle A^p x,x\rangle - \langle Ax,x\rangle^{(1-\nu)p}\langle A^{\nu p}x,x\rangle
$$

for any $x \in H$ with $\|x\| = 1$, where $r = \min\{1-\nu, \nu\}$ and $R = \max\{1-\nu, \nu\}$.

If we take in (4.5) $\nu = \frac{1}{2}$, then we get the inequality

$$\frac{1}{2}\left(\left\langle A^{p/2}x, x\right\rangle - \langle Ax, x\rangle^{p/2}\right)^2 \leq \langle A^p x, x\rangle - \langle Ax, x\rangle^{p/2}\left\langle A^{p/2}x, x\right\rangle \tag{4.6}$$

for any $x \in H$ with $\|x\| = 1$.

If we take $p = 1$ in (4.5), then we get

$$r\left(\langle Ax, x\rangle^{1/2} - \left\langle A^{1/2}x, x\right\rangle\right)^2 \leq \langle Ax, x\rangle - \langle Ax, x\rangle^{1-\nu}\langle A^{\nu}x, x\rangle \tag{4.7}$$

for any $x \in H$ with $\|x\| = 1$.

If $A \leq M1_H$, then $\langle Ax, x\rangle^{1-\nu} \leq M^{1-\nu}$ for $\nu \in (0,1)$ and $x \in H$ with $\|x\| = 1$, and since

$$\langle Ax, x\rangle - \langle Ax, x\rangle^{1-\nu}\langle A^{\nu}x, x\rangle = \langle Ax, x\rangle^{1-\nu}\left(\langle Ax, x\rangle^{\nu} - \langle A^{\nu}x, x\rangle\right)$$

then by (4.7) we get

$$\frac{r}{M^{1-\nu}}\left(\langle Ax, x\rangle^{1/2} - \left\langle A^{1/2}x, x\right\rangle\right)^2 \leq \langle Ax, x\rangle^{\nu} - \langle A^{\nu}x, x\rangle \tag{4.8}$$

for any $x \in H$ with $\|x\| = 1$.

Finally, if we use the logarithmic inequality (1.13) we can state the following result as well:

Theorem 4.2. *With the assumptions of Theorem 3.1 we have*

$$\begin{aligned} &\frac{1}{2}\nu(1-\nu)\gamma\left(\ln\left(\langle f(A)x, x\rangle\right) - \langle \ln f(A)x, x\rangle\right)^2 \\ &\leq \frac{1}{2}\nu(1-\nu)\gamma \\ &\times\left(\left\langle \ln^2 f(A)x, x\right\rangle - 2\langle \ln f(A)x, x\rangle \ln\left(\langle f(A)x, x\rangle\right) + \ln^2\left(\langle f(A)x, x\rangle\right)\right) \\ &\leq (1-\nu)f\left(\langle Ax, x\rangle\right) + \nu\langle f(A)x, x\rangle - f^{1-\nu}\left(\langle Ax, x\rangle\right)\langle f^{\nu}(A)x, x\rangle \\ &\leq \frac{1}{2}\nu(1-\nu)\Delta \\ &\times\left(\left\langle \ln^2 f(A)x, x\right\rangle - 2\langle \ln f(A)x, x\rangle \ln\left(\langle f(A)x, x\rangle\right) + \ln^2\left(\langle f(A)x, x\rangle\right)\right) \end{aligned} \tag{4.9}$$

for any $\nu \in [0,1]$ *and for any* $x \in H$ *with* $\|x\| = 1$, *where* $r = \min\{1-\nu, \nu\}$ *and* $R = \max\{1-\nu, \nu\}$.

Moreover, if f *is convex on* $[m, M]$, *then*

$$\begin{aligned} &\frac{1}{2}\nu(1-\nu)\gamma(\ln(\langle f(A)x, x\rangle) - \langle \ln f(A)x, x\rangle)^2 \\ &\leq \langle f(A)x, x\rangle - f^{1-\nu}(\langle Ax, x\rangle)\langle f^{\nu}(A)x, x\rangle \end{aligned} \tag{4.10}$$

and

$$\begin{aligned} &f^{1-\nu}(\langle Ax, x\rangle)[f^{\nu}(\langle Ax, x\rangle) - \langle f^{\nu}(A)x, x\rangle] \\ &\leq \frac{1}{2}\nu(1-\nu)\Delta \\ &\times \left(\left\langle \ln^2 f(A)x, x\right\rangle - 2\langle \ln f(A)x, x\rangle \ln(\langle f(A)x, x\rangle) + \ln^2(\langle f(A)x, x\rangle)\right) \end{aligned} \tag{4.11}$$

for any $\nu \in [0,1]$ *and for any* $x \in H$ *with* $\|x\| = 1$.

If f *is concave on* $[m, M]$, *then*

$$\begin{aligned} &\frac{1}{2}\nu(1-\nu)\gamma(\ln(\langle f(A)x, x\rangle) - \langle \ln f(A)x, x\rangle)^2 \\ &\leq f^{1-\nu}(\langle Ax, x\rangle)[f^{\nu}(\langle Ax, x\rangle) - \langle f^{\nu}(A)x, x\rangle] \end{aligned} \tag{4.12}$$

for any $\nu \in [0,1]$ *and for any* $x \in H$ *with* $\|x\| = 1$.

Acknowledgement

The author would like to thank the anonymous referees for their valuable suggestions that have been implemented in the final version of the manuscript.

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