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ON THE NORMS OF SOME SPECIAL MATRICES WITH PADOVAN AND PELL-PADOVAN-LIKE SEQUENCE

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Abstract. The focus of this paper is to define some special matrices like $r-$circulant, circulant, semi-circulant, Hankel and Toeplitz matrices with the help of integer sequences. In particular, this work is focusing on obtaining norms of the aforementioned types of matrices that are involved with Pell-Padovan-like sequences. Furthermore, the upper and lower bounds of spectral norms of those matrices have been also determined.

1. Introduction and Preliminaries

Toeplitz matrices arise in many different theoretical and applicative fields, such as their applications in the mathematical modeling of all problems where some sort of shift-invariant occurs in terms of space or time. This type of matrices is widely used in the computation of spline functions, time series analysis, signal and image processing, queueing theory, polynomial and power series computations and in many other areas, where, the outcome of numerical solutions of certain differential and integral equations has been modeled by Toeplitz matrices, see for example [7] and [8]. Many articles have been written about the estimation of the spectral norms of Toeplitz matrices which have connections to signal and image processing, time series analysis and many other problems [9].

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The Fibonacci sequence in [3] is defined by

$$F_n = F_{n-1} + F_{n-2} \tag{1.1}$$

with initial conditions $F_0 = 0$ and $F_1 = 1$. Kalman [2] generalized Fibonacci sequence as follows:

$$F_{n+k} = a_0F_n + a_1F_{n+1} + a_2F_{n+2} + \ldots + a_kF_{n+k-1}. \tag{1.2}$$

In [18], third order recurrence relation is given, which is defined as Padovan sequence

$$P_n = P_{n-2} + P_{n-3} \tag{1.3}$$

with the initial conditions $P_0 = P_1 = P_2 = 1$.
The generalized form of the sequence (1.3) is given as

$$P_n = rP_{n-2} + sP_{n-3} \tag{1.4}$$

for every integer $n \geq 1$ with the initial conditions $P_0 = a, P_1 = b, P_2 = c$ where a, b, c, r and s are non- negative integers. Let us consider $a = 1$, $b = 0$, $c = 2$, $r = 2$ and $s = 1$ in (1.4), we have a third order recurrence relation which is defined as Pell-Padovan -like sequence [1] . For every integer $n \geq 3$, this sequence satisfies the folllowing

$$Q_n = 2Q_{n-2} + Q_{n-3} \tag{1.5}$$

with initial conditions $Q_0 = 1, Q_1 = 0, Q_2 = 2$.
Many papers were published on some special matrices associated with the Fibonacci and Lucas numbers [6, 10, 11, 12, 14, 15, 16]. These special two numbers are particularly popular with astonishing properties [17]. Many authors investigated the norms of special matrices with these numbers. For example, Solak [15] has found the norms of circulant matrices with these numbers. In [11] authors have found the upper and lower bounds of some special matrices with tribonacci sequence.

In this paper, circulant, $r-$circulant, semi -circulant, Hankel and Toeplitz matrices associated with third order recurrence relation specially Padovan and Pell-Padovan-like sequence will be investigated. Furthermore, the upper and lowers bounds of those special matrices are obtained. This work is initiated by stating some related preliminaries to our study.

Let U_i be any integer sequence. We define some special matrices on this sequence, in particular the ones which are defined in (1.3) and (1.5).

Matrix $A = A_r = (a_{ij}) \in M_{n,n}(\mathbb{C})$ is called $r-$circulant on any integer sequence U_i if it is of the form

$$a_{ij} = \begin{cases} \mathsf{U}_{j-i} & j \geq i \\ r\mathsf{U}_{n+j-i} & j < i \end{cases} \tag{1.6}$$

where $r \in \mathbb{C}$. If r=1, then matrix A is called circulant.

Matrix $A = (a_{ij}) \in M_{n,n}(\mathbb{C})$ is called semi-circulant on U_i if it is of the form

$$a_{ij} = \begin{cases} \mathsf{U}_{j-i+1} & i \leq j \\ 0 & \text{otherwise.} \end{cases}$$

Hankel matrix on integer sequence U_i is defined as $\mathsf{H} = (\mathsf{h}_{ij}) \in M_{n,n}(\mathbb{C})$, where $\mathsf{h}_{ij} = \mathsf{U}_{i+j-1}$.

Similarly, matrix $A = (a_{ij}) \in M_{n,n}(\mathbb{C})$ is Toeplitz matrix on integer sequence if it is of the form $a_{ij} = U_{i-j}$

The ℓ_p norm of a matrix $A = (a_{ij}) \in M_{n,n}(\mathbb{C})$ is defined by

$$\|A\|_p = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^p \right)^{1/p} \quad (1 \leq p \leq \infty).$$

If $p = \infty$, then $\|A\|_\infty = \lim_{p\to\infty} \|A\|_p = \max_{i,j} |A_{ij}|$.

The Euclidean (Frobenius) norm of the matrix A is defined as

$$\|A\|_F = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2 \right)^{1/2}.$$

The spectral norm of the matrix A is given as

$$\|A\|_2 = \sqrt{\max_{1\leq i\leq n} |\gamma_i|},$$

where γ_i are the eigenvalues of the matrix $\left(\bar{A}\right)^t A$.

The following inequality between Euclidean and spectral norm holds [19]

$$\frac{1}{\sqrt{n}}\|A\|_F \leq \|A\|_2 \leq \|A\|_F. \tag{1.7}$$

Definition 1.1. [13] Let $A = (a_{ij})$ and $B = (b_{ij})$ be $m \times n$ matrices. Then, the Hadamard product of A and B is given by

$$A \circ B = (a_{ij}b_{ij}).$$

Definition 1.2. [15] The maximum column length norm $c_1(.)$ and maximum row length norm $r_1(.)$ for $m \times n$ matrix $A = (a_{ij})$ is defined $c_1(A) = \sqrt{\max_j \sum_i |a_{ij}|^2}$ and $r_1(A) = \sqrt{\max_i \sum_j |a_{ij}|^2}$ respectively.

Theorem 1.1. [5] *Let $A = (a_{ij})$, $B = (b_{ij})$ and $C = (c_{ij})$ be $p \times q$ matrices. If $C = A \circ B$, then $\|C\|_2 \leq r_1(A)c_1(B)$.*

The following lemmas describes the properties Padovan sequence.

Lemma 1.1. *The sum of first n terms of Padovan sequence is given by:*

$$\sum_{m=0}^{n} P_m = P_{n+5} - 2.$$

Lemma 1.2. *The sum of square of first n terms of Padovan sequence is given as:*

$$\sum_{m=0}^{n} P_m^2 = P_{n+2}^2 - P_{n-1}^2 - P_{n-3}^2.$$

Lemma 1.3. *For every $n > 0$ the following identity is hold.*

$$\sum_{k=1}^{n}\sum_{m=1}^{k} P_m^2 = M_k = P_{n+1}^2 + 2P_n^2 + P_{n-3}^2 + 2P_{n-1}^2 + P_{n-2}^2 - n - 4.$$

Proof.

$$\sum_{k=1}^{n}\sum_{m=1}^{k} P_m^2 = \sum_{k=1}^{n}\left(P_{k+2}^2 - P_{k-1}^2 - P_{k-3}^2 - 1\right)$$

$$\sum_{k=1}^{n}\sum_{m=1}^{k} P_m^2 = \sum_{k=1}^{n} P_{n+2}^2 - \sum_{k=1}^{n} P_{n-1}^2 - \sum_{k=1}^{n} P_{n-3}^2 - n,$$

by using lemma (1.2), we have

$$\sum_{k=1}^{n}\sum_{m=1}^{k} P_m^2 = P_{n+1}^2 + 2P_n^2 + P_{n-3}^2 + 2P_{n-1}^2 + P_{n-2}^2 - n - 4.$$

□

There is a connection between Pell-Padovan -Like sequence and the Fibonacci sequence given in the following lemmas.

Lemma 1.4. [1] *For every* $n \geq 0$ *, we have* $Q_n = F_n + (-1)^n$ *where* F_n *is the sequence defined in (1.1).*

Lemma 1.5. *For every* $n > 0$ *the following identity is holds:*

$$\sum_{i=0}^{n-1} Q_i^2 = \left(F_{n-1}F_n + 2(-1)^{n-1}F_{n-2} + n - 2\right).$$

Lemma 1.6. [1] *The Sum of first n terms of Pell-Padovan-Like numbers is given by:*

$$\sum_{i=0}^{n} Q_i = \frac{1}{2}\left[F_{n+2} + F_{n+1} + F_n + (-1)^n - 1\right].$$

Lemma 1.7. *The sum of square of first n terms of Pell-Padovan-like numbers is.*

$$\sum_{k=1}^{n}\sum_{i=1}^{k} Q_i^2 = \left(\frac{F_{n+1}^2 + F_nF_{n+2} + n(n+1) - 4n + 4(-1)^{n+1}(F_{n-1} - F_n) + 3}{2}\right).$$

Proof. Using lemma (1.4)

$$\sum_{k=1}^{n}\sum_{i=1}^{k} Q_i^2 = \sum_{i=1}^{k}\left(F_i + (-1)^i\right)^2$$

$$\sum_{k=1}^{n}\sum_{i=1}^{k} Q_i^2 = \sum_{k}^{n}\sum_{i=1}^{k}\left(F_i^2 + 1 + 2(-1)^i F_i\right)$$

$$\sum_{k=1}^{n}\sum_{i=1}^{k} Q_i^2 = \sum_{k=1}^{n}\left(\sum_{i=1}^{n} F_i^2 + k + 2\sum_{i=1}^{n}(-1)^i F_i\right)$$

we know that

$$\sum_{i=1}^{k}(-1)^{i+1}F_i = (-1)^{k+1}F_{k-1} + 1 \tag{1.8}$$

so we have

$$\sum_{k=1}^{n}\sum_{i=1}^{k} Q_i^2 = \sum_{k=1}^{n}\left(F_k F_{k+1} + 2(-1)^k F_{k-1} + k - 2\right)$$

again using the results we have

$$\sum_{k=1}^{n}\sum_{i=1}^{k} Q_i^2 = \left(\frac{F_{n+1}^2 + F_n F_{n+2} + n(n+1) - 4n + 4(-1)^{n+1}(F_{n-1} - F_n) + 3}{2}\right).$$

□

Lemma 1.8. *For any $n > 0$ the following identity is hold*

$$\sum_{k=1}^{n}\sum_{i=1}^{k} Q_{-k}^2 = \frac{F_{n+1}^2 + F_{n+2}(F_n - 4) - 2F_{n+1} + 4n + 4n(n+1) + 5}{2}.$$

Proof. From lemma (1.4) we have

$$Q_{-k} = (-1)^{k+1}F_k + (-1)^k$$

implies that

$$Q_{-k}^2 = \left(F_k^2 - 2F_k + 1\right)$$

Taking Double sum on both sides

$$\sum_{i=1}^{n}\sum_{k=1}^{i} Q_{-k}^2 = \sum_{k=1}^{n}\sum_{i=1}^{k}\left(F_k^2 - 2F_k + 1\right)$$

$$\sum_{k=1}^{n}\sum_{i=1}^{k} Q_{-i}^2 = \sum_{k=1}^{n}\left(\sum_{i=1}^{k} F_i^2 - 2\sum_{i=1}^{k} F_i + \sum_{i=1}^{k} 1\right)$$

$$\sum_{i=1}^{n}\sum_{i=1}^{k} Q_{-i}^2 = \sum_{i=1}^{n} \left(F_k F_{k+1} - 2\left(F_{k+2}-1\right) + k\right)$$

$$\sum_{k=1}^{n}\sum_{i=1}^{k} Q_{-k}^2 = \left(\left(\frac{F_{n+1}^2 + F_n F_{n+2} - 1}{2}\right) - 2\left(2F_{n+2} + F_{n+1} - 3\right) + 2n + n(n+1)\right)$$

$$\sum_{k=1}^{n}\sum_{i=1}^{k} Q_{-k}^2 = \frac{F_{n+1}^2 + F_{n+2}(F_n - 4) - 2F_{n+1} + 4n + 4n(n+1) + 5}{2}.$$

□

2. Padovan sequence

In this section we will obtain the norms of $r-$circulant, circulant, semi-circulant and Hankel matrices associated with the Padovan sequence (1.3).

Theorem 2.1. *Let $A = A_r(P_0, P_{1,}...P_{n-1})$ be r-circulant matrix.*
If $|r| \geq 1$, then $\sqrt{P_{n+1}^2 - P_{n-2}^2 - P_{n-4}^2} \leq \|A\|_2 \leq |r|\left(P_{n+1}^2 - P_{n-2}^2 - P_{n-4}^2\right)$
If $|r| < 1$, then $|r|\sqrt{P_{n+1}^2 - P_{n-2}^2 - P_{n-4}^2} \leq \|A\|_2 \leq \sqrt{n\left(P_{n+1}^2 - P_{n-2}^2 - P_{n-4}^2\right)}$.

Proof. The $r-$circulant matrix A is given as:

$$A = \begin{bmatrix} P_0 & P_1 & P_2 & \cdots & P_{n-1} \\ rP_{n-1} & P_0 & P_1 & \cdots & P_{n-2} \\ rP_{n-2} & rP_{n-1} & P_0 & \cdots & P_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ rP_1 & rP_2 & rP_3 & \cdots & P_0 \end{bmatrix}$$

and from the definition of Euclidean norm, we have

$$\|A\|_F^2 = \sum_{k=0}^{n-1}(n-k)\,P_k^2 + \sum_{k=1}^{n-1} k|r|^2 P_k^2 \tag{2.1}$$

Here we have two ceases depending on r. Case 1. when $|r| \geq 1$

$$\|A\|_F^2 \geq \sum_{k=0}^{n-1}(n-k)P_k^2 + \sum_{k=1}^{n-1} kP_k^2 = n\sum_{k=0}^{n-1} P_k^2$$

Using lemma (1.5), we obtain

$$\|A\|_F^2 \geq n\sum_{k=1}^{n-1} P_k^2 = n(P_{n+1}^2 - P_{n-2}^2 - P_{n-4}^2)$$

by inequality (1.7)

$$\|A\|_2 \geq \sqrt{P_{n+1}^2 - P_{n-2}^2 - P_{n-4}^2} \tag{2.2}$$

On the other hand, let us define two new matrices C and D

$$C = \begin{bmatrix} rP_0 & 1 & 1 & \cdots & 1 \\ rP_{n-1} & rP_0 & 1 & \cdots & 1 \\ rP_{n-2} & rP_{n-1} & rP_0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ rP_1 & rP_2 & rP_3 & \cdots & rP_0 \end{bmatrix} \quad D = \begin{bmatrix} P_0 & P_1 & P_2 & \cdots & P_{n-1} \\ 1 & P_0 & P_1 & \cdots & P_{n-2} \\ 1 & 1 & P_0 & \cdots & P_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & P_0 \end{bmatrix}$$

Such that $A = C \circ D$, then by definition (1.2)

$$r_1(C) = \max_{i \leq i \leq n} \sqrt{\sum_{j=1}^{n} |c_{ij}|^2} = \sqrt{\sum_{j=1}^{n} |c_{nj}|^2} = \sqrt{|r|^2 \sum_{k=0}^{n-1} P_k^2} = |r| \sqrt{P_{n+1}^2 - P_{n-2}^2 - P_{n-4}^2}$$

$$c_1(D) = \max_{1 \leq j \leq n} \sqrt{\sum_{i=1}^{n} |d_{ij}|^2} = \sqrt{\sum_{i=1}^{n} |c_{nj}|^2} = \sqrt{\sum_{k=0}^{n-1} P_k^2} = \sqrt{P_{n+1}^2 - P_{n-2}^2 - P_{n-4}^2}$$

Now using theorem (1.3), we get

$$\|A\|_2 \leq r_1(C)c_1(D) = |r| \left(P_{n+1}^2 - P_{n-2}^2 - P_{n-4}^2\right) \tag{2.3}$$

combine equations (2.2) and (2.3), we have

$$\sqrt{P_{n+1}^2 - P_{n-2}^2 - P_{n-4}^2} \leq \|A\|_2 \leq |r| \left(P_{n+1}^2 - P_{n-2}^2 - P_{n-4}^2\right)$$

Case 2. When $|r| \leq 1$, Then we have

$$\|A\|_F^2 \geq \sum_{k=0}^{n-1} (n-k)\, |r|^2 P_k^2 + \sum_{k=0}^{n-1} k|r|^2 P_k^2 = n\sum_{k=0}^{n-1} |r|^2 P_k^2$$

$$n|r|^2 \sum_{k=0}^{n-1} P_k^2 = n|r|^2 \left(P_{n+1}^2 - P_{n-2}^2 - P_{n-4}^2\right)$$

$$\frac{1}{\sqrt{n}}\|A\|_F \geq |r| \sqrt{P_{n+1}^2 - P_{n-2}^2 - P_{n-4}^2}$$

Then, by inequality (1.7)

$$\|A\|_2 \geq |r| \sqrt{P_{n+1}^2 - P_{n-2}^2 - P_{n-4}^2}. \tag{2.4}$$

Let us define two new matrices C' and D' as

$$C' = \begin{bmatrix} P_0 & 1 & 1 & \cdots & 1 \\ r & P_0 & 1 & \cdots & 1 \\ r & r & P_0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r & r & r & \cdots & P_0 \end{bmatrix}, \ D' = \begin{bmatrix} P_0 & P_1 & P_2 & \cdots & P_{n-1} \\ P_{n-1} & P_0 & P_1 & \cdots & P_{n-2} \\ P_{n-2} & P_{n-1} & P_0 & \cdots & P_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ P_1 & P_2 & P_3 & \cdots & P_0 \end{bmatrix}$$

Such that $A = C' \circ D'$, then we obtain

$$r_1(C') = \max_{1\leq j\leq n} \sqrt{\sum_{i=1}^{n} |c'_{ij}|^2} = \sqrt{P_0^2 + (n-1)} = \sqrt{n}$$

$$c_1(D') = \max_{1\leq i\leq n} \sqrt{\sum_{j=1}^{n} |d'_{ij}|^2} = \sqrt{\sum_{k=0}^{n-1} P_k^2} = \sqrt{P_{n+1}^2 - P_{n-2}^2 - P_{n-4}^2}$$

$$\|A\|_2 \leq r_1(C')c_1(D') = \sqrt{n}\sqrt{P_{n+1}^2 - P_{n-2}^2 - P_{n-4}^2}$$

$$\|A\|_2 \leq \sqrt{n\left(P_{n+1}^2 - P_{n-2}^2 - P_{n-4}^2\right)} \tag{2.5}$$

using equations (2.4) and (2.5), we have

$$|r| \sqrt{P_{n+1}^2 - P_{n-2}^2 - P_{n-4}^2} \leq \|A\|_2 \leq \sqrt{n\left(P_{n+1}^2 - P_{n-2}^2 - P_{n-4}^2\right)}.$$

□

Theorem 2.2. *Let A be a circulant matrix and $\|A\|_F = \sqrt{n\left(P_{n+1}^2 - P_{n-2}^2 - P_{n-4}^2\right)}$. Then*

$$\sqrt{\left(P_{n+1}^2 - P_{n-2}^2 - P_{n-4}^2\right)} \leq \|A\|_2 \leq \left(P_{n+1}^2 - P_{n-2}^2 - P_{n-4}^2\right).$$

Proof. Since circulant matrix on Padovan sequence is defined by

$$A = \begin{bmatrix} P_0 & P_1 & P_2 & \cdots & P_{n-1} \\ P_{n-1} & P_0 & P_1 & \cdots & P_{n-2} \\ P_{n-2} & P_{n-1} & P_0 & \cdots & P_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ P_1 & P_2 & P_3 & \cdots & P_0 \end{bmatrix}.$$

Let matrices B and C be defined as

$$B = \begin{cases} b_{ij} = P_{(\bmod (j-i,n))} & i \leq j \\ b_{ij} = 1 & i < j \end{cases} \text{ and, } C = \begin{cases} c_{ij} = P_{(\bmod (j-i,n))} & i < j \\ c_{ij} = 1 & i \geq j \end{cases}$$

It is easy to see that $A = B \circ C$, then by definition (1.2)

$$r_1(B) = \max_i \sqrt{\sum_{j=1}^{n} |b_{ij}|^2} = \sqrt{\sum_{i=0}^{n-1} P_i^2} = \sqrt{P_{n+1}^2 - P_{n-2}^2 - P_{n-4}^2}$$

$$c_1(C) = \max_j \sqrt{\sum_{i=1}^{n} |c_{ij}|^2} = \sqrt{1 + \sum_{i=1}^{n-1} P_i^2} = \sqrt{P_{n+1}^2 - P_{n-2}^2 - P_{n-4}^2}$$

by theorem (1.3),we get

$$\|A\|_2 \leq P_{n+1}^2 - P_{n-2}^2 - P_{n-4}^2. \tag{2.6}$$

For the Euclidean norm of the matrix A, we have

$$\|A\|_F = \sqrt{n\left(P_{n+1}^2 - P_{n-2}^2 - P_{n-4}^2\right)}$$

by inequality (1.9)

$$\sqrt{P_{n+1}^2 - P_{n-2}^2 - P_{n-4}^2} \leq \|A\|_2 \tag{2.7}$$

so by equations (2.6) and (2.7), we obtain

$$\sqrt{P_{n+1}^2 - P_{n-2}^2 - P_{n-4}^2} \leq \|A\|_2 \leq P_{n+1}^2 - P_{n-2}^2 - P_{n-4}^2.$$

□

Theorem 2.3. *The Euclidean norm of $n \times n$ semi-circulant matrix $A = (a_{ij})$ with the Padovan numbers is given as*

$$\|A\|_F^2 = P_{n+1}^2 + 2P_n^2 + P_{n-3}^2 + 2P_{n-1}^2 + P_{n-2}^2 - n - 4.$$

Proof. For the semicirculant matrix $A = (a_{ij})$ with the Padovan numbers we have

$$a_{ij} = \begin{cases} P_{j-i+1} & i \leq j \\ 0 & otherwise. \end{cases}$$

From the definition of Euclidean norm, we have

$$\|A\|_F^2 = \sum_{j=1}^{n} \sum_{i=1}^{j} (P_{j-i+1})^2 = \sum_{j=1}^{n} \sum_{k=1}^{j} \left(P_k{}^2\right).$$

Using lemma (1.6) , we have

$$\|A\|_E^2 = P_{n+1}^2 + 2P_n^2 + P_{n-3}^2 + 2P_{n-1}^2 + P_{n-2}^2 - n - 4.$$

□

Theorem 2.4. *Let $A = (a_{ij})$ be $n \times n$ Hankel matrix with $a_{ij} = P_{i+j-1}$. Then $\|A\|_1 = \|A\|_\infty = P_{2n+4} - P_{n+4}$.*

Proof. From the definition of the matrix A , we can write

$$\|A\|_1 = \max_{i \leq j \leq n} \sum_{i=1}^{n} |a_{ij}| = \max_{1 \leq j \leq n} \{|a_{1j}| + |a_{2j}| + |a_{3j}| + \cdots + |a_{nj}|\}$$

$$\|A\|_1 = P_n + P_{n+1} + P_{n+2} + \cdots + P_{2n-1}$$

$$\|A\|_1 = \sum_{i=1}^{2n-1} P_i - \sum_{i=1}^{n-1} P_i$$

by lemma (1.4), we have

$$\|A\|_1 = P_{2n+4} - P_{n+4}$$

similarly the row norm of the matrix A can be computed as

$$\|A\|_\infty = \max_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}| = \|A\|_1 = P_{2n+4} - P_{n+4}.$$

□

Theorem 2.5. *If $A = (a_{ij})$ is $n \times n$ Hankel matrix with $a_{ij} = P_{i+j-1}$, then*

$$\|A\|_F = \sqrt{P^2_{2n+1} - P^2_{2n-2} - P^2_{2n-4} - 2P^2_{n+1} + 2P^2_{n-2} + 2P^2_{n-4} + 1}.$$

Proof. Since

$$A = \begin{bmatrix} P_1 & P_2 & P_3 & \cdots & P_{n-1} & P_n \\ P_2 & P_3 & P_4 & \cdots & P_n & P_{n+1} \\ P_3 & P_4 & P_5 & \cdots & P_{n+1} & P_{n+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ P_{n-1} & P_n & P_{n+1} & \cdots & P_{2n-3} & P_{2n-2} \\ P_n & P_{n+1} & P_{n+2} & \cdots & P_{2n-2} & P_{2n-1} \end{bmatrix}$$

$$\|A\|_F = \left(\sum_{i=1}^{n}\sum_{j=1}^{n} |a_{ij}|^2\right)^{\frac{1}{2}} = \left(\sum_{k=1}^{n} P_k^2 + \sum_{k=2}^{n+1} P_k^2 + \cdots + \sum_{k=2}^{2n-1} P_k^2\right)^{\frac{1}{2}}$$

$$\|A\|_F = \left(\left(\sum_{k=1}^{n} P_k^2 + \sum_{k=1}^{n+1} P_k^2 + \cdots + \sum_{k=1}^{2n-1} P_k^2\right) - \left(\sum_{k=1}^{n-1}\sum_{i=1}^{k} P_i^2\right)\right)^{\frac{1}{2}}$$

$$\|A\|_F = \left(\left(\sum_{k=1}^{n} P_k^2 + \sum_{k=1}^{n+1} P_k^2 + \cdots + \sum_{k=1}^{2n-1} P_k^2\right) - \left(\sum_{k=1}^{n-1}\sum_{i=1}^{k} P_i^2\right)\right)^{\frac{1}{2}}$$

$$\|A\|_F = \sqrt{(M_n + M_{n+1} + M_{n+2} + \cdots M_{2n-1}) - \sum_{k=1}^{n-1} M_k}$$

where M_k is defined in lemma (1.6)

$$\|A\|_F = \sqrt{\sum_{k=1}^{2n-1} M_k - 2\sum_{k=1}^{n-1} M_k}$$

$$\|A\|_F = \sqrt{P_{2n+1}^2 - P_{2n-2}^2 - P_{2n-4}^2 - 2P_{n+1}^2 + 2P_{n-2}^2 + 2P_{n-4}^2 + 1}$$

□

Theorem 2.6. *If $A = (a_{ij})$ is $n \times n$ Hankel matrix with $a_{ij} = P_{i+j-1}$ then,*

$$\frac{1}{\sqrt{n}}\|A\|_F \le \|A\|_2 \le P_{n+2}^2 - P_{n-1}^2 - P_{n-3}^2 - 1.$$

Proof. From theorem (2.5),and inequality (1.7)

$$\frac{1}{\sqrt{n}}\|A\|_F \le \|A\|_2. \tag{2.8}$$

Let us define two new matrices

$$U_n = \begin{cases} P_{i+j-1} & i \le j \\ 1 & i > j \end{cases} \quad \text{and } V_n = \begin{cases} P_{i+j-1} & i > j \\ 1 & i \le j. \end{cases}$$

It is easy to see that $A = U_n \circ V_n$. Thus we obtain the result from definition (1.2)

$$r_1(U_n) = \max_i \sqrt{\sum_j |u_{ij}|^2} = \sqrt{\sum_{i=1}^{n} P_i^2} = \sqrt{P_{n+2}^2 - P_{n-1}^2 - P_{n-3}^2 - 1}$$

and

$$c_1(V_n) = \max_j \sqrt{\sum_i |v_{ij}|^2} = \sqrt{1 + \sum_{i=2}^{n} P_i^2} = \sqrt{P_{n+2}^2 - P_{n-1}^2 - P_{n-3}^2 - 1}.$$

Using the theorem (1.3)

$$\|A\|_2 \le P_{n+2}^2 - P_{n-1}^2 - P_{n-3}^2 - 1. \tag{2.9}$$

By combine (2.8) and (2.9), we have get the required result. □

Example 2.1. *Now, we validate our results for particular values of n, see table below*

Matrices	$n = 5$	$n = 6$	$n = 9$
r− circulant	$\lvert r\rvert \geq 1,\ \sqrt{11} \leq \lvert\lvert A\rvert\rvert_2 \leq 11\lvert r\rvert$	$\lvert r\rvert \geq 1,\ \sqrt{20} \leq \lvert\lvert A\rvert\rvert_2 \leq 20\lvert r\rvert$	$\lvert r\rvert \geq 1,\ \sqrt{110} \leq \lvert\lvert A\rvert\rvert_2 \leq 110\lvert r\rvert$
	$\lvert r\rvert \leq 1,\ \lvert r\rvert\sqrt{11} \leq \lvert\lvert A\rvert\rvert_2 \leq \sqrt{55}$	$\lvert r\rvert \leq 1,\ \lvert r\rvert\sqrt{20} \leq \lvert\lvert A\rvert\rvert_2 \leq \sqrt{120}$	$\lvert r\rvert \leq 1,\ \lvert r\rvert\sqrt{110} \leq \lvert\lvert A\rvert\rvert_2 \leq \sqrt{990}$
circulant	$\sqrt{11} \leq \lvert\lvert A\rvert\rvert_2 \leq 11$	$\sqrt{20} \leq \lvert\lvert A\rvert\rvert_2 \leq 20$	$\sqrt{110} \leq \lvert\lvert A\rvert\rvert_2 \leq 110$
semi-circulant	$\lvert\lvert A\rvert\rvert_F = \sqrt{38}$	$\lvert\lvert A\rvert\rvert_F = \sqrt{73}$	$\lvert\lvert A\rvert\rvert_F = \sqrt{430}$
Hankel	$\lvert\lvert A\rvert\rvert_1 = \lvert\lvert A\rvert\rvert_\infty = 28$	$\lvert\lvert A\rvert\rvert_1 = \lvert\lvert A\rvert\rvert_\infty = 53$	$\lvert\lvert A\rvert\rvert_1 = \lvert\lvert A\rvert\rvert_\infty = 323$
	$\lvert\lvert A\rvert\rvert_F = \sqrt{170}$	$\lvert\lvert A\rvert\rvert_F = \sqrt{552}$	$\lvert\lvert A\rvert\rvert_F = \sqrt{16988}$
	$\sqrt{34} \leq \lvert\lvert A\rvert\rvert_2 \leq 19$	$\sqrt{92} \leq \lvert\lvert A\rvert\rvert_2 \leq 35$	$\sqrt{\frac{16988}{9}} \leq \lvert\lvert A\rvert\rvert_2 \leq 190$

3. Pell -Padovan -like sequence

In this section we will obtain norms of $r-$circulant, circulant, semi-circulant, Hankel and Toeplitz matrices with Pell-Padovan- like sequence (1.5).

Theorem 3.1. *Let $B = B_r(Q_0, Q_1, ...Q_{n-1})$ be $r-$circulant matrix.*
If $|r| \geq 1$, then $\sqrt{\left(F_{n-1}F_n + 2(-1)^{n-1}F_{n-2} + n - 2\right)} \leq \|B\|_2 \leq |r| \left(F_{n-1}F_n + 2(-1)^{n-1}F_{n-2} + n - 2\right)$
If $|r| < 1$, then $|r|\sqrt{\left(F_{n-1}F_n + 2(-1)^{n-1}F_{n-2} + n - 2\right)} \leq \|B\|_2 \leq \sqrt{n\left(F_{n-1}F_n + 2(-1)^{n-1}F_{n-2} + n - 2\right)}$

Proof. The $r-$circulant matrix B is of the form

$$B = \begin{bmatrix} Q_0 & Q_1 & Q_2 & \cdots & Q_{n-1} \\ rQ_{n-1} & Q_0 & Q_1 & \cdots & Q_{n-2} \\ rQ_{n-2} & rQ_{n-1} & Q_0 & \cdots & Q_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ rQ_1 & rQ_2 & rQ_3 & \cdots & Q_0 \end{bmatrix}.$$

Then by definition of Euclidean norm

$$\|B\|_F^2 = \sum_{k=0}^{n-1} (n-k)\, Q_k^2 + \sum_{k=1}^{n-1} k|r|^2 Q_k^2. \tag{3.1}$$

Here we have two ceases depending on r.

Case 1. when $|r| \geq 1$

$$\|B\|_F^2 \geq \sum_{k=0}^{n-1} (n-k)Q_k^2 + \sum_{k=1}^{n-1} kQ_k^2 = n \sum_{k=0}^{n-1} Q_k^2.$$

Using he lemma (1.8), we get

$$\|B\|_F^2 \geq n \sum_{k=0}^{n-1} Q_k^2 = n\left(F_{n-1}F_n + 2(-1)^{n-1}F_{n-2} + n - 2\right)$$

$$\frac{1}{\sqrt{n}}\|B\|_F \geq \sqrt{\left(F_{n-1}F_n + 2(-1)^{n-1}F_{n-2} + n - 2\right)}$$

by inequality (1.7), we have

$$\|B\|_2 \geq \sqrt{\left(F_{n-1}F_n + 2(-1)^{n-1}F_{n-2} + n - 2\right)}. \tag{3.2}$$

On the other hand, let the matrices C and D be defined as

$$C = \begin{bmatrix} rQ_0 & 1 & 1 & \cdots & 1 \\ rQ_{n-1} & rQ_0 & 1 & \cdots & 1 \\ rQ_{n-2} & rQ_{n-1} & rQ_0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ rQ_1 & rQ_2 & rQ_3 & \cdots & rQ_0 \end{bmatrix}, D = \begin{bmatrix} Q_0 & Q_1 & Q_2 & \cdots & Q_{n-1} \\ 1 & Q_0 & Q_1 & \cdots & Q_{n-2} \\ 1 & 1 & Q_0 & \cdots & Q_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & Q_0 \end{bmatrix}$$

Such that $B = C \circ D$, then by definition (1.2) we have

$$r_1(C) = \max_{i \leq i \leq n} \sqrt{\sum_{j=1}^{n} |c_{ij}|^2} = \sqrt{\sum_{j=1}^{n} |c_{nj}|^2}$$

$$= \sqrt{|r|^2 \sum_{k=0}^{n-1} Q_k^2} = |r| \sqrt{\left(F_{n-1}F_n + 2(-1)^{n-1}F_{n-2} + n - 2\right)}$$

$$c_1(D) = \max_{1 \leq j \leq n} \sqrt{\sum_{i=1}^{n} |d_{ij}|^2} = \sqrt{\sum_{i=1}^{n} |c_{nj}|^2} = \sqrt{\sum_{k=0}^{n-1} Q_k^2} = \sqrt{\left(F_{n-1}F_n + 2(-1)^{n-1}F_{n-2} + n - 2\right)}$$

using theorem (1.3), we have

$$\|B\|_2 \leq r_1(C)c_1(D) = |r| \left(F_{n-1}F_n + 2(-1)^{n-1}F_{n-2} + n - 2\right)$$

$$\|B\|_2 \leq |r| \left(F_{n-1}F_n + 2(-1)^{n-1}F_{n-2} + n - 2\right) \tag{3.3}$$

Now using the equations (3.2) and (3.3), we get the required result

$$\sqrt{\left(F_{n-1}F_n + 2(-1)^{n-1}F_{n-2} + n - 2\right)} \leq \|B\|_2 \leq |r| \left(F_{n-1}F_n + 2(-1)^{n-1}F_{n-2} + n - 2\right)$$

Case 2.When $|r| \leq 1$, Then we have

$$\|B\|_F^2 \geq \sum_{k=0}^{n-1} (n-k)\,|r|^2 Q_k^2 + \sum_{k=0}^{n-1} k|r|^2 Q_k^2 = n \sum_{k=0}^{n-1} |r|^2 Q_k^2$$

by using the lemma (1.8), we have

$$n|r|^2 \sum_{k=0}^{n-1} Q_k^2 = n|r|^2 \left(F_{n-1}F_n + 2(-1)^{n-1}F_{n-2} + n - 2\right).$$

Thus

$$\frac{1}{\sqrt{n}}\|B\|_F \geq |r|\sqrt{\left(F_{n-1}F_n + 2(-1)^{n-1}F_{n-2} + n - 2\right)}$$

by inequality (1.7)

$$\|B\|_2 \geq |r| \sqrt{\left(F_{n-1}F_n + 2(-1)^{n-1}F_{n-2} + n - 2\right)} \tag{3.4}$$

Let us define two new matrices C' and D' be as

$$C' = \begin{bmatrix} Q_0 & 1 & 1 & \cdots & 1 \\ r & Q_0 & 1 & \cdots & 1 \\ r & r & Q_0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r & r & r & \cdots & Q_0 \end{bmatrix} \text{ and, } D' = \begin{bmatrix} Q_0 & Q_1 & Q_2 & \cdots & Q_{n-1} \\ Q_{n-1} & Q_0 & P_1 & \cdots & Q_{n-2} \\ Q_{n-2} & Q_{n-1} & Q_0 & \cdots & Q_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ Q_1 & Q_2 & Q_3 & \cdots & Q_0 \end{bmatrix}$$

It is clear that $B = C' \circ D'$, then we obtain by definition (1.2)

$$r_1(C') = \max_{1 \leq i \leq n} \sqrt{\sum_{j=1}^{n} |c'_{ij}|^2} = \sqrt{Q_0^2 + (n-1)} = \sqrt{n}$$

$$c_1(D') = \max_{1\le j\le n}\sqrt{\sum_{i=1}^{n}|d'_{ij}|^2} = \sqrt{\sum_{k=0}^{n-1}Q_k^2} = \sqrt{\left(F_{n-1}F_n + 2(-1)^{n-1}F_{n-2} + n - 2\right)}.$$

So by theorem (1.3) we obtain

$$\|B\|_2 \le r_1(C')c_1(D') = \sqrt{n}\sqrt{\left(F_{n-1}F_n + 2(-1)^{n-1}F_{n-2} + n - 2\right)}$$

$$\|B\|_2 \le \sqrt{n\left(F_{n-1}F_n + 2(-1)^{n-1}F_{n-2} + n - 2\right)} \tag{3.5}$$

So we have the result by using the equations (3.4) and (3.5)

$$|r|\sqrt{\left(F_{n-1}F_n + 2(-1)^{n-1}F_{n-2} + n - 2\right)} \le \|B\|_2 \le \sqrt{n\left(F_{n-1}F_n + 2(-1)^{n-1}F_{n-2} + n - 2\right)}$$

□

Theorem 3.2. *Let B be a circulant matrix such that*

$$\|B\|_F = \sqrt{n\left(F_{n-1}F_n + 2(-1)^{n-1}F_{n-2} + n - 2\right))}$$

then

$$\sqrt{\left(F_{n-1}F_n + 2(-1)^{n-1}F_{n-2} + n - 2\right)} \le \|B\|_2$$

and

$$\|B\|_2 \le \sqrt{\left(F_{n-1}F_n + 2(-1)^{n-1}F_{n-2} + n - 2\right)}\sqrt{1 + \left(F_{n-1}F_n + 2(-1)^{n-1}F_{n-2} + n - 2\right)}.$$

Proof. Since circulant matrix for Pell-Padovan-Like sequence is defined by

$$B = \begin{bmatrix} Q_0 & Q_1 & Q_2 & \cdots & Q_{n-1} \\ Q_{n-1} & Q_0 & Q_1 & \cdots & Q_{n-2} \\ Q_{n-2} & Q_{n-1} & Q_0 & \cdots & Q_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ Q_1 & Q_2 & Q_3 & \cdots & Q_0 \end{bmatrix}$$

Let matrices B'' and C'' be defined as

$$B'' = \begin{cases} b_{ij} = Q_{(j-i, mod\ n)} & i \leq j \\ b_{ij} = 1 & i < j \end{cases}, \; C'' = \begin{cases} c_{ij} = Q_{(j-i, mod\ n)} & i < j \\ c_{ij} = 1 & i \geq j \end{cases}$$

such that $B = B'' \circ C''$ then by definition (1.2), we have

$$r_1(B'') = \max_i \sqrt{\sum_{j=1}^{n} |b_{ij}|^2} = \sqrt{\sum_{i=0}^{n-1} Q_i^2} = \sqrt{\left(F_{n-1}F_n + 2(-1)^{n-1}F_{n-2} + n - 2\right)}$$

$$c_1(C'') = \max_j \sqrt{\sum_{i=1}^{n} |c_{ij}|^2} = \sqrt{1 + \sum_{i=1}^{n-1} Q_i^2} = \sqrt{1 + \left(F_{n-1}F_n + 2(-1)^{n-1}F_{n-2} + n - 2\right)}$$

by theorem (1.3), we get

(3.6)

$$\|B\|_2 \leq \sqrt{\left(F_{n-1}F_n + 2(-1)^{n-1}F_{n-2} + n - 2\right)}\sqrt{1 + \left(F_{n-1}F_n + 2(-1)^{n-1}F_{n-2} + n - 2\right)}.$$

For the Euclidean norm of the matrix B, we have

$$\|B\|_F = \sqrt{n\left(F_{n-1}F_n + 2(-1)^{n-1}F_{n-2} + n - 2\right))}$$

by inequality (1.7)

$$\sqrt{\left(F_{n-1}F_n + 2(-1)^{n-1}F_{n-2} + n - 2\right)} \leq \|B\|_2 \tag{3.7}$$

which complete the proof. □

Theorem 3.3. *The Euclidean norm of $n \times n$ semi-circulant matrix $B = (b_{ij})$ with the Pell- Padovan- like numbers numbers is given as*

$$\|B\|_F^2 = \frac{F_{n+1}^2 + F_nF_{n+2} + n(n-3) + 3 - 4(-1)^{n+1}F_{n-2}}{2}.$$

Proof. For the semi-circulant matrix $B = (b_{ij})$ with the Pell- Padovan- like numbers we have

$$b_{ij} = \begin{cases} Q_{j-i+1} & i \leq j \\ 0 & otherwise. \end{cases}$$

From the definition of Euclidean norm, we have
$\|B\|_F^2 = \sum\limits_{j=1}^{n}\sum\limits_{j=1}^{j}(Q_{j-i+1})^2 = \sum\limits_{j=1}^{n}\left(\sum\limits_{k=1}^{j} Q_k^2\right)$
Using lemma (1.7), we have

$$\|B\|_F^2 = \left(\frac{F_{n+1} + F_n F_{n+2} + n(n+1) - 4n + 4(-1)^{n+1}(F_{n-1} - F_n) + 3}{2}\right).$$

□

Theorem 3.4. *Let $B = (b_{ij})$ be $n \times n$ Hankel matrix with $b_{ij} = Q_{i+j-1}$. Then we have*

$$\|B\|_1 = \frac{F_{2n+1} + F_{2n} + F_{2n-1} - 1 - 2F_{n+1} + (-1)^n}{2}.$$

Proof. From the definition of the matrix B , we can write

$$\|B\|_1 = \max_{i \le j \le n} \sum_{i=1}^{n} |a_{ij}| = \max_{1 \le j \le n} \{|a_{1j}| + |a_{2j}| + |a_{3j}| \ldots |a_{nj}|\}$$

$$\|B\|_1 = Q_n + Q_{n+1} + Q_{n+2} + \cdots + Q_{2n-1}$$

by lemma (1.9), we have

$$\|B\|_1 = \sum_{i=1}^{2n-1} Q_i - \sum_{i=1}^{n-1} Q_i$$

$$\|B\|_1 = \frac{F_{2n+1} + F_{2n} + F_{2n-1} - 1 - 2F_{n+1} + (-1)^n}{2}$$

similarly the row norm of the matrix B can be computed as

$$\|B\|_\infty = \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}| = \frac{F_{2n+1} + F_{2n} + F_{2n-1} - 1 - 2F_{n+1} + (-1)^n}{2}.$$

□

Theorem 3.5. *If $B = (b_{ij})$ is $n \times n$ Hankel matrix with $b_{ij} = Q_{i+j-1}$, then we have*

$$\|B\|_F = \sqrt{U_{2n-1} - U_{n-1}}$$

where $U_n = \sum\limits_{k=1}^{n}\sum\limits_{i=1}^{k} Q_i^2$ is given in lemma 1.7.

Proof. The Hankel matrix B on Pell-Padovan-like sequence is given by

$$B = \begin{bmatrix} Q_1 & Q_2 & Q_3 & \cdots & Q_{n-1} & Q_n \\ Q_2 & Q_3 & Q_4 & \cdots & Q_n & Q_{n+1} \\ Q_3 & Q_4 & Q_5 & \cdots & Q_{n+1} & Q_{n+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ Q_{n-1} & Q_n & Q_{n+1} & \cdots & Q_{2n-3} & Q_{2n-2} \\ Q_n & Q_{n+1} & Q_{n+2} & \cdots & Q_{2n-2} & Q_{2n-1} \end{bmatrix}$$

$$\|A\|_F = \left(\sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}|^2\right)^{\frac{1}{2}} = \left(\sum_{k=1}^{n} Q_k^2 + \sum_{k=2}^{n+1} Q_k^2 + \cdots + \sum_{k=2}^{2n-1} Q_k^2\right)^{\frac{1}{2}}$$

$$\|A\|_F = \left(\left(\sum_{k=1}^{n} Q_k^2 + \sum_{k=1}^{n+1} Q_k^2 + \cdots + \sum_{k=1}^{2n-1} Q_k^2\right) - \left(\sum_{k=1}^{n-1}\sum_{i=1}^{k} Q_i^2\right)\right)^{\frac{1}{2}}$$

$$\|A\|_F = \left(\left(\sum_{k=1}^{2n-1}\sum_{i}^{k} Q_i^2\right) - 2\left(\sum_{k=1}^{n-1}\sum_{i}^{k} Q_i^2\right)\right)^{\frac{1}{2}}$$

$$\|A\|_F = \sqrt{U_{2n-1} - U_{n-1}}.$$

□

Theorem 3.6. *If $B = b_{ij}$ is $n \times n$ Hankel matrix with $b_{ij} = Q_{i+j-1}$ then, we have*

$$\frac{1}{\sqrt{n}}\|B\|_F \leq \|B\|_2 \leq \sqrt{(F_nF_{n+1} + 2(-1)^nF_{n-1} + n - 2)}\sqrt{(F_nF_{n+1} + 2(-1)^nF_{n-1} + n - 1)}$$

where $\|B\|_F$ is defined in theorem (3.5).

Proof. From Theorem (1.3), and equation inequality (1.7)

$$\frac{1}{\sqrt{n}}\|A\|_F \leq \|A\|_2 \tag{3.8}$$

Let us define two new matrices

$$U_n = \begin{cases} Q_{i+j-1} & i \leq j \\ 1 & i > j \end{cases} \quad \text{and } V_n = \begin{cases} Q_{i+j-1} & i > j \\ 1 & i \leq j. \end{cases}$$

It can easily seen that $B = U_n \circ V_n$. Thus we obtain the result from definition (1.2)

$$r_1(U_n) = \max_i \sqrt{\sum_j |u_{ij}|^2} = \sqrt{\sum_{i=1}^{n} Q_i^2} = \sqrt{F_n F_{n+1} + 2(-1)^n F_{n-1} + n - 2}$$

and

$$c_1(V_n) = \max_j \sqrt{\sum_i |v_{ij}|^2} = \sqrt{1 + \sum_{i=2}^{n} Q_i^2} = \sqrt{F_n F_{n+1} + 2(-1)^n F_{n-1} + n - 1}.$$

Using the inequality(1.7)

$$\|B\|_2 \leq \sqrt{(F_n F_{n+1} + 2(-1)^n F_{n-1} + n - 2)}\sqrt{(F_n F_{n+1} + 2(-1)^n F_{n-1} + n - 1)}.$$

□

Theorem 3.7. *The bounds of spectral norms of Toeplitz matrix B for Pell-Padovan-like sequence are given as*

$$\|B\|_2 \geq \sqrt{\frac{2n + 2F_n^2 + F_{n-1}F_{n+1} + 4(-1)^n(F_{n-2} - F_{n-1}) + F_{n+1}(F_{n-1} - 4) - 2F_n + 5n(n-1) + 8}{2n}}$$

and

$$\|B\|_2 \leq \sqrt{(F_{n-1}F_n - 2(F_{n+1} - 1) + n)\left(F_{n-1}F_n + 2(-1)^{n-1}F_{n-2} + n - 4\right)}$$

where $\|.\|_2$ *is the spectral norm and* Q_n *is the Pell-Padovan-like sequence.*

Proof. The matrix B is of the form

$$B = \begin{bmatrix} Q_0 & Q_{-1} & Q_{-2} & \cdots & Q_{2-n} & Q_{2-n} \\ Q_1 & Q_0 & Q_{-1} & \cdots & Q_{3-n} & Q_{2-n} \\ Q_2 & Q_1 & Q_0 & \cdots & U_{4-n} & Q_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ Q_{n-2} & Q_{n-3} & Q_{n-4} & \cdots & Q_0 & Q_{-1} \\ Q_{n-1} & Q_{n-2} & Q_{n-3} & \cdots & Q_1 & Q_0 \end{bmatrix}$$

By using the equations (1.10) and (1.11)

$$\|B\|_F^2 = nQ_0^2 + \sum_{i=1}^{n-1}\sum_{k=1}^{i} Q_k^2 + \sum_{i=1}^{n-1}\sum_{k=1}^{i} Q_{-k}^2$$

$$\|B\|_F^2 = n + \left(\frac{F_n^2+F_{n-1}F_{n+1}+n(n-1)-4(n-1)+4(-1)^n(F_{n-2}-F_{n-1})+3}{2}\right) + \left(\frac{F_n^2+F_{n+1}(F_{n-1}-4)-2F_n+4(n-1)+4n(n-1)+5}{2}\right)$$

$$\|B\|_F^2 = \frac{2n+2F_n^2+F_{n-1}F_{n+1}+4(-1)^n(F_{n-2}-F_{n-1})+F_{n+1}(F_{n-1}-4)-2F_n+5n(n-1)+8}{2}$$

$$\|B\|_2 \geq \sqrt{\frac{2n+2F_n^2+F_{n-1}F_{n+1}+4(-1)^n(F_{n-2}-F_{n-1})+F_{n+1}(F_{n-1}-4)-2F_n+5n(n-1)+8}{2n}}.$$

Consider the matrices

$$C = \begin{cases} c_{ij} = 1 & j = 1 \\ c_{ij} = Q_{i-j} & j \neq 1 \end{cases} \text{ and } D = \begin{cases} d_{ij} = 1 & j \neq 1 \\ d_{ij} = Q_{i-j} & j = 1 \end{cases}$$

such that $B = C \circ D$, Then using definition 1.1 and theorem 1.1

$$r_1(C) = \max_i \sqrt{\sum_j (c_{ij})^2} = \sqrt{1 + \sum_{k=1}^{n-1} Q_{-k}^2} = \sqrt{F_{n-1}F_n - 2(F_{k+1} - 1) + n}$$

and

$$c_1(D) = \max_j \sqrt{\sum_i (d_{ij})^2} = \sqrt{\sum_{k=0}^{n-1} Q_k^2} = \sqrt{F_{n-1}F_n + 2(-1)^{n-1}F_{n-2} + n - 4}$$

$$\|B\|_2 \leq \sqrt{(F_{n-1}F_n - 2(F_{n+1} - 1) + n)\left(F_{n-1}F_n + 2(-1)^{n-1}F_{n-2} + n - 4\right)}.$$

□

Example 3.1. *Some special values of n will be taken to give the results about the aforementioned matrices with integers sequences entries.*

Matrices	$n=5$	$n=8$	$n=10$
r− circulant	$\lvert r\rvert \geq 1,\ \sqrt{22} \leq \lVert B\rVert_2 \leq 22\lvert r\rvert$	$\lvert r\rvert \geq 1,\ \sqrt{263} \leq \lVert B\rVert_2 \leq 263\lvert r\rvert$	$\lvert r\rvert \geq 1,\ \sqrt{1836} \leq \lVert B\rVert_2 \leq 1836\lvert r\rvert$
	$\lvert r\rvert \leq 1,\ \lvert r\rvert\sqrt{22} \leq \lVert B\rVert_2 \leq \sqrt{110}$	$\lvert r\rvert \leq 1,\ \lvert r\rvert\sqrt{263} \leq \lVert B\rVert_2 \leq \sqrt{2104}$	$\lvert r\rvert \leq 1,\ \lvert r\rvert\sqrt{1936} \leq \lVert B\rVert_2 \leq \sqrt{18360}$
circulant	$\sqrt{22} \leq \lVert B\rVert_2 \leq \sqrt{506}$	$\sqrt{263} \leq \lVert B\rVert_2 \leq \sqrt{69432}$	$\sqrt{1936} \leq \lVert B\rVert_2 \leq \sqrt{3372732}$
semi-circulant	$\lVert B\rVert_F = \sqrt{67}$	$\lVert B\rVert_F = \sqrt{1193}$	$\lVert B\rVert_F = \sqrt{7999}$
Hankel	$\lVert B\rVert_1 = \lVert B\rVert_\infty = 80$	$\lVert B\rVert_1 = \lVert B\rVert_\infty = 1563$	$\lVert B\rVert_1 = \lVert B\rVert_\infty = 10857$
	$\lVert B\rVert_F = \sqrt{1514}$	$\lVert B\rVert_F = \sqrt{3338484}$	$\lVert B\rVert_F = \sqrt{22881403}$
	$\sqrt{\frac{67}{5}} \leq \lVert B\rVert_2 \leq \sqrt{506}$	$\sqrt{\frac{1193}{8}} \leq \lVert B\rVert_2 \leq \sqrt{69432}$	$\sqrt{\frac{7999}{10}} \leq \lVert B\rVert_2 \leq \sqrt{3372732}$
Toeplitz	$\sqrt{\frac{89}{5}} \leq \lVert B\rVert_2 \leq \sqrt{120}$	$\sqrt{117} \leq \lVert B\rVert_2 \leq \sqrt{56115}$	$\sqrt{\frac{6031}{10}} \leq \lVert B\rVert_2 \leq \sqrt{3125136}$

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