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SOME NEW RESULTS ON CONTROLLABILITY AND OBSERVABILITY FOR IMPULSIVE DYNAMIC SYSTEMS

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Abstract. This paper introduces a new transition matrix for impulsive dynamic systems on time scales and establishes some properties of them for the study of the controllability and observability of such systems.

1. Introduction

Many interesting natural phenomenons are represented by smooth differential equations. But the situation becomes quite different when a physical phenomenon has sudden changes in its state as mechanical systems with impact, biological systems like heartbeats, blood flows, population dynamics [2, 29], chemistry, engineering and control theory [3, 11]. Mathematical models of such processes are systems of differential equations that undergo instantaneous changes in the state are called impulsive systems.

We describe an impulsive differential equation by three components: a continuous-time differential equation, which governs the state of the system between impulsive; an impulse equation, which models an impulsive jump defined by a jump function at the instant an impulse occurs; and a jump criterion, which defines a set of jump events in which the impulsive equation is active. Impulsive differential equations involving impulse effect appear as a natural description of observed evolution phenomena of

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several real-world problems. The theory of impulsive systems has been given extensive attention [17, 23, 32].

In recent years, some research dealing with the study of controllability and observability for impulsive systems [24, 27, 33, 34]. Interest in impulsive control systems has grown in recent years due to its theoretical and practical significanc [3, 8, 15, 25, 34]. Some authors studied the stability, controllability, and observability for dynamical systems on time scales [4, 5, 9, 10, 13, 17, 30, 31], but the few authors have studied the controllability and observability of impulsive dynamic systems on time scales [12, 24, 27, 28]. Some authors studied and established new results on controllability for Volterra integro-dynamic systems [22, 35].

Nevertheless, the necessary and sufficient conditions on controllability and observability were not addressed for the impulsive adjoint dynamic system. Moreover, the Gramian matrices for controllability and observability are independent impulsive conditions.

We should note that research in this paper is strongly motivated by the work of Lupulescu [28], and [26]. In [28], the authors examine the following dynamic system

$$\begin{cases} x^{\Delta} = A_k(t)\,x + B_k(t)\,u, \ t \in [t_{k-1}, t_k)_{\mathbb{T}_0}, \\ x\left(t_k^+\right) = (I + c_k)\,x(t_k)\,, t = t_{k,}\ k = 1, 2, \cdots, \\ x(t_0) = x_0, \end{cases}$$

with scalars impulse c_k.

In this paper, we give some results of controllability and observability for an impulsive dynamic system of the form

$$\begin{cases} x^{\Delta} = A_k(t)\,x + B_k(t)\,u(t)\,, \ t \in [t_{k-1}, t_k)_{\mathbb{T}_0}, \\ x\left(t_k^+\right) = (I + C_k)\,x(t_k)\,, \ t = t_{k,}\ k = 1, 2, \cdots, \\ y(t) = D_k(t)\,x + E_k(t)\,u(t)\,, \\ x(t_0) = x_0, \end{cases} \tag{1.1}$$

and its adjoint dynamic system

$$(1.2) \qquad \begin{cases} x\left(t\right)^{\Delta} = -A_k^T\left(t\right)x^{\sigma}\left(t\right) + B_k\left(t\right)u\left(t\right), \ t \in [t_{k-1}, t_k)_{\mathbb{T}_0}, \\ x\left(t_k^+\right) = \left(I + C_k\right)x\left(t_k\right), t = t_{k,} \ k = 1, 2, \cdots, \\ y\left(t\right) = D_k\left(t\right)x + E_k\left(t\right)u\left(t\right), \\ x\left(t_0\right) = x_0, \end{cases}$$

with the following conditions:

(i) Time scale $\mathbb{T}$ is unbounded above with bounded graininess (i.e. $\sup \mathbb{T} = \infty$ and $\mu(t) < \infty$), $[t_{k-1}, t_k)_{\mathbb{T}_0} \subset \mathbb{T}_0 := [t_0, \infty) \cap \mathbb{T}$.

(ii) $t_0 < t_1 < t_2 < \cdots t_k < \cdots$, with $\lim_{k\to\infty} t_k = \infty$, where , $t_k \in \mathbb{T}_0$ are right-dense.

(iii) $x(t_k^+) := \lim_{h\to 0^+} x(t_k + h)$, $x(t_k^-) := \lim_{h\to 0^+} x(t_k - h)$

(iv) $A_k(\cdot) \in C_{rd}\mathcal{R}(\mathbb{T}_0, M_n(\mathbb{R}))$, $B_k(\cdot) \in C_{rd}\mathcal{R}(\mathbb{T}_0, M_{n\times m}(\mathbb{R}))$, $C_k(\cdot) \in M_n(\mathbb{R})$, $D_k(\cdot) \in C_{rd}\mathcal{R}(\mathbb{T}_0, M_{p\times n}(\mathbb{R}))$, $E_k(\cdot) \in C_{rd}\mathcal{R}(\mathbb{T}_0, M_{p\times m}(\mathbb{R}))$, $x(\cdot) \in \mathbb{R}^n$ is the state variable, and $u(\cdot) \in \mathbb{R}^m$ is the control input and $y(\cdot) \in \mathbb{R}^p$ is the output.

The primary purpose of this paper is to derive Gramian matrices with relates impulsive conditions. The fundamental difficulty is to drive conditions for impulsive systems in time-varying coefficient matrices.

We organize the rest of this paper: Section 2 presents the preliminary results. Section 3 and Section 4 investigate the controllability and observability of linear impulsive dynamic systems and its adjoint systems, respectively. We present some numerical examples to show the effectiveness of the proposed methods. We present the conclusion in the last section.

2. Preliminaries:

In what follows, we recall some notions about time scale analysis. We can find an extensive study of the analysis on time scales in [1, 6, 7].

A **time scale**, denoted by $\mathbb{T}$, is an arbitrary, non-empty closed subset of real numbers. The operator $\sigma : \mathbb{T} \to \mathbb{T}$ called the **forward jump operator** is defined by $\sigma(t) := \inf\{s \in \mathbb{T}, s > t\}$. The **step size function** $\mu : \mathbb{T} \to \mathbb{R}_+$ is given by $\mu(t) := \sigma(t) - t$. We say a point $t \in \mathbb{T}$ is **right dense** if $\sigma(t) = t$ i.e. $(\mu(t) = 0)$, and **right scattered** if $\mu(t) > t$ i.e. $(\mu(t) > 0)$. The operator $\rho : \mathbb{T} \to \mathbb{T}$ called the **backward jump operator** is defined by $\rho(t) := \sup\{s \in \mathbb{T}, s < t\}$. A point $t \in \mathbb{T}$ is said to be **left dense** if $\rho(t) = t$ and **left scattered** if $\rho(t) < t$. A point $t \in \mathbb{T}$ is called **dense** if it is left and right dense at the same time. That is $\rho(t) = t = \sigma(t)$. A point $t \in \mathbb{T}$ is called **isolated** if $\rho(t) < t < \sigma(t)$. If $\mathbb{T}$ has a left-scattered maximum M, then $\mathbb{T}^k = \mathbb{T} - \{M\}$; otherwise set $\mathbb{T}^k = \mathbb{T}$.

Example 2.1. *Let $\mathbb{T} = \mathbb{R}$, then for any $t \in \mathbb{T}$*

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\} = \inf(t, \infty) = t$$

and

$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\} = \sup(-\infty, t) = t.$$

So every point of $\mathbb{T}$ is dense. Also $\mu(t) = 0$ for every $t \in \mathbb{T}$.

Let $\mathbb{T} = \mathbb{Z}$ then for any point $t \in \mathbb{T}$

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\} = \inf\{t+1, t+2, t+3, ...\} = t+1 > t$$

and

$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\} = \sup\{..., t-3, t-2, t-1\} = t-1 < t.$$

So every point $t \in \mathbb{T}$ is an isolated point. Also in that case $\mu(t) = 1$.

If $\mathbb{T} = \{2^n : n \in \mathbb{Z}\} \cup \{0\}$ then for any $t = 2^n \neq 0 \in \mathbb{T}$

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\} = \inf\left\{2^{n+1}, 2^{n+2}, 2^{n+3}, ...\right\} = 2^{n+1} > t$$

and

$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\} = \sup\left\{..., 2^{n-3}, 2^{n-2}, 2^{n-1}\right\} = 2^{n-1} < t.$$

So every $t \neq 0 \in \mathbb{T}$ *is an isolated point of* $\mathbb{T}$. *But for* $t = 0 \in \mathbb{T}$ *we have*

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > 0\} = \inf\{2^n : n \in \mathbb{T}\} = 0$$

hence $0 \in \mathbb{T}$ *is right dense point.*

The **delta derivative** of a function $f : \mathbb{T} \to \mathbb{R}$ at a point $t \in \mathbb{T}^k$ is defined by

$$f^{\Delta}(t) = \lim_{\substack{s \to t \\ s \neq \sigma(t)}} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s}.$$

A function f is called **rd-continuous** provided that it is continuous at right dense points in $\mathbb{T}$, and has a finite limit at left-dense points, and the set of rd-continuous functions are denoted by $C_{rd}(\mathbb{T}, \mathbb{R})$. The set of functions $C^1_{rd}(\mathbb{T}, \mathbb{R})$ includes the functions f whose derivative is in $C_{rd}(\mathbb{T}, \mathbb{R})$ too.

For $s, t \in \mathbb{T}$ and a function $f \in C_{rd}(\mathbb{T}, \mathbb{R})$, the Δ-integral is defined to be

$$\int_s^t f(\tau)\Delta\tau = F(t) - F(s),$$

where $F \in C^1_{rd}(\mathbb{T}, \mathbb{R})$ is an anti-derivative of f, i.e., $F^{\Delta} = f$ on $\mathbb{T}^k$.

A function $f \in C_{rd}(\mathbb{T}, \mathbb{R})$ is called **regressive** if $1 + \mu(t)f(t) \neq 0$ for all $t \in \mathbb{T}^k$, and $f \in C_{rd}(\mathbb{T}, \mathbb{R})$ is called positively regressive if $1 + \mu(t)f(t) > 0$ on $\mathbb{T}^k$. The set of regressive functions and the set of positively regressive functions are denoted by $C_{rd}\mathcal{R}(\mathbb{T}, \mathbb{R})$ and $C_{rd}\mathcal{R}^+(\mathbb{T}, \mathbb{R})$, respectively.

Let $f \in C_{rd}\mathcal{R}(\mathbb{T}, \mathbb{R})$ and $s \in \mathbb{T}$, then the **generalized exponential function** $e_f(\cdot, s)$ on a time scale $\mathbb{T}$ is defined to be the unique solution of the following initial value problem

$$\begin{cases} x^{\Delta}(t) = f(t)x(t) \\ x(s) = 1. \end{cases}$$

For $h \in \mathbb{R}^+$, set $\mathbb{C}_h := \{z \in \mathbb{C} : z \neq -1/h\}$, $\mathbb{Z}_h := \{z \in \mathbb{C} : -\pi/h < Im(z) \leq \pi/h\}$, and $\mathbb{C}_0 := \mathbb{Z}_0 := \mathbb{C}$. For $h \in \mathbb{R}^+_0$ and $z \in \mathbb{C}_h$, the cylinder transformation $\xi_h : \mathbb{C}_h \to \mathbb{Z}_h$

is defined by

$$\xi_h(z) := \begin{cases} z, & h = 0 \\ \frac{1}{h}\mathrm{Log}(1+zh), & h > 0, \end{cases}$$

and we can also write the exponential function in the form

$$e_f(t,s) := \exp\left\{ \int_s^t \xi_{\mu(\tau)}(f(\tau))\Delta\tau \right\} \text{ for } s,t \in \mathbb{T}.$$

A function $f : \mathbb{T} \to \mathbb{R}^n$ is piecewise rd-continuous (we write $f \in C_{prd}(\mathbb{T}, \mathbb{R}^n)$) if it is regulated and if it is rd-continuous at all, except possibly at finitely many, right-dense points $t \in \mathbb{T}$.

We denote by $C^1_{rd}(\mathbb{T}, \mathbb{R}^n)$ the set of all functions $f : \mathbb{T} \to \mathbb{R}^n$ that are differentiable on $\mathbb{T}$ and its delta-derivative $f^\Delta \in C_{rd}(\mathbb{T}, \mathbb{R}^n)$. The set of rd-continuous (respectively rd-continuous and regressive) matrix-valued functions $A : \mathbb{T} \to M_n(\mathbb{R})$ is denoted by $C_{rd}(\mathbb{T}, M_n(\mathbb{R}))$ (respectively by $C_{rd}\mathcal{R}(\mathbb{T}, M_n(\mathbb{R}))$). We recall that a matrix-valued function A is said to be regressive if $I + \mu(t)A(t)$ is invertible for all $t \in \mathbb{T}^k$, where I is the $n \times n$ identity matrix.

In order to define the solution of

$$\begin{cases} x^\Delta = A_k(t)\, x, \ t \in [t_{k-1}, t_k)_{\mathbb{T}_0}, \\ x\left(t_k^+\right) = (I + C_k)\, x(t_k), \ t = t_k,\ k = 1, 2, \cdots, \\ x(t_0) = x_0, \end{cases} \tag{2.1}$$

we introduce the following spaces

$$\Omega := \left\{ \begin{array}{l} x : \mathbb{T}_0 \to \mathbb{R}^n; x \in C\left((t_k, t_{k+1})_{\mathbb{T}_0}, \mathbb{R}^n\right), k = 0, 1, \cdots, \ x\left(t_k^+\right) \\ \text{and } x\left(t_k^-\right) \text{ exist with } x\left(t_k^-\right) = x(t_k), k = 0, 1, \cdots, \end{array} \right\}$$

and

$$\Omega^{(1)} := \left\{ x \in \Omega : x \in C^1\left((t_k, t_{k+1})_{\mathbb{T}_0}, \mathbb{R}^n\right), k = 0, 1, \cdots \right\},$$

where $C\left((t_k, t_{k+1})_{\mathbb{T}_0}, \mathbb{R}^n\right)$ is the set of all continuous functions on $(t_k, t_{k+1})_{\mathbb{T}_0}$ and $C^1\left((t_k, t_{k+1}), \mathbb{R}^n\right)$ is the set of all continuously differentiable functions on $(t_k, t_{k+1})_{\mathbb{T}_0}, k = 0, 1, \cdots$.

A function $x \in \Omega^{(1)}$ is said to be a solution of (2.1), if it satisfies $x^{\Delta}(t) = A_k(t)\, x(t)$, everywhere on $\mathbb{T}_\tau \setminus \left\{\tau, t_{k(\tau)}, t_{k(\tau)+1}, \cdots\right\}$ and for each $j = k(\tau), k(\tau)+1, \cdots$ satisfies the impulsive conditions $x\left(t_j^+\right) = (I + C_j)\, x(t_j)$ and the initial condition $x(\tau) = x_0$, where $k(\tau) := \min\{k = 1, 2, \cdots : \tau < t_k\}$.

Consider the following system on time scales:

$$x^{\Delta} = A(t)\, x(t), \tag{2.2}$$

where $A(\cdot) \in C_{rd}\mathcal{R}(\mathbb{T}, M_n(\mathbb{R}))$. This is a homogenous linear dynamic system on time scales. Now we present some auxiliary propositions to prove our major results.

Proposition 2.1. [6] *If $A(\cdot) \in C_{rd}\mathcal{R}(\mathbb{T}, M_n(\mathbb{R}))$ and $h \in C_{rd}(\mathbb{T}, \mathbb{R}^n)$, then for each $(\tau, \eta) \in \mathbb{T} \times \mathbb{R}^n$ the initial value problem*

$$x^{\Delta} = A(t)\, x + h(t),\ x(\tau) = \eta,$$

has a unique solution given by

$$x(t) = \Phi_A(t, \tau)\, \eta + \int_{\tau}^{t} \Phi_A(t, \sigma(s))\, h(s)\, \Delta s,\ t \geq \tau,$$

where $\Phi_A(t, \tau)$ is the transition matrix at initial time $\tau \in \mathbb{T}$.

Along with (2.2), consider its adjoint equation

$$y^{\Delta} = -A^T(t)\, y^{\sigma}. \tag{2.3}$$

If $A(\cdot) \in C_{rd}\mathcal{R}(\mathbb{T}_+, M_n(\mathbb{R}))$ and $h \in C_{rd}(\mathbb{T}_+, \mathbb{R}^n)$, then the initial value problem $y^{\Delta} = -A(t)y^{\sigma}$, $y(\tau) = \eta$, has a unique solution $y : \mathbb{T}_0 \to \mathbb{R}^n$ is given by $y(t) = \Phi_{\ominus A^T}(t, \tau)\eta$, $t \geq \tau$.

Proposition 2.2. [6] *If $A \in C_{rd}\mathcal{R}(\mathbb{T}_+, M_n(\mathbb{R}))$, and $h \in C_{rd}(\mathbb{T}_0, \mathbb{R}^n)$, then for each $(\tau, \eta) \in \mathbb{T}_0 \times \mathbb{R}^n$ the initial value problem*

$$y^{\Delta} = -A^T(t)y^{\sigma} + h(t),\ y(\tau) = \eta,$$

has a unique solution $y : \mathbb{T}_0 \to \mathbb{R}^n$, is given by

$$y(t) = \Phi_{\ominus A^T}(t,\tau)\eta + \int_\tau^t \Phi_{\ominus A^T}(t,s)h\left(s\right)\Delta s,\ t \in \mathbb{T}_0.$$

If $A \in M_n(\mathbb{R})$ is a constant matrix, then we use the notation $e_A(t,\tau)$ instead of $\Phi_A(t,\tau)$.

Proposition 2.3. [6] *For the system (2.2) with $A \in M_n(\mathbb{R})$ constant matrix, there are scalar functions $\gamma_0(t,\tau), \cdots, \gamma_{n-1}(t,\tau) \in C^{\infty}_{rd}(\mathbb{T}_0, \mathbb{R})$ such that the unique solution has representations*

$$e_A(t,\tau) = \sum_{j=0}^{n-1} \gamma_j(t,\tau)\left(A\right)^j.$$

Let us define a matrix $S_{A_k}\left(t,\tau\right), t \in [t_{k-1}, t_k)_{\mathbb{T}_0}$ associated with $\{C_k, t_k\}_{k=1}^{\infty}$,given by:

$$S_{A_k}\left(t,\tau\right) := \begin{cases} \Phi_{A_k}\left(t,\tau\right), \text{ if } t_{k-1} < \tau < t < t_k. \\ \Phi_{A_k}\left(t, t_k^+\right)\left(I + C_{k-1}\right)\Phi_{A_{k-1}}\left(t_k,\tau\right), \\ \text{if } t_{k-1} \le \tau < t_k < t < t_{k+1}. \\ \Phi_{A_k}\left(t, t_{k-1}^+\right)\left[\prod_{\tau < t_j \le t}\left(I + C_j\right)\Phi_{A_{k-1}}\left(t_j, t_{j-1}^+\right)\right] \\ \times\left(I + C_i\right)\Phi_{A_i}\left(t_i,\tau\right), \\ \text{if } t_{i-1} \le \tau < t_i < \cdots < t_k < t < t_{k+1}, \end{cases} \tag{2.4}$$

where $\Phi_{A_k}\left(t,\tau\right),\ 0 \le \tau \le t$, is the transition matrix of system (2.2) at initial time $\tau \in \mathbb{T}_0$.

If $A_k(\cdot) = A_k$ and $B_k(\cdot) = B_k$ are constants matrices, then we use the notation $\tilde{S}_{A_k}(t,\tau)$ instead of $S_{A_k}(t,\tau)$ and written as:

$$\tilde{S}_{A_k}(t,\tau) := \begin{cases} e_{A_k}(t,\tau), \text{ if } t_{k-1} < t < \tau \leq t_k. \\ e_{A_k}\left(t, t_k^+\right)(I + C_{k-1})\, e_{A_{k-1}}(t_k, \tau), \\ \text{if } t_{k-1} \leq \tau < t_k < t < t_{k+1}. \\ e_{A_k}\left(t, t_{k-1}^+\right)\left[\prod_{\tau < t_j \leq t} (I + C_j)\, e_{A_{k-1}}\left(t_j, t_{j-1}^+\right)\right] \\ \times (I + C_i)\, e_{A_i}(t_i, \tau), \\ \text{if } t_{i-1} \leq \tau < t_i < \cdots < t_k < t < t_{k+1} \end{cases} \tag{2.5}$$

Remark 2.1. *If $A_k(I + C_k) = (I + C_k)A_k$, for all k, then*

$$S_{A_k}(t,\tau)(I + C_k) = (I + C_k)S_{A_k}(t,\tau),$$

and also

$$e_{A_k}(t,\tau)(I + C_k) = (I + C_k)e_{A_k}(t,\tau).$$

Remark 2.2. *By equations (2.4) and (2.5), for $t_{i-1} \leq \tau < t_i < \cdots < t_k < t < t_{k+1}$, we obtain*

$$S_{A_k}(t,\tau) = \Phi_{A_k}\left(t, t_k^+\right)(I + C_k)\, S_{A_k}(t_k,\tau), \tag{2.6}$$

and

$$\tilde{S}_{A_k}(t,\tau) = e_{A_k}\left(t, t_k^+\right)(I + C_k)\, \tilde{S}_{A_k}(t_k,\tau). \tag{2.7}$$

Moreover, the following properties hold:

(i) $S_{A_k}\left(t_k^+,\tau\right) = (I + C_k)S_{A_k}(t_k,\tau),\ t_k \geq \tau,\ k = 1, 2, \cdots.$

(ii) $S_{A_k}\left(t, t_k^+\right) = S_{A_k}(t, t_k)(I + C_k)^{-1},\ t_k \leq t,\ k = 1, 2, \cdots.$

(iii) $S_{A_k}\left(t, t_k^+\right) S_{A_k}\left(t_k^+,\tau\right) = S_{A_k}(t,\tau),\ t_0 < \tau \leq t_k \leq t,\ k = 1, 2, \cdots.$

By using mathematical induction we have the following results for the solutions of systems (1.1) and (1.2).

Theorem 2.1. *If* $A_k(\cdot) \in C_{rd}\mathcal{R}(\mathbb{T}_0, M_n(\mathbb{R}))$*. Then for each* $(t_0, x_0) \in \mathbb{T}_0 \times \mathbb{R}^n$*, the initial value problem (1.1) has a unique solution given by*

$$x(t) = \begin{cases} S_{A_k}(t, t_0)\, x_0 + \sum\limits_{i=1}^{k-1} \int\limits_{t_{i-1}}^{t_i} S_{A_k}(t, \sigma(\tau))\, B_i(\tau)\, u(\tau)\, \Delta\tau \\ + \int\limits_{t_{k-1}}^{t} S_{A_k}(t, \sigma(\tau))\, B_k(\tau)\, u(\tau)\, \Delta\tau. \end{cases} \tag{2.8}$$

Remark 2.3. *If* $\mathbb{T} = \mathbb{R}$, $A_k(t) = A(t)$, *and* $B_k(t) = 0$, *then we obtain results of* [34].

Theorem 2.2. *If* $A_k(\cdot) \in C_{rd}R(\mathbb{T}_0, M_n(\mathbb{R}))$, $B_k(t)$ *and* $C_k \in M_{n\times m}(\mathbb{R})$, $x \in \mathbb{R}^n$ *is the state variable, and* $u \in \mathbb{R}^m$ *is the control input for* $k = 1, 2, \cdots$*. Then for each* $(t_0, x_0) \in \mathbb{T}_0 \times \mathbb{R}^n$*, the initial value problem (1.2) has a unique solution given by*

$$x(t) = \begin{cases} S^T_{A_k}(t_0, t)\, x_0 + \sum\limits_{i=1}^{k-1} \int\limits_{t_{i-1}}^{t_i} S^T_{A_k}(\tau, t)\, B_i(\tau)\, u(\tau)\, \Delta\tau \\ \qquad + \int\limits_{t_{k-1}}^{t} S^T_{A_k}(\tau, t)\, B_k(\tau)\, u(\tau)\, \Delta\tau. \end{cases} \tag{2.9}$$

3. Controllability

Definition 3.1. The impulsive system (1.1) (or (1.2))is called controllable on $[t_0, t_f]_{\mathbb{T}_0}$, with $t_f > t_0$, if given any initial state $x_0 \in \mathbb{R}^n$, there is a piecewise rd-continuous input signal $u(\cdot) : [t_0, t_f]_{\mathbb{T}_0} \to \mathbb{R}^m$such that the corresponding solution of (1.1) or (1.2)satisfies $x(t_f) = 0$.

Let us define the following Gramian matrices for (1.1), which are adopted from [10]:

$$
\begin{aligned}
&G_1 := G_1(t_0, t_f, t_f) = \int_{t_0}^{t_f} S_{A_1}(t_f, \sigma(\tau)) B_1(\tau) B_1^T(\tau) \\
&\times S_{A_1}^T(t_f, \sigma(\tau)) \Delta\tau,\ \ t_f \in [t_0, t_1]_{\mathbb{T}_0}. \\
&\text{For } 2 \le l \le k-2, \text{ and } t_f \in [t_{l-1}, t_l)_{\mathbb{T}_0} \\
&G_l := G_l(t_{l-1}, t_f, t_f) = \int_{t_{l-1}}^{t_f} S_{A_l}(t_f, \sigma(\tau)) B_l(\tau) B_l^T(\tau) S_{A_l}^T(t_f, \sigma(\tau)) \Delta\tau. \\
&G_{k-1} := G_{k-1}(t_{k-2}, t_f, t_f) = \int_{t_{k-2}}^{t_f} S_{A_{k-1}}(t_f, \sigma(\tau)) B_{k-1}(\tau) \\
&\times B_{k-1}^T(\tau) S_{A_{k-1}}^T(t_f, \sigma(\tau)) \Delta\tau, t_f \in [t_{k-2}, t_{k-1})_{\mathbb{T}_0}. \\
&G_k := G_k(t_{k-1}, t_f, t_f) = \int_{t_{k-1}}^{t_f} S_{A_k}(t_f, \sigma(\tau)) B_k(\tau) \\
&\times B_k^T(\tau) S_{A_k}^T(t_f, \sigma(\tau)) \Delta\tau,\ \ t_f \in [t_{k-1}, t_k)_{\mathbb{T}_0}.
\end{aligned}
\tag{3.1}
$$

Similarly, for the adjoint system (1.2), the Gramian matrices are as follows:

$$
\begin{aligned}
&\bar{G}_1 := \int_{t_0}^{t_f} S_{A_1}^T(\tau, t_f) B_1(\tau) B_1^T(\tau) S_{A_1}(\tau, t_f) \Delta\tau,\ \ t \in [t_0, t_1]_{\mathbb{T}_0}. \\
&\text{For } 2 \le l \le k-2, \text{ and } t_f \in [t_{l-1}, t_l)_{\mathbb{T}_0} \\
&\bar{G}_l := \int_{t_{l-1}}^{t_f} S_{A_l}^T(\tau, t_f) B_l(\tau) B_l^T(\tau) S_{A_l}(\tau, t_f) \Delta\tau. \\
&\bar{G}_{k-1} := \int_{t_{k-2}}^{t_f} S_{A_{k-1}}^T(\tau, t_f) B_{k-1}(\tau) \\
&\times B_{k-1}^T(\tau) S_{A_{k-1}}(\tau, t_f) \Delta\tau,\ \ t_f \in [t_{k-2}, t_{k-1})_{\mathbb{T}_0}. \\
&\bar{G}_k := \int_{t_{k-1}}^{t_f} S_{A_k}^T(\tau, t_f) B_k(\tau) B_k^T(\tau) S_{A_k}(\tau, t_f) \Delta\tau,\ \ t_f \in [t_{k-1}, t_k)_{\mathbb{T}_0}.
\end{aligned}
\tag{3.2}
$$

Theorem 3.1. *(I) If for all $l \in \{1, 2, \cdots, k\}$, rank $(G_l) = n$, then the impulsive system (1.1) is controllable on $([t_0, t_f]_{\mathbb{T}_0} (t_f \in [t_{k-1}, t_k)_{\mathbb{T}_0})$.*

(II) If the impulsive system (1.1) is controllable on $[t_0, t_f]_{\mathbb{T}} (t_f \in [t_{k-1}, t_k)_{\mathbb{T}_0})$, and assume that $(I + C_i)$ are invertible for $i = 1, 2, ..., k$, then rank $(G_1 \cdots G_k) = n$.

Proof. (I) Let $l \in \{1, 2, 3, \cdots, k\}$ be such that $rank\,(G_l) = n$, that is $G\,(t_0, t_{l-1}, t_l)$ is invertible. Then for a given $x_0 \in \mathbb{R}^n$, choose a control function $u\,(t)$ define as

$$u\,(t) := \begin{cases} -B_1^T\,(\tau)\, S_{A_1}^T\,(t_f, \sigma\,(\tau))\, G_1^{-1} S_{A_1}\,(t_f, t_0)\, x_0; \;\; k = 1 \\ -S_{A_l}\,(t_f, t_0)\, B_l^T\,(\tau)\, S_{A_l}^T\,(t_f, \sigma\,(\tau))\, G_l^{-1} x_0, \;\text{ if } t \in [t_{l-1}, t_l)_{\mathbb{T}_0} \text{ for } 2 \leq l \leq k-2 \\ 0, \;\text{ if } t \in [t_0, t_f]_{\mathbb{T}_0} \backslash [t_{l-1}, t_l)_{\mathbb{T}_0} \\ -S_{A_{k-1}}\,(t_f, t_0)\, B_{k-1}^T\,(\tau)\, S_{A_{k-1}}^T\,(t_f, \sigma\,(\tau))\, G_{k-1}^{-1} x_0, \;\text{ if } t \in [t_{k-2}, t_{k-1})_{\mathbb{T}_0} \\ 0, \;\text{ if } t \in [t_0, t_f]_{\mathbb{T}_0} \backslash [t_{k-2}, t_{k-1})_{\mathbb{T}_0} \\ -S_{A_k}\,(t_f, t_0)\, B_k^T\,(\tau)\, S_{A_k}^T\,(t_f, \sigma\,(\tau))\, G_k^{-1} x_0, \;\text{ if } t \in [t_{k-1}, t_k)_{\mathbb{T}_0} \\ 0, \;\text{ if } t \in [t_0, t_f]_{\mathbb{T}_0} \backslash [t_{k-1}, t_k)_{\mathbb{T}_0}. \end{cases}$$

Subtituting $t = t_f$ and input $u\,(t)$ in the solution of equation (2.8), we obtain

$$\begin{aligned} x\,(t_f) &= S_{A_1}\,(t_f, t_0)\, x_0 - \int_{t_0}^{t_f} S_{A_1}\,(t_f, \sigma\,(t))\, B_1\,(\tau)\, B_1^T\,(\tau)\, S_{A_1}^T\,(t_f, \sigma\,(\tau)) \\ &\quad \times G_1^{-1} S_{A_1}\,(t_f, t_0)\, x_0 \Delta\tau \\ &= S_{A_1}\,(t_f, t_0)\, x_0 - \int_{t_0}^{t_f} S_{A_1}\,(t_f, \sigma\,(t))\, B_1\,(\tau)\, B_1^T\,(\tau)\, S_{A_1}^T\,(t_f, \sigma\,(\tau))\, \Delta\tau \\ &\quad \times G_1^{-1} S_{A_1}\,(t_f, t_0)\, x_0 \\ &= S_{A_1}\,(t_f, t_0)\, x_0 - G_1 G_1^{-1} S_{A_1}\,(t_f, t_0)\, x_0 \\ &= 0, \end{aligned}$$

and for $t \in [t_{l-1}, t_l)_{\mathbb{T}_0}$, $2 \leq l \leq k-2$, we obtain

$$\begin{aligned} x\,(t_f) &= S_{A_l}\,(t_f, t_0)\, x_0 - G_l G_l^{-1} S_{A_l}\,(t_f, t_0)\, x_0 \\ &= 0. \end{aligned}$$

Similarly, for all other cases, we have $x\,(t_f) = 0$. Thus the system (1.1) is controllable on $[t_0, t_f]_{\mathbb{T}_0}\,(t_f \in [t_{k-1}, t_k)_{\mathbb{T}_0})$

(II) Suppose that (1.1) is controllable on $[t_0, t_f]_{\mathbb{T}_0}$ $(t_f \in [t_{k-1}, t_k)_{\mathbb{T}_0})$. We have to prove that $rank\,(G_0\ G_1\ ...\ G_k) = n$. Suppose the contrary that $rank\,(G_0\ G_1\ ...\ G_k) < n$, then there exist a nonzero $x_\alpha \in \mathbb{R}^n$ such that

$$x_\alpha^T G_i x_\alpha = 0,\ i = 1, 2, ..., k.$$

For $i = 1$, it follows that

$$\int_{t_0}^{t_f} x_\alpha^T S_{A_1}(t_f, \sigma(\tau))\, B_1(\tau)\, B_1^T(\tau)\, S_{A_1}^T(t_f, \sigma(\tau))\, x_\alpha \Delta\tau = 0.$$

As the integrand in this expression is the nonnegative rd-continuous function, so we obtain

$$\left\| B_1^T(t)\, S_{A_1}^T(t_f, \sigma(t))\, x_\alpha \right\|^2 = 0,$$

which follows that

$$x_\alpha^T S_{A_1}(t_f, \sigma(t))\, B_1(t) = 0; t \in [t_0, t_1)_{\mathbb{T}_0}. \tag{3.3}$$

For $2 \le l \le k-2$

$$\int_{t_{l-1}}^{t_f} x_\alpha^T S_{A_l}(t_f, \sigma(\tau))\, B_l(\tau)\, B_l^T(\tau)\, S_{A_l}^T(t_f, \sigma(\tau))\, x_\alpha \Delta\tau = 0,$$

it follows that

$$x_\alpha^T S_{A_l}(t_f, \sigma(t))\, B_l(t) = 0; t \in [t_{l-1}, t_l)_{\mathbb{T}_0}. \tag{3.4}$$

Next for $t \in [t_{k-2}, t_{k-1})_{\mathbb{T}_0}$

$$x_\alpha^T S_{A_{k-1}}(t_f, \sigma(t))\, B_{k-1}(t) = 0, \tag{3.5}$$

similarly for $t \in [t_{k-1}, t_k)_{\mathbb{T}_0}$

$$x_\alpha^T S_{A_k}(t_f, \sigma(t))\, B_k(t) = 0. \tag{3.6}$$

Since the impulsive system (1.1) is controllable on $[t_0, t_f]_{\mathbb{T}_0}$, so choosing $x_0 = x_\alpha$, there exist a piecewise rd-continuous input $u(\cdot)$ such that

$$
\begin{aligned}
0 = x(t_f) = S_{A_k}(t_f, t_0)\, x_\alpha + \sum_{i=1}^{k-1} \int_{t_{i-1}}^{t_i} S_{A_k}(t_f, \sigma(\tau))\, B_i(\tau)\, u(\tau)\, \Delta\tau \\
+ \int_{t_{k-1}}^{t_f} S_{A_k}(t_f, \sigma(\tau))\, B_k(\tau)\, u(\tau)\, \Delta\tau.
\end{aligned} \tag{3.7}
$$

Multiply through by $S_{A_k}(t_0, t_f)\, x_\alpha^T$ to the equation (3.7), we obtain

$$
\begin{aligned}
x_\alpha^T x_\alpha &= 0, \\
\|x_\alpha\|^2 &= 0,
\end{aligned}
$$

which contradicts that $x_\alpha \neq 0$ and so, we conclude that

$$
rank(G_0\ G_1\ \cdots\ G_k) = n.
$$

□

Let us define new matrices:

$$
W_i = \left[B_i\ A_i B_i\ \cdots\ A_i^{n-1} B_i\right] \text{ for } i = 1, 2, \cdots, k. \tag{3.8}
$$

Theorem 3.2. *If $A_k(t) = A_k$ and $B_k(t) = B_k$ are constants matrices. Then the impulsive system (1.1) is controllable on $[t_0, t_f]_{\mathbb{T}_0}$ $(t_f \in [t_{k-1}, t_k)_{\mathbb{T}_0})$ if and only if*

$$
rank\,(W_1\ W_2\ \cdots\ W_k) = n. \tag{3.9}
$$

Proof. Suppose that the system (1.1) is controllable on $[t_0, t_f]_{\mathbb{T}_0}$ $(t_f \in [t_{k-1}, t_k)_{\mathbb{T}_0})$. If the rank condition (3.9) does not hold, then there exists nonzero $x_\alpha \in \mathbb{R}^n$ such that

$$
x_\alpha^T A_i^j B_i = 0 \tag{3.10}
$$

for $i = 1, 2, \cdots, k,\ j = 0, 1, 2, \cdots, n-1$.

By using (3.1) for constant A_k and B_k and Proposition 2.3, we obtain

$$\begin{aligned} x_\alpha^T G_1(t_0,t_1,t_f) &= \int_{t_0}^{t_f} x_\alpha^T e_{A_1}(t_f,\sigma(\tau)) B_1 B_1^T e_{A_1}^T(t_f,\sigma(\tau))\Delta\tau \\ &= 0. \end{aligned}$$

For $2 \le l \le k-2$

$$x_\alpha^T G_l(t_{l-1},t_f,t_f) = \int_{t_{l-1}}^{t_f} x_\alpha^T \tilde{S}_{A_l}(t_f,\sigma(\tau)) B_l B_l^T \tilde{S}_{A_l}^T(t_f,\sigma(\tau))\Delta\tau,$$

by using the Remark 2.1, it follows that

$$\begin{aligned} x_\alpha^T G_l(t_{l-1},t_f,t_f) &= \int_{t_{l-1}}^{t_f} x_\alpha^T (I+C_l) e_{A_l}\left(t_f,t_l^+\right) \tilde{S}_{A_l}(t_l,\sigma(\tau)) \\ &\quad \times B_l B_l^T \tilde{S}_{A_l}^T(t_f,\sigma(\tau))\Delta\tau. \end{aligned}$$

Again using Remark 2.1 and Proposition 2.3, we have

$$\begin{aligned} x_\alpha^T G_l(t_{l-1},t_f,t_f) &= \int_{t_{l-1}}^{t_f} (I+C_l)\sum_{j=0}^{n-1}\gamma_{ij}\left(t_f,t_l^+\right) x_\alpha^T \tilde{S}_{A_l}(t_l,\sigma(\tau)) A_l^j B_l B_l^T \\ &\quad \times \tilde{S}_{A_l}^T(t_f,\sigma(\tau))\Delta\tau \\ &= 0. \end{aligned}$$

Similarly,

$$\begin{aligned} x_\alpha^T G_{k-1}(t_{k-2},t_f,t_f) &= \int_{t_{k-2}}^{t_f} x_\alpha^T \tilde{S}_{A_{k-1}}(t_f,\sigma(\tau)) B_{k-1} B_{k-1}^T \tilde{S}_{A_{k-1}}^T(t_f,\sigma(\tau))\Delta\tau \\ &= 0, \end{aligned}$$

and

$$\begin{aligned} x_\alpha^T G_k\left(t_{k-1}, t_f, t_f\right) &= \int_{t_{k-1}}^{t_f} x_\alpha^T \tilde{S}_{A_k}\left(t_f, \sigma\left(\tau\right)\right) B_k\left(\tau\right) B_k^T\left(\tau\right) \tilde{S}_{A_k}^T\left(t_f, \sigma\left(\tau\right)\right) \Delta\tau \\ &= 0. \end{aligned}$$

which is a contraction to (II) $[t_0, t_f]_{\mathbb{T}_0}\left(t_f \in (t_{k-1}, t_k]_{\mathbb{T}_0}\right)$, then it follows from the Thm 3.1 that the Gramian matrices defined above are not invertible. Thus there exists nonzero $x_\alpha \in \mathbb{R}^n$ such that

$$0 = x_\alpha^T G_1\left(t_0, t_f, t_f\right) x_\alpha = \int_{t_0}^{t_f} x_\alpha^T \tilde{S}_{A_1}\left(t_f, \sigma\left(\tau\right)\right) B_1 B_1^T \tilde{S}_{A_1}^T\left(t_f, \sigma\left(\tau\right)\right) x_\alpha \Delta\tau.$$

Exactly as in proof of Theorem 3.1, it follows that

$$x_\alpha^T \tilde{S}_{A_1}\left(t_f, \sigma\left(t\right)\right) B_1 = 0,\ t \in [t_0, t_1]_{\mathbb{T}_0}, \tag{3.11}$$

for $2 \leq l \leq k-2$

$$x_\alpha^T \tilde{S}_{A_l}\left(t_f, \sigma\left(t\right)\right) B_l = 0,\ t \in [t_{l-1}, t_l)_{\mathbb{T}_0}. \tag{3.12}$$

Similarly,

$$x_\alpha^T \tilde{S}_{A_{k-1}}\left(t_f, \sigma\left(t\right)\right) B_{k-1} = 0,\ t \in [t_{k-2}, t_{k-1})_{\mathbb{T}_0} \tag{3.13}$$

and

$$x_\alpha^T \tilde{S}_{A_k}\left(t_f, \sigma\left(t\right)\right) B_k = 0,\ t \in [t_{k-1}, t_f)_{\mathbb{T}_0}. \tag{3.14}$$

By continuity of $\tilde{S}_{A_i}\left(t_i, \cdot\right)$ and density of $\sigma\left([t_{i-1}, t_i]_{\mathbb{T}_0}\right)$ in the interval $[\sigma\left(t_{i-1}\right), \sigma\left(t_i\right))_{\mathbb{T}_0} = [t_{i-1}, t_i)_{\mathbb{T}_0}$ for all $t \in [t_{i-1}, t_i)_{\mathbb{T}_0}$, we obtain

$$x_\alpha^T \tilde{S}_{A_i}\left(t_i, \tau\right) B_i = 0\ \tau \in [t_{i-1}, t_i)_{\mathbb{T}_0}, i = 1, 2, \cdots, k-1. \tag{3.15}$$

Also, by continuity of $\tilde{S}_{A_k}(t_f,\cdot)$ and density of $\sigma\left([t_{k-1},t_f]_{\mathbb{T}_0}\right)$ in the interval $[\sigma(t_{k-1}),\sigma(t_f))_{\mathbb{T}_0}=[t_{k-1},t_f)_{\mathbb{T}_0}$ for all $t\in[t_{k-1},t_f)_{\mathbb{T}_0}$, we obtain that

$$x_\alpha^T\tilde{S}_{A_k}(t_f,t)\,B_k=0,\ t\in[t_{k-1},t_f)_{\mathbb{T}_0}. \tag{3.16}$$

In particular, if we are taking $t=t_i$ in (3.15) and $t=t_f$ in (3.16), then it follows that $x_\alpha^TB_i=0$ for $i=1,2,\cdots,k$. Since $\tilde{S}_{A_i}(t_i,\cdot)$ is delta differentiable [26], then subsequent derivatives and density arguments as above gives

$$(-1)^j\,x_\alpha^T\tilde{S}_{A_i}(t_i,t)\,A_i^jB_i=0;\ t\in[t_{i-1},t_i)_{\mathbb{T}_0}, \tag{3.17}$$

for $i=1,2,\cdots,k-1$, and $j=0,1,2,\cdots,n-1$. Similarly,

$$(-1)^j\,x_\alpha^T\tilde{S}_{A_k}(t_f,t)\,A_k^jB_k=0;\ t\in[t_{k-1},t_f)_{\mathbb{T}_0} \tag{3.18}$$

for $j=1,2,\cdots,n-1$. If we take $t=t_i$ in (3.17) and $t=t_f$ in (3.18), then it follows that $x_\alpha^TA_i^jB_i=0$ for $i=1,2,\cdots,k$, and $j=0,1,2,\cdots,n-1$. Therefore,

$$x_\alpha^T\left[B_i\ A_iB_i\cdots A_i^{n-1}B_i\right]=0.$$

Which implies that the rank condition (3.9) fails, which gives contradiction. So the impulsive system (1.1) is controllable on $[t_0,t_f]_{\mathbb{T}_0}\,(t_f\in[t_{k-1},t_k)_{\mathbb{T}_0})$. □

Our next results are for controllability of the adjoint system (1.2) for both time-variant and time-invariant cases. Proofs are like the proofs of Theorems 3.1 and 3.2.

Using the following control function $u(\cdot)$

$$u(t) := \begin{cases} -B_1^T(t) S_{A_1}(t,t_f) \bar{G}_1^{-1} S_{A_1}^T(t_0,t_f) x_0; \ k=1 \\ -B_l^T(t) S_{A_l}(t,t_f) \bar{G}_l^{-1} S_{A_l}^T(t_0,t_f) x_0, \ \text{for } 2 \le l \le k-2; t \in [t_{l-1},t_l]_{\mathbb{T}_0} \\ 0 \text{ if } t \in [t_0,t_f]_{\mathbb{T}_0} \backslash [t_{l-1},t_l)_{\mathbb{T}_0} \\ -B_{k-1}^T(t) S_{A_{k-1}}(t,t_f) \bar{G}_{k-1}^{-1} S_{A_{k-1}}^T(t_0,t_f) x_0 \text{ if } t \in [t_{k-2},t_{k-1}]_{\mathbb{T}_0} \\ 0 \text{ if } t \in [t_0,t_f]_{\mathbb{T}_0} \backslash [t_{k-2},t_{k-1})_{\mathbb{T}_0} \\ -B_k^T(t) S_{A_k}(t,t_f) \bar{G}_k^{-1} S_{A_k}^T(t_0,t_f) x_0 \text{ if } t \in [t_{k-1},t_k]_{\mathbb{T}_0} \\ 0 \text{ if } t \in [t_0,t_f]_{\mathbb{T}_0} \backslash [t_{k-1},t_k)_{\mathbb{T}_0}. \end{cases}$$

Theorem 3.3. *(I) If for all $l \in \{1,2,\cdots,k\}$ rank $\left(\bar{G}_l\right) = n$, then the impulsive adjoint system (1.2) is controllable on*$[t_0,t_f]_{\mathbb{T}_0}$ $(t_f \in [t_{k-1},t_k)_{\mathbb{T}_0})$.

(II) If the impulsive adjoint system (1.2) is controllable on $[t_0,t_f]_{\mathbb{T}_0}$ $(t_f \in [t_{k-1},t_k)_{\mathbb{T}_0})$, *and assume that $(I+C_i)$ are invertible for $i=1,2,\cdots,k$, then rank* $\left(\bar{G}_1 \ \bar{G}_2 \ \cdots \ \bar{G}_k\right) = n$.

Let us define the following matrices for adjoint dynamic system (1.2)

$$\bar{W}_i := \left[B_i^T \ B_i^T A_i \ \cdots \ B_i^T A_i^{n-1}\right] \text{ for } i=1,2,\cdots,k.$$

Theorem 3.4. *The time invariant impulsive system (1.2) is controllable on* $[t_0,t_f]_{\mathbb{T}_0}$ $(t_f \in [t_{k-1},t_k)_{\mathbb{T}_0})$ *if and only if*

$$rank\left(\bar{W}_1 \bar{W}_2 \cdots \bar{W}_k\right) = n \tag{3.19}$$

Example 3.1. *Consider the following time-invariant impulsive dynamic system:*

$$\begin{cases} x^{\Delta} = A_k(t)x + B_k(t)u, \ t \in [t_{k-1},t_k)_{\mathbb{T}_0}, \\ x\left(t_k^+\right) = (I+C_k)x(t_k), \ t = t_k, \ k=1,2,3, \\ x(0) = x_0, \end{cases} \tag{3.20}$$

where

$$A_1 = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}, \; B_1 = \begin{pmatrix} 3 \\ 0 \end{pmatrix}, \tag{3.21}$$

$$A_2 = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}, \; B_2 = \begin{pmatrix} 0 \\ 3 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} -3 & -2 \\ 3 & 4 \end{pmatrix}, \; B_3 = \begin{pmatrix} 0 \\ -2 \end{pmatrix}.$$

We have to compute $rank[W_1 \; W_2 \; W_3]$, *where*

$$W_1 = [B_1 \; A_1B_1] = \begin{pmatrix} 3 & 6 \\ 0 & 3 \end{pmatrix}$$

$$W_2 = [B_2 \; A_2B_2], = \begin{pmatrix} 0 & 6 \\ 3 & 9 \end{pmatrix}$$

similarly,

$$W_3 = \begin{pmatrix} 0 & 4 \\ -2 & -8 \end{pmatrix}.$$

By using equation (3.19) we obtain $rank[W_1 \; W_2 \; W_3] = 2$. *It follows that the system (3.20) is controllable.*

Example 3.2. *Let us consider the following population model with impulse*

$$\begin{aligned} P^{\Delta}(t) &= r_k P(t) + c_k U(t), \; t \neq t_k \\ P\left(t_k^+\right) &= (r_{k+1} - r_k) P(t_k) \\ P(0) &= P_0, \end{aligned}$$

where $P(t)$ *is the rate of population growth between two consecutive impulsive points and* $U(t)$ *is the control input. Such a model can describe the evaluation of Cicada*

Magicicada Septendecim. Using the Theorem 3.2 it is easy to see that the system is controllable.

4. Observability:

In this section, we establish the results of observability for the systems (1.1) and it's adjoint system (1.2) for both time-variant and time-invariant cases.

Definition 4.1. The impulsive system (1.1) (or system (1.2))is said to be completely observable on $[t_0, t_f]_{\mathbb{T}}\,(t_f > t_0)$ if any initial state $x(t_0) = x_0, \in \mathbb{R}^n$, is uniquely determined by the corresponding system input $u(t)$ and the system output $y(t)$ for $t \in [t_0, t_f]_{\mathbb{T}}$.

Theorem 4.1. *For $i = 1, 2, \cdots, k$, the impulsive system (1.1) is observable on $[t_0, t_f]_{\mathbb{T}_0}\,(t_f \in [t_{k-1}, t_k)_{\mathbb{T}_0})$, if and only if the matrix*

$$M(t_0, t_f) := M(t_0, t_0, t_1) + \sum_{i=1}^{k-1} M(t_0, t_{i-1}, t_i) + M(t_0, t_{k-1}, t_f)$$

is invertible, where

$$M(t_0, t_0, t_1) := \int_{t_0}^{t_1} S_{A_1}^T(\tau, t_0)\, D_1^T(\tau)\, D_1(\tau)\, S_{A_1}(\tau, t_0)\, \Delta\tau,$$

$$M(t_0, t_{i-1}, t_i) := \int_{t_{i-1}}^{t_i} S_{A_i}^T(\tau, t_0)\, D_i^T(\tau)\, D_i(\tau)\, S_{A_i}(\tau, t_0)\, \Delta\tau,\ i = 2, 3, \cdots, k-1,$$

and

$$M(t_0, t_{k-1}, t_f) = \int_{t_{k-1}}^{t_f} S_{A_k}^T(\tau, t_0)\, D_k^T(\tau)\, D_k(\tau)\, S_{A_k}(\tau, t_0)\, \Delta\tau.$$

Proof. Suppose that $M(t_0, t_f)$ is invertible. By using equation (2.8) and system (1.1), we obtain

$$y(t) = D_1(t)\, S_{A_1}(t, t_0)\, x_0 + D_1(t) \int_{t_0}^{t_1} S_{A_1}(t, \sigma(\tau))\, B_1(\tau)\, u(\tau)\, \Delta\tau + D_1(t)\, u(t)$$

for $t \in [t_0, t_1]_{\mathbb{T}_0}$, and

$$y(t) = D_l(t) S_{A_l}(t, t_0) x_0 + D_l(t) \sum_{i=1}^{l-1} \int_{t_{i-1}}^{t_i} S_{A_l}(t, \sigma(\tau)) B_i(\tau) u(\tau) \Delta\tau$$
$$+ D_l(t) \int_{t_{l-1}}^{t} S_{A_l}(t, \sigma(\tau)) B_l(\tau) u(\tau) \Delta\tau + E_l(t) u(t)$$

for $t \in [t_{l-1}, t_l)_{\mathbb{T}_0},\ l = 2, \cdots, k$.

From Def 4.1, the observability of system (1.1) is equivalent to the following output

$$y(t) = \begin{cases} D_1(t) S_{A_1}(t, t_0) x_0 \ t \in [t_0, t_1]_{\mathbb{T}_0} \\ D_l(t) S_{A_l}(t, t_0) x_0,\ t \in [t_{l-1}, t_l)_{\mathbb{T}_0},\ l = 2, ..., k, \end{cases} \tag{4.1}$$

as $u(t) = 0$.

Now, multiply with $S_{A_l}^T(t, t_0) D_l^T(t)$ both sides of above equation and integrate from t_0 to t_f, we get

$$\begin{aligned} \int_{t_0}^{t_f} S_{A_l}^T(t, t_0) D_l^T(t) y(t) \Delta\tau = \Big[\int_{t_0}^{t_1} S_{A_1}^T(\tau, t_0) D_1^T(\tau) D_1(\tau) S_{A_1}(\tau, t_0) \Delta\tau \\ + \sum_{i=2}^{k-1} \int_{t_{i-1}}^{t_i} S_{A_i}^T(\tau, t_0) D_i^T(\tau) D_i(\tau) S_{A_i}(\tau, t_0) \Delta\tau \\ + \int_{t_{k-1}}^{t_f} S_{A_k}^T(\tau, t_0) D_k^T(\tau) D_k(\tau) S_{A_k}(\tau, t_0) \Delta\tau \Big] x_0, \end{aligned} \tag{4.2}$$

which follows

$$\begin{aligned} \int_{t_0}^{t_f} S_{A_l}^T(t, t_0) D_l^T(t) y(t) \Delta\tau = \\ \left[M(t_0, t_0, t_1) + \sum_{i=2}^{k-1} M(t_0, t_{i-1}, t_i) + M(t_0, t_{k-1}, t_f) \right] x_0. \end{aligned} \tag{4.3}$$

Since the matrix $M(t_0, t_f)$ is invertible, and it can easily be seen that left side of equation (4.3) depends on $y(t),\ t \in [t_0, t_f]_{\mathbb{T}_0}$. So from equation(4.3), we deduce that $x(t_0) = x_0$ is uniquely determined by the corresponding system output $y(t)$ for $t \in [t_0, t_f]_{\mathbb{T}_0}$.

Conversely, suppose that the matrix $M(t_0, t_f)$ is not invertible, then there exist a nonzero $x_\alpha \in \mathbb{R}^n$, such that

$$x_\alpha^T M(t_0, t_f) x_\alpha = 0. \tag{4.4}$$

Since $M(t_0, t_0, t_1), M(t_0, t_{i-1}, t_i),\ i = 2, ..., k-1$, and $M(t_0, t_{k-1}, t_f)$ are positive semidefinite matrices, we have

$$\begin{cases} x_\alpha^T M(t_0, t_0, t_1) x_\alpha = 0, \\ x_\alpha^T M(t_0, t_{i-1}, t_i) x_\alpha = 0, i = 2, \cdots, k-1, \\ x_\alpha^T M(t_0, t_{k-1}, t_f) x_\alpha = 0. \end{cases} \tag{4.5}$$

Choose $x_0 = x_\alpha$, and using the equations (4.1), which follows that

$$\begin{aligned} &\int_{t_0}^{t_f} y^T(\tau) y(\tau) \Delta\tau = \int_{t_0}^{t_1} x_\alpha^T S_{A_1}^T(\tau, t_0) D_1^T(\tau) D_1(\tau) S_{A_1}(\tau, t_0) x_\alpha \Delta\tau \\ &+ \sum_{i=2}^{k-1} \int_{t_{i-1}}^{t_i} x_\alpha^T S_{A_i}^T(\tau, t_0) D_i^T(\tau) D_i(\tau) S_{A_i}(\tau, t_0) x_\alpha \Delta\tau \\ &+ \int_{t_{k-1}}^{t_f} x_\alpha^T S_{A_k}^T(\tau, t_0) D_k^T(\tau) D_k(\tau) S_{A_k}(\tau, t_0) x_\alpha \Delta\tau. \end{aligned}$$

Furthermore, we have

$$\begin{aligned} &\int_{t_0}^{t_f} y^T(\tau) y(\tau) \Delta\tau = x_\alpha^T M(t_0, t_0, t_1) x_\alpha \\ &+ \sum_{i=2}^{k-1} x_\alpha^T M(t_0, t_{i-1}, t_i) x_\alpha + x_\alpha^T M(t_0, t_{k-1}, t_f) x_\alpha. \end{aligned}$$

By using equation (4.5) we obtain

$$\int_{t_0}^{t_f} \|y(\tau)\|^2 \Delta\tau = 0,$$

it follows that $y(t) = 0$ for all $i = 1, 2, \cdots, k$. Which contradict Definition 4.1, so the given matrix $M(t_0, t_f)$ is invertible. □

The next result gives a sufficient and necessary criterion for a time-invariant case. For an impulsive system (1.1), let us defined the following matrix:

$$\bar{V} := \begin{bmatrix} V_1 \\ \vdots \\ V_k \end{bmatrix},$$

where

$$V_i := \begin{bmatrix} D_i \\ \vdots \\ D_i A_i \\ D_i A_i^{n-1} \end{bmatrix}, \ i = 1, 2, \cdots, k.$$

Theorem 4.2. *Assume that* $A_k(t) = A_k$, *and* $D_k(t) = D_k$, *are constant matrices. Then impulsive system (1.1) is observable on* $[t_0, t_f]_{\mathbb{T}_0}$ $(t_f \in (t_{k-1}, t_k]_{\mathbb{T}_0})$, *if and only if rank* $(\bar{V}) = n$.

Proof. Suppose that $rank\left(\bar{V}\right) < n$. Then there is a nonzero vector x_α such that $\bar{V} x_\alpha = 0$. It implies that

$$D_i A_i^j x_\alpha = 0, i = 1, 2, \cdots, k, \ j = 0, 1, \cdots, n-1. \tag{4.6}$$

By using equations (4.5) and (4.6)

$$\begin{aligned} M(t_0, t_0, t_1) x_\alpha &= \int\limits_{t_0}^{t_1} \tilde{S}_{A_1}^T(\tau, t_0) D_1^T D_1 \tilde{S}_{A_1}(\tau, t_0) x_\alpha \Delta\tau \\ &= 0. \end{aligned}$$

By the same arguments, for $i = 2, \cdots, k-1$, we have

$$M(t_0, t_{i-1}, t_i) x_\alpha = \int_{t_{i-1}}^{t_i} \tilde{S}_{A_i}^T(\tau, t_0) D_i^T D_i \tilde{S}_{A_i}(\tau, t_0) x_\alpha \Delta\tau$$
$$= \int_{t_{i-1}}^{t_i} \tilde{S}_{A_i}^T(\tau, t_0) D_i^T D_i e_{A_i}\left(\tau, t_i^+\right)(I + C_i) \tilde{S}_{A_i}(t_i, t_0) x_\alpha \Delta\tau.$$

From Proposition 2.3, equation (4.6) and Remark 2.1, we obtain

$$\begin{aligned} M(t_0, t_{i-1}, t_i) x_\alpha &= \int_{t_{i-1}}^{t_i} \tilde{S}_{A_i}^T(\tau, t_0) D_i^T (I + C_i) \\ &\quad \times \sum_{j=0}^{n-1} \gamma_{ij}\left(\tau, t_i^+\right) D_i A_i^j \tilde{S}_{A_i}(\tau, t_0) x_\alpha \tilde{S}_{A_i}(t_i, \tau) \Delta\tau \\ &= 0. \end{aligned}$$

Similarly,

$$M(t_0, t_{k-1}, t_f) x_\alpha = 0,$$

and so we obtain $M(t_0, t_f)\, x_\alpha = 0$. Since x_α is nonzero, the matrix is not invertible, then system (1.1) is not observable which is contradiction to assumption. So, $rank\left(\bar{V}\right) = n$.

Conversely, suppose that $rank\left(\bar{V}\right) = n$, and we have to prove that the impulsive system (1.1) is observable on $[t_0, t_f]_{\mathbb{T}_0}$ $(t_f \in [t_{k-1}, t_k)_{\mathbb{T}_0})$.

Otherwise, it follows that the matrix $M(t_0, t_f)$ is not invertible, then there exists a nonzero vector x_α such that $x_\alpha^T M(t_0, t_f)\, x_\alpha = 0$. From Theorem 4.1, we obtain

$$D_1 \tilde{S}_{A_1}(t, t_0)\, x_\alpha = 0, \tag{4.7}$$

for $i = 2, \cdots, k-1$

$$D_i \tilde{S}_{A_i}(t, t_0)\, x_\alpha = 0, \tag{4.8}$$

and

$$D_k \tilde{S}_{A_k}(t, t_0)\, x_\alpha = 0. \tag{4.9}$$

Obviously, at $t = t_0$, we have $D_i x_\alpha = 0, i = 1, 2, ..., k$, and delta differentiating equations (4.7),(4.8) and (4.9), we obtain

$$D_i A_i^j x_\alpha = 0, i = 1, \cdots, k \text{ and } j = 0, 1, \cdots, n-1.$$

Therefore, we have $\bar{V} x_\alpha = 0$, which implies that $rank\left(\bar{V}\right) < n$ which leads to contradiction. So system (1.1) is observable. The proof is completed. □

Example 4.1. *Consider the following impulsive time-invariant dynamic system*

$$\begin{aligned} & x^\Delta = A_k(t)\, x + B_k(t)\, u(t), \ t \in [t_{k-1}, t_k)_{\mathbb{T}_0}, \\ & x\left(t_k^+\right) = (I + C_k)\, x(t_k), t = t_k, \ k = 1, 2, 3, \\ & y(t) = D_k(t)\, x + E_k(t)\, u(t), \\ & x(0) = x_0, \end{aligned} \tag{4.10}$$

with

$$
\begin{aligned}
A_1 &= \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}, \ D_1 = \begin{pmatrix} 2 & 3 \end{pmatrix}, \\
A_2 &= \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}, \ D_2 = \begin{pmatrix} 0 & 1 \end{pmatrix}, \\
A_3 &= \begin{pmatrix} -3 & -2 \\ 3 & 4 \end{pmatrix}, \ D_3 = \begin{pmatrix} -2 & 1 \end{pmatrix}.
\end{aligned}
$$

We have to compute the following matrices

$$
V_i = \begin{bmatrix} D_i \\ D_i A_i \end{bmatrix}, \ i = 1, 2, 3.
$$

So that

$$
\begin{aligned}
V_1 &= \begin{pmatrix} 2 & 3 \\ 7 & 9 \end{pmatrix}, \\
V_2 &= \begin{pmatrix} 0 & 1 \\ 0 & 3 \end{pmatrix}, \\
V_3 &= \begin{pmatrix} -2 & 1 \\ 9 & 8 \end{pmatrix}.
\end{aligned}
$$

Now we compute $\bar{V}$, defined as

$$
\bar{V} = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{pmatrix} 2 & 3 \\ 7 & 9 \\ 0 & 1 \\ 0 & 3 \\ -2 & 1 \\ 9 & 8 \end{pmatrix}
$$

We obtain rank $\left(\bar{V}\right) = 2$. Therefore, the system (4.10) is observable.

Our next results are for complete observability for the adjoint system (1.2).

Theorem 4.3. *For $i = 1, 2, \cdots, k$, the impulsive system (1.2) is observable on $[t_0, t_f]_{\mathbb{T}_0}$ $(t_f \in (t_{k-1}, t_k]_{\mathbb{T}_0})$ if and only if the matrix*

$$\bar{M}(t_0, t_f) := \bar{M}(t_0, t_0, t_1) + \sum_{i=1}^{k-1} \bar{M}(t_0, t_{i-1}, t_i) + \bar{M}(t_0, t_{k-1}, t_f)$$

is invertible, where

$$\bar{M}(t_0, t_0, t_1) = \int_{t_0}^{t_1} S_{A_1}(t_0, \tau) D_1^T(\tau) D_1(\tau) S_{A_1}^T(t_0, \tau) \Delta\tau$$

$$\bar{M}(t_0, t_{i-1}, t_i) = \int_{t_{i-1}}^{t_i} S_{A_i}(t_0, \tau) D_i^T(\tau) D_i(\tau) S_{A_i}^T(t_0, \tau) \Delta\tau, \ i = 2, 3, \cdots, k-1$$

and

$$\bar{M}(t_0, t_{k-1}, t_f) = \int_{t_{k-1}}^{t_f} S_{A_k}(t_0, \tau) D_k^T(\tau) D_k(\tau) S_{A_k}^T(t_0, \tau) \Delta\tau.$$

For the time invariant varsion of theorem (4.3) we define the following matrices:

$$\tilde{V} := \begin{bmatrix} \tilde{V}_1 \\ \vdots \\ \tilde{V}_k \end{bmatrix}$$

And

$$\tilde{V}_i := \begin{bmatrix} D_i^T \\ \vdots \\ A_i D_i^T \\ A_i^{n-1} D_i^T \end{bmatrix}, i = 1, 2, \cdots, k.$$

Theorem 4.4. *Assume that $A_k(t) = A_k$, and $D_k(t) = D_k$, are constant matrices. Then impulsive system (1.2) is observable on $[t_0, t_f]_{\mathbb{T}_0}$ $(t_f \in [t_{k-1}, t_k)_{\mathbb{T}_0})$ if and only if rank $\left(\tilde{V}\right) = n$.*

Remark 4.1. *If* $\mathbb{T} = \mathbb{R}$, *then we obtain results of* [36] *and* [16] *for* $F_k = 0$. *If* $A_k(t) = A(t)$ *and* $B_k(t) = B(t)$, *then we obtain the results of* [27] *and in* [14] *if* $\mathbb{T} = \mathbb{R}$. *We can find the nonimpulsive versions on time scales in* [4, 10]. *Most of our results are new for discrete time scales.*

Remark 4.2. *The Gramian matrices for time-varying sysems in* [28] *are without impulsive, however, our controllability and observability criteria for time-varying systems depend on impulsive behavior.*

5. Conclusion

In this paper, we addressed the controllability and observability criteria for linear impulsive and its adjoint time-varying systems on time scales. We established several necessary and sufficient conditions for state controllability and observability of such systems, respectively. A comparison with some existing results shows the lower conservativeness of the proposed results. As we have shown that we consider a large class of systems, the results generalize some known results in [4, 10, 14, 16].

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