Jordan Journal of Mathematics and Statistics (JJMS) 14(1), 2021, pp 97 - 109

GENERALIZED DISTRIBUTION ASSOCIATED WITH QUASI-SUBORDINATION IN TERMS OF ERROR FUNCTION AND BELL NUMBERS

S. O. OLATUNJI [(1)] AND S. ALTINKAYA [(2)]

Abstract. Generalized distribution is a statistical tools used in geometric function theory in recent time because of its application to real life problems. In this present work, the generalized distribution associated with quasi-subordination in terms of error function and bell numbers were studied. The first few coefficient bounds were obtained which are used to obtain the Fekete-Szegö inequality.

1. Introduction

Let Γ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic and univalent in the open unit disk $U = \{z : |z| < 1\}$ with condition $f(0) = 0$ and $f'(0) = 1$. The well-known subclasses of (1.1) are starlike and convex functions which satisfies $Re(\frac{zf'(z)}{f(z)}) > 0$ and $Re(1 + \frac{zf''(z)}{f'(z)}) > 0$ respectively. For two analytic functions f and g such that $f(0) = g(0)$, we say f is subordinate to g in U written $f(z) \prec g(z)$, if there exists a Schwartz function $w(z)$(analytic in U with $w(0) = 0$ and $|w(z)| \leq |z|$) such that $f(z) = g(w(z)),\ z \in U$. Furthermore, if

2000 *Mathematics Subject Classification.* 30C45.
Key words and phrases. Univalent function, analytic function, error function, bell-numbers,quasi-subordination, distribution series, probability, q-starlike, q-convex.
Copyright © Deanship of Research and Graduate Studies, Yarmouk University, Irbid, Jordan.
Received: May 15, 2019 Accepted: March 3, 2020 .

g is univalent in U, then we have the following equivalence

$$f(z) \prec g(z) \longleftrightarrow f(0) = g(0) \quad and \quad f(U) \subset g(U), \tag{1.2}$$

(see [22]).
Also, f is said to be quasi-subordinate to g in U and denoted as $f(z) \prec_q g(z), z \in U$, if there exists an analytic function $\varphi(z)$ with $|\varphi(z)| \leq 1 (z \in U)$ such that $\frac{f(z)}{\varphi(z)}$ is analytic in U and

$$\frac{f(z)}{\varphi(z)} \prec g(z) (z \in U), \tag{1.3}$$

that is there exists a Schwartz function $\omega(z)$ such that $f(z) = \psi(z)g(w(z)), z \in U$, (see concepts in [30]).

MacGregor [18] revealed that if $\varphi(z) \equiv 1 (z \in U)$, then the quasi-subordination $\prec_q$ becomes the usual subordination, and for the Schwartz function $\omega(z) = z (z \in U)$, the quasi-subordination $\prec_q$ becomes the majorization "$<<$". In this case

$$f(z) \prec_q g(z) \longrightarrow f(z) = \varphi(z) g(\omega(z)) \longrightarrow f(z) << g(z).$$

Several authors have engaged themselves on quasi-subordination for different subclasses of functions and their interesting results are too voluminous to discuss. Just to mention but a few [13, 14, 19, 20, 21, 23, 24, 25, 31].

Recently, Porwal [26] studied and introduced the generalized discrete probability distribution in geometric function theory. He obtained the results for moments, mean, variance and moment generating function.
Let S denote the sum of the convergent series of the form

$$S = \sum_{n=0}^{\infty} a_n, \tag{1.4}$$

where $a_n \geq 0$ for $n \in N$. The generalized discrete probability distribution whose probability mass function is given as

$$p(n) = \frac{a_n}{S}, \quad n = 0, 1, 2, 3, ..., \tag{1.5}$$

$p(n)$ is a probability mass function because $p(n) \geq 0$ and $\sum_n p_n = 1$.
Also, denote

$$\psi(x) = \sum_{n=0}^{\infty} a_n x^n, \tag{1.6}$$

then from $S = \sum_{n=0}^{\infty} a_n$, series $\psi(x)$ is convergent for both $|x| < 1$ and $x = 1$.
If X is a discrete random variable that takes values $x_1, x_2, ...$ associated with probabilities $P_1, P_2, ...$then expected X denoted by $E(X)$ is defined as

$$E(X) = \sum_{n=0}^{\infty} P_n x_n. \tag{1.7}$$

The moment of a discrete probability distribution (r^{th}) about $x = 0$ is defined by

$$\mu'_r = E(X^r), \tag{1.8}$$

where μ'_1 is the mean of the distribution and the variance is given as

$$\mu'_2 - (\mu'_1)^2. \tag{1.9}$$

Moment about the origin is given as

$$Mean \;=\; \mu'_1 = \frac{\psi'}{S} \tag{1.10}$$

and

$$Variance \;=\; \mu'_2 - (\mu'_1)^2 = \frac{1}{S}\Big[\psi''(1) + \psi'(1) - \frac{(\psi'(1))^2}{S}\Big]. \tag{1.11}$$

The moment generating function of a random variable X is denoted by

$$M_x(t) = E(e^{X(t)}) \tag{1.12}$$

and the moment generating function of generalized discrete probability is given as

$$M_x(t) = \frac{\psi(e^{(t)})}{S}. \tag{1.13}$$

By specializing a_n, several distributions like Logarithmic distribution, Poison distribution, Binomial distribution, Zeta distribution, Geometric distribution, Bernoulli distribution and so on will be obtained which has been studied by scholars. The reviewer can see these in Baricz [4] and [5] .

The aim of the authors is the introduction of power series whose coefficient are probabilities of generalized distribution of the form

$$K_\phi(z) = z + \sum_{n=2}^{\infty} \frac{b_{n-1}}{S} z^n, \tag{1.14}$$

where $S = \sum_{n=0}^{\infty} a_n$.

The fractional $q-$calculus is a geometrical function theory instrument used to investigate and construct various subclasses of analytic functions. Researchers have studied $q-$calculus in terms of derivatives and integrals and their results are in literature.

For $0 < q < 1$, Jackson's q-derivative of a function $f \in \Gamma$ is given as follows

$$D_q f(z) = \begin{cases} \dfrac{f(z) - f(qz)}{(1-q)z}, & \text{if } z \neq 0 \\ f'(0), & \text{if } z = 0 \end{cases} \tag{1.15}$$

and $D_q^2 f(z) = D_q(D_q f(z))$ (see [15]).

From (1.15), one may have

$$D_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1}, \tag{1.16}$$

where $[n]_q = \dfrac{1-q^n}{1-q}$ and n is the basic number, if $q \longrightarrow 1^-$, $[n]_q \longrightarrow n$. (See details in [2] [10] [16], [28] and [29]).

An error function is a special function because it shows up anywhere the normal curve appears. It occurs in diffusion for transportation, very useful in physics, chemistry, biology, mass flow and so on. Error function occurs in quantum mechanics to estimate the probability of observing a particle in a particular region.

The function of the form

$$erf(z)=\frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}dt=\frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n z^{2n+1}}{(2n+1)n!}, \tag{1.17}$$

was introduced by Abramowitz and Stegun [1]. The properties and inequalities of error function were studied by Alzer [3] , Coman [11], Elbert [12] and so on. Ramachandran *et al.* [28] and [29] simplified (1.17) to obtain

$$Erf(z)=z+\sum_{n=2}^{\infty}\frac{(-1)^{n-1}}{(2n-1)(n-1)!}z^n \tag{1.18}$$

and studied

$$\mathcal{F}=(f*Erf)(z)=z+\sum_{n=2}^{\infty}\frac{(-1)^{n-1}}{(2n-1)(n-1)!}a_n z^n, \tag{1.19}$$

which is the convolution (Hadamard Product) of (1.1) and (1.19) define the class of starlike and convex functions interms of subordination which satisfies $\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)} \prec P_k(z)$ and $1+\frac{z\mathcal{F}''(z)}{\mathcal{F}'(z)} \prec P_k(z)$ where $P_k(z)$ is the canonical region.

The convolution of (1.14) and (1.18) gives

$$D_q(K_\phi * Erf)(z)=1+\sum_{n=2}^{\infty}\frac{(-1)^{n-1}[n]_q b_{n-1}}{(2n-1)(n-1)!S}z^{n-1}. \tag{1.20}$$

For a fixed non-negative integer n, the Bell numbers B_n count the possible disjoint partitions of a set with n elements into non-empty subsets or, equivalently, the number of equivalence relations on it. The Bell numbers B_n satisfy a recurrence relation involving binomial coefficients $B_{n+1}=\sum_{k=0}^{n}\binom{n}{k}B_k$, where $B_0=B_1=1$, $B_2=2$, $B_3=5$, $B_4=15$ and $B_5=52$. (one refers the reviewer to [6], [7],[8],[9]) and [27].

The function

$$Q(z)=e^{e^z-1}=\sum_{n=o}^{\infty}B_n\frac{z^n}{n!}=1+z+z^2+\frac{5}{6}z^3+\frac{5}{8}z^4+... \tag{1.21}$$

were considered by Kumar *et al.*[17] which is starlike with respect to 1 and its coefficients generate the Bell numbers.

Motivated by earlier work by [21], [23], [28], and [9], in this work, the authors obtained the first initial bounds for the class of generalized distribution associated with quasi-subordination in terms of error function and Bell-numbers and $\varphi(z)$ analytic in U be of the form

$$\varphi(z) = d_0 + d_1 z + d_2 z^2 + ... \tag{1.22}$$

For the purpose of this investigation, the following Lemmas and definition shall be considered.

Lemma 1: If a function $p \in P$ is given by

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \quad z \in U, \tag{1.23}$$

then $|p_n| \leq 2(k \in \mathbb{N})$, where P is the class of Caratheodory functions analytic in U for which $p(0) = 1$ and $Rep(z) > 0, z \in U$.

Lemma 2: et the Schwartz function $\omega(z)$ be given by

$$\omega(z) = \omega_1(z) + \omega_2 z^2 + \omega_3 z^3 + ... \quad (z \in U), \tag{1.24}$$

then

$$|\omega_1| \leq 1, \quad |\omega_2 - t\omega_1^2| \leq 1 + (|t| - 1)|\omega_1|^2 \leq max 1, |t|,$$

where $t \in \mathbb{C}$.

Definition: Let $Q(z) \in P$ be univalent in $Q(U)$ symmetrical about the real axis and $Q'(z) > 0$. For a function $\mathcal{F} \in \Gamma$ is said to be in the class $\phi S_q(Q, \beta)$ if

$$\frac{\frac{z(D_q(K_\phi * Erf)(z))}{(K_\phi * Erf)(z)} - \beta}{1 - \beta} - 1 \prec_q Q(z) - 1, \tag{1.25}$$

where $0 \leq \beta < 1$ and other parameters as define above.

2. Main Results

Theorem 2.1. *Let $Erf \in \Gamma$ of the form* (1.18) *belong to the class $\phi S_q(Q,\beta)$, then*

$$|\frac{b_1}{S}| \leq \frac{3(1-\beta)}{|1-[2]_q|} \tag{2.1}$$

and for some $\mu \in \mathbb{C}$

$$|\frac{b_2}{S} - \mu\frac{b_1^2}{S^2}| \leq \frac{10(1-\beta)}{|1-[3]_q|} max\left\{1, \left|1 - \frac{(1-\beta)[10(1-[2]_q) - 9(1-[3]_q)\mu]}{10(1-[2]_q)^2}\right|\right\}, \tag{2.2}$$

where $0 < q < 1$ and $0 \leq \beta < 1$.

Proof. Let $Erf \in \phi S_q(Q,\beta)$, then for a Schwartz function $\omega(z)$ and for an analytic $\varphi(z)$ given by (1.22), we have

$$\frac{\frac{z(D_q(K_\phi * Erf)(z))}{(K_\phi * Erf)(z)} - \beta}{1-\beta} - 1 = \varphi(z)(Q(\omega(z)) - 1), \quad z \in U. \tag{2.3}$$

In view of (1.21), one will get

(2.4)

$$\varphi(z)(Q(\omega(z))-1) = (d_0+d_1z+d_2z^2+...)(\omega_1z+(\omega_1^2+\omega_2)+...) = d_0\omega_1z+d_0(\omega_1^2+\omega_2)+\omega_1d_1+...$$

Using the series expansion of $K_\phi * \mathcal{F}$ from (2.3), one will obtain

$$\frac{\frac{z(D_q(K_\phi * Erf)(z))}{(K_\phi * Erf)(z)} - \beta}{1-\beta} - 1 = \frac{\frac{b_1(1-[2]_q)}{3S}}{1-\beta}z + \frac{\frac{b_1^2(1-[2]_q)}{9S^2} - \frac{b_2(1-[3]_q)}{10S}}{1-\beta}z^2 + ... \tag{2.5}$$

From the expansions (2.4) and (2.5), on equating the coefficients of z and z^2 in (2.3), one find that

$$\frac{b_1(1-[2]_q)}{3S} = (1-\beta)d_0\omega_1 \tag{2.6}$$

and

$$\frac{b_1^2(1-[2]_q)}{9S^2} - \frac{b_2(1-[3]_q)}{10S} = (1-\beta)[d_0(\omega_1^2+\omega_2)+\omega_1d_1]. \tag{2.7}$$

Now (2.6) yields

$$\frac{b_1}{S} = \frac{3(1-\beta)d_0\omega_1}{1-[2]_q} \tag{2.8}$$

which in view of (2.7) gives that

$$\frac{b_2(1-[3]_q)}{10S} = \frac{(1-\beta)^2 d_0^2\omega_1^2}{1-[2]_q} - (1-\beta)[d_0(\omega_1^2+\omega_2)+\omega_1 d_1] \tag{2.9}$$

and therefore

$$\frac{b_2}{S} = \frac{10(1-\beta)}{(1-[3]_q)}\left[-\omega_1 d_1 - d_0\left\{\omega_2 + \left(-\frac{(1-\beta)d_0}{1-[2]_q}+1\right)\omega_1^2\right\}\right]. \tag{2.10}$$

For some $\mu \in \mathbb{C}$, we obtain from (2.8) and (2.9).

$$\frac{b_2}{S} - \mu\frac{b_1^2}{S^2} = \frac{10(1-\beta)}{1-[3]_q}\left[-\omega_1 d_1 - (\omega_2+\omega_1^2)d_0 + \left(\frac{10(1-[2]_q)-9(1-[3]_q)\mu}{10(1-[2]_q)^2}\right)(1-\beta)d_0^2\omega_1^2\right]. \tag{2.11}$$

Since, $\varphi(z)$ given by (1.22) is analytic and bounded in U, therefore, on using [22], we obtain for some $y(|y| \leq 1)$

$$|d_0| \leq 1 \quad \textit{and} \quad d_1 = (1-d_0^2)y. \tag{2.12}$$

On putting the value of d_1 from (2.12) into (2.11), one may get

$$\begin{aligned}\frac{b_2}{S} - \mu\frac{b_1^2}{S^2} = \frac{10(1-\beta)}{1-[3]_q}\big[&-y\omega_1 - (\omega_2+\omega_1^2)d_0 \\ &+ \left(\frac{(1-\beta)[10(1-[2]_q)-9(1-[3]_q)\mu]}{10(1-[2]_q)^2}\omega_1^2 + y\omega_1\right)d_0^2\bigg].\end{aligned} \tag{2.13}$$

If $d_0 = 0$ in (2.13), one may have

$$\left|\frac{b_2}{S} - \mu\frac{b_1^2}{S^2}\right| \leq \frac{10(1-\beta)}{|1-[3]_q|}. \tag{2.14}$$

But if $d_0 \neq 0$, let us then suppose that

$$F(d_0) = -y\omega_1 - (\omega_2+\omega_1^2)d_0 + \left(\frac{(1-\beta)[10(1-[2]_q)-9(1-[3]_q)\mu]}{10(1-[2]_q)^2}\omega_1^2 + y\omega_1\right)d_0^2 \tag{2.15}$$

which is a polynomial in d_0 and hence in $|d_0| \leq 1$ and maximum of $|F(d_0)|$ is attained at $d_0 = e^{i\theta}(0 \leq \theta < 2\pi)$. We find that max $F(e^{i\theta}) = |F(1)|$ and

(2.16)
$$\left|\frac{b_2}{S} - \mu\frac{b_1^2}{S^2}\right| \leq \frac{10(1-\beta)}{|1-[3]_q|}\left|-\omega_2 + \left(\frac{(1-\beta)[10(1-[2]_q) - 9(1-[3]_q)\mu]}{10(1-[2]_q)^2} - 1\right)\omega b_1^2\right|,$$

which on using Lemma 2 shows that

(2.17)
$$\left|\frac{b_2}{S} - \mu\frac{b_1^2}{S^2}\right| \leq \frac{10(1-\beta)}{|1-[3]_q|}\max\left\{1, \left|\frac{(1-\beta)[10(1-[2]_q) - 9(1-[3]_q)\mu]}{10(1-[2]_q)^2} - 1\right|\right\},$$

and this last above inequality together with (2.14) thus establishes the result. □

Theorem 2.2. *Let* $Erf \in \Gamma$ *of the form* (1.18) *belong to the class* $\phi S_q(Q, \beta)$*, then*

$$\left|\frac{b_1}{S}\right| \leq \frac{3(1-\beta)}{|1-[2]_q|} \tag{2.18}$$

and for some $\mu \in \mathbb{C}$

$$\left|\frac{b_2}{S} - \mu\frac{b_1^2}{S^2}\right| \leq \frac{10(1-\beta)}{|1-[3]_q|} max\left\{1, \left|1 - \frac{(1-\beta)[10(1-[2]_q) - 9(1-[3]_q)\mu]}{10(1-[2]_q)^2}\right|\right\} \tag{2.19}$$

where $0 < q < 1$ *and* $0 \leq \beta < 1$.

Proof. Let $Erf \in \phi S_q(Q, \beta)$. Similar to the proof of Theorem 2.1, if $\varphi(z) \equiv 1$, then (1.22) evidently implies that $d_0 = 1$ and $d_n = 0$, $n \in \mathbb{N}$, hence in view of (2.8) and (2.11) and Lemma 2, the desired result were obtained. □

The next theorem devoted for the majorization.

Theorem 2.3. *If a function* $Erf \in \phi S_q(Q, \beta)$ *of the form* (1.18) *belong satisfies*

$$\frac{\frac{z(D_q(K_\phi * \mathcal{F})(z))}{(K_\phi * \mathcal{F})(z)} - \beta}{1-\beta} - 1 << Q(z) - 1, \quad z \in U, \tag{2.20}$$

then

$$\left|\frac{b_1}{S}\right| \leq \frac{3(1-\beta)}{|1-[2]_q|} \tag{2.21}$$

and for some $\mu \in \mathbb{C}$

(2.22)
$$\left|\frac{a_2}{S} - \mu\frac{b_1^2}{S^2}\right| \leq \frac{10(1-\beta)}{|1-[3]_q|} max\left\{1, \left|1 - \frac{(1-\beta)[10(1-[2]_q) - 9(1-[3]_q)\mu]}{10(1-[2]_q)^2}\right|\right\},$$

where $0 < q < 1$ *and* $0 \leq \beta < 1$.

Proof. Following the proof of Theorem 1, if $\omega(z) \equiv z$ in (1.24), so that $\omega_1 = 1$ and $\omega_n = 0$, $n = 2, 3, ...$, then in view of (2.8) and (2.11), we have

$$\left|\frac{b_1}{S}\right| \leq \frac{3(1-\beta)}{|1-[2]_q|} \tag{2.23}$$

and

(2.24)
$$\frac{b_2}{S} - \mu\frac{b_1^2}{S^2} = \frac{10(1-\beta)}{1-[3]_q}\left[-d_1 - d_0 + \left(\frac{(1-\beta)[10(1-[2]_q) - 9(1-[3]_q)\mu]}{10(1-[2]_q)^2}\right)d_0^2\right].$$

On putting the value of d_1 from

(2.25)
$$\frac{b_2}{S} - \mu\frac{b_1^2}{S^2} = \frac{10(1-\beta)}{1-[3]_q}\left[-y - d_0 + \left(\frac{(1-\beta)[10(1-[2]_q) - 9(1-[3]_q)\mu]}{10(1-[2]_q)^2} + y\right)d_0^2\right].$$

If $d_0 = 0$ in (2.25), one obtain

$$\left|\frac{b_2}{S} - \mu\frac{a_1^2}{S^2}\right| \leq \frac{10(1-\beta)}{|1-[3]_q|}, \tag{2.26}$$

and if $d_0 \neq 0$, let

$$G(d_0) := -y - d_0 + \left(\frac{(1-\beta)[10(1-[2]_q) - 9(1-[3]_q)\mu]}{10(1-[2]_q)^2} + y\right)d_0^2 \tag{2.27}$$

which being a polynomial in d_0 and hence in $|d_0| \leq 1$ and maximum of $|F(d_0)|$ is attained at $d_0 = e^{i\theta}(0 \leq \theta < 2\pi)$. We find that $\max G(e^{i\theta}) = |G(1)|$ and consequently

$$\left|\frac{b_2}{S} - \mu\frac{b_1^2}{S^2}\right| \leq \frac{10(1-\beta)}{1-[3]_q}\left|\frac{(1-\beta)[10(1-[2]_q) - 9(1-[3]_q)\mu]}{10(1-[2]_q)^2} - 1\right|. \tag{2.28}$$

which together with (2.26) establishes the desired result of Theorem 3. □

Acknowledgement

We would like to appreciate the reviewers for their comments, contributions and their time spent to review the work thoroughly.

References

[1] M. Abramowitz, I. A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables*, Dorer Publications Inc. New York, 1965.

[2] S. Altinkaya, S. Yalcin, Estimates on Coefficients of a General Subclass of Bi-univalent Functions Associated with Symmetric q-derivative Operator by Means of the Chebyshev Polynomials, *Asia Pacific Journal of Mathematics*, **4**(2017), 90–99.

[3] H. Alzer, Error Functions Inequalities, *Advances in Computational Mathematics*, **33**(2010), 349-379.

[4] A. Baricz, Geometric Properties of Generalized Bessel Functions, *Publicationes Mathematicae*, **73**(2008), 155–178.

[5] A. Baricz, Geometric Properties of Generalized Bessel Functions of Complex Order, *Mathematica*, **48**(2006), 13–18.

[6] E. T. Bell, The Iterated Exponential Integers, *Annals of Mathematics*, **39**(1938), 539–557.

[7] E. T. Bell, Exponential Polynomials, *Annals of Mathematics*, **35**(1934), 258–277.

[8] B. C. Berndt, Reaches His Hand from His Grave to Snatch Your Theorems from You, *Asia Pacific Mathematics Newsletter*, **1**(2011), 8–13.

[9] E. R. Canifield, Engel's Inequality for Bell-numbers *Journal of Combinatorial Theory, Series A*, **72**(1995), 184–187.

[10] A. Cetinkaya, Y. Kahramaner, YU. Polatoglu, Fekete-Szegö Inequalities for q-starlike and q-convex Functions, *Acta. Universitatis Apulensis*, **53**(2018), 55–64.

[11] D. Coman, The Radius of Starlikeness for Error Function, *Stud. Univ. Babes, Bolyal Math.*, **36**(1991), 13–16.

[12] A. Elbert, A. Laforgia, The Zeros of the Complementary Error Function, *Numerical Algorithms*, **49**(2008), 153–157.

[13] R. El-Ashwah, S. Kanas, Fekete-Szegö Inequalities for Quasi-subordination Functions Classes of Complex Order, *Kyungpook Mathematical Journal*, **55**(2005), 679–688.

[14] S. P. Goyal, O. Singh, R. Mukharjee, Certain Results on a Subclass on Analytic and Bi-univalent Functions Associated with Coefficient Estimates and Quasi-subordination, *Palestine Journal of Mathematics*, **5**(2016), 79–85.

[15] F. H. Jackson, On q-functions and a Certain Difference Operator, *Transaction of the Royal Society of Edinburgh*, **46**(1908), 253–281.

[16] S. Kanas, S. Altinkaya, S. Yalcin, Subclass of k-uniformly Starlike Functions Defined by Symmetric q-derivative Operator, *Ukrains' kyi Matematychnyi Zhurnal*, **70**(2018), 1499–1510.

[17] V. Kumar, N. E. Cho, V. Ravichandran, H. M. Srivastava, Sharp Coefficient Bounds for Starlike Functions Associated with the Bell Numbers, *Mathematica Slovaca*, **69**(2019), 1053–1064.

[18] T. H. MacGregor, Majorization by Univalent Functions, *Duke Mathematical Journal*, **34**(1967), 95–102.

[19] N. Magesh, V. K. Balaji, J. Yamini, Certain Subclasses of Bi-starlike and Bi-convex Functions Based on Quasi-subordination, *Abstract and Applied Analysis*, **2016**(2016), 6pages.

[20] H. M. Mohd, M. Darus, Fekete-Szegö Problems for Quasi-subordination Classes, *Abstract and Applied Analysis*, **2012**(2012), Hindawi, 14pp.

[21] G. Murugusundaramoorthy, S. O. Olatunji, O. A. Fadipe-Joseph, Fekete-Szegö Problems for Analytic Functions in the Space of Logistic Sigmoid Functions Based on Quasi-subordination, *International Journal of Nonlinear Analysis and Application*, **9**(2018), 55–68.

[22] Z. Nehari, *Conformal Mappings*, McGraw-Hill, New-York (1952).

[23] S. O. Olatunji, E. J. Dansu, A. Abidemi, On a Sakaguchi Type Class of Analytic Functions Associated with Quasi-subordination in the Space of Modified Sigmoid Functions, *Electronic Journal of Mathematical Analysis and Application*, **5**(2017), 97–105.

[24] S. O. Olatunji, O. A. Fadipe-Joseph, Initial Chebyshev Polynomial Class of Analytic Function Based on Quasi-subordination, *Palestine Journal of Mathematics*, **8**(2019), 227–233.

[25] C. Pommerenke, *Univalent Functions*, Vandenhoeck and Ruprecht, Gottingen, 1975.

[26] S. Porwal, Generalized Distribution and Its Geometric Properties Associated with Univalent Functions, *Journal of Complex Analysis*, **2018**(2018), 5pp.

[27] F. Qi, Some Inequalities for Bell-numbers, *Proceedings-Mathematical Sciences*, **127**(2017), 551–564.

[28] C. Ramachandran, L. M. Vanitha, K. Stanislawa, Certain Results on q-starlike and q-convex Error Functions, *Mathematica Slovaca*, **68**(2018), 3614–368.

[29] C. Ramachandran, K. Dhanalakshmi, L. Vanitha, Hankel Determinant for a Subclass of Analytic Functions Associated with Error Functions Bounded by Conical Regions, *International Journal of Mathematical Analysis*, **11**(2017), 571–581.

[30] M. S. Robertson, Quasi-subordination and Coefficients Conjectures, *Bulletin of American Mathematical Society*, **76**(1970), 1–9.

[31] P. Sharma, R. K. Raina, On a Sakaguchi Type Class of Analytic Functions Associated with Quasi-subordination, *Rikkyo Daigaku Sugaku Zasshi*, **64**(2015), 59–70.

(1) Department of Mathematical Sciences, Federal University of Technology, P.M.B.704, Akure, Nigeria.

Email address: olatunjiso@futa.edu.ng

(2) Department of Mathematics, Faculty of Arts and Science, Uludag University, 16059, Bursa, Turkey.

Email address: sahsene@uludag.edu.tr