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GENERALIZED LAGUERRE POLYNOMIAL BOUNDS FOR SUBCLASS OF BI-UNIVALENT FUNCTIONS

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Abstract. In the present paper, we propose to introduce a new subclass of bi-univalent analytic functions $\mathrm{T}_{\Sigma}(\lambda, \gamma)$ $(0 < \lambda \leq 1,\ \gamma \geq 0)$ which is defined by making use of the generalized Laguerre polynomials in the open unit disk ∇. We derive upper bounds for the coefficients $|a_2|$, $|a_3|$ and discuss Fekete-Szegö problem for the functions belonging to the new introduced class $\mathrm{T}_{\Sigma}(\lambda, \gamma)$.

1. Introduction and Motivation

Let $\mathbb{C}$ denote the set of complex numbers and let $H(\nabla)$ be the class of functions which are analytic in the open unit disk $\nabla := \{z \in \mathbb{C} : |z| < 1\}$. Let Λ be the class of all functions $f \in H(\nabla)$ satisfying the normalization condition $f(0) = f'(0) - 1 = 0$. Thus, a function $f \in \Lambda$ has the following Taylor-Maclurian series expansion:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in \nabla). \tag{1.1}$$

We denote by $\mathcal{S}$, the subclass of all functions in Λ that are univalent in ∇.

For real number $\beta > -1$, the polynomial solution $y(x)$ of the differential equation (see [14])

$$xy'' + (1 + \beta - x)y' + ny = 0, \tag{1.2}$$

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where n is non-negative integers, is called generalized Laguerre polynomial or associated Laguerre polynomial and it is denoted by $L_n^{\beta}(x)$. It has many applications in areas of mathematical physics and quantum mechanics; for example in the integration of Helmholtz's equation in paraboloidal coordinates, in the theory of propagation of electromagnetic oscillations along long lines and so on. These polynomials satisfy certain recurrence relations, namely,

$$L_{n+1}^{\beta}(x) = \frac{2n+1+\beta-x}{n+1} L_n^{\beta}(x) - \frac{n+\beta}{n+1} L_{n-1}^{\beta}(x) \quad (n \geq 1) \tag{1.3}$$

with the initial conditions

$$L_0^{\beta}(x) = 1, \ L_1^{\beta}(x) = 1 + \beta - x. \tag{1.4}$$

It can easily derived from (1.3) that

$$\begin{aligned} L_2^{\beta}(x) &= \frac{x^2}{2} - (\beta+2)x + \frac{(\beta+1)(\beta+2)}{2}, \\ L_3^{\beta}(x) &= -\frac{x^3}{6} + \frac{(\beta+3)x^2}{2} - \frac{(\beta+2)(\beta+3)}{2}x + \frac{(\beta+1)(\beta+2)(\beta+3)}{6}, \end{aligned} \tag{1.5}$$

and so on.
It may be noted that the simply Laguerre polynomials are the special case $\beta = 0$ of generalized Laguerre polynomial i.e. $L_n^0(x) = L_n(x)$.

Result 1.1. *(see [14]) Let $F(x, z)$ be the generating function of the generalized Laguerre polynomial $L_n^{\beta}(x)$. Then*

$$F(x,z) = \sum_{n=0}^{\infty} L_n^{\beta}(x) z^n = \frac{e^{-\frac{xz}{1-z}}}{(1-z)^{\beta+1}} \quad (x \in \mathbb{R}, \ z \in \nabla). \tag{1.6}$$

Now we recall the subordination between two analytic functions as follows:

Definition 1.1. Let f and g be two analytic functions in ∇. Then f is subordinate to g if there exists an analytic function w, satisfying the condition of Schwarz lemma (i.e $w(0) = 0$, $|w(z)| < 1$ $(z \in \nabla)$) such that $f(z) = g(w(z))$. We denote this subordination by $f(z) \prec g(z)$ $(z \in \nabla)$. In particular, if the function g is univalent in ∇, then (see [5, 16])

$$f(z) \prec g(z) \ (z \in \nabla) \Longleftrightarrow f(0) = g(0) \text{ and } f(\nabla) \subset g(\nabla).$$

The Koebe One-Quarter theorem (see [9]) asserts that the image of ∇ under every functions in the normalized univalent function class $\mathcal{S}$ contains the disk $|z| < 1/4$. Therefore, the inverse of $f \in \Lambda$ is an univalent analytic function f^{-1} in a disk $\Delta_\rho = \{z : z \in \mathbb{C} \text{ and } |z| < \rho, \ \rho \geq \frac{1}{4}\}$. Then we have $f^{-1}(f(z)) = z$, $(z \in \nabla)$ and $f(f^{-1}(w)) = w$ $(w \in \Delta_\rho)$. From (1.1), we have

$$g(w) = f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots . \tag{1.7}$$

Definition 1.2. A function $f \in \Lambda$ is said to be bi-univalent in ∇ if both f and f^{-1} are univalent in ∇. Let Σ be denote the class of bi-univalent function in ∇ given by (1.1).

A good amount of literature are available for the bounds of the coefficient estimate for the class of bi-univalent functions. The concept of bi-univalent analytic functions was introduced by Lewin [15] in 1967 and he showed that $|a_2| < 1.51$. Subsequently, Brannan and Clunie [3] conjectured that $|a_2| \leq \sqrt{2}$ for $f \in \Sigma$. Netanyahu [17] on the other hand, showed that $\max_{f\in\Sigma} |a_2| = \frac{4}{3}$. Brannan and Taha [4] introduced certain subclasses of the bi-univalent function class Σ similar to the familiar subclasses $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ of starlike and convex functions of order α $(0 < \alpha < 1)$ respectively. The classes $\mathcal{S}^*_\Sigma(\alpha)$ and $\mathcal{K}_\Sigma(\alpha)$ of bi-univalent starlike functions of order α and bi-convex functions of order α, corresponding to the function classes $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$, were also introduced analogously. For each of the function classes $\mathcal{S}^*_\Sigma(\alpha)$ and $\mathcal{K}_\Sigma(\alpha)$, they found non-sharp estimates on the initial coefficients.

The coefficient estimate problems for bi-univalent function is about half a century old.

Any good results on this topic is recognized world-wide as a significant contribution. But the problem remained in hibernation for about thirty years until it was freshly discussed by Srivastava et al.[20]. From the work of Srivastava et al.[20], we choose to recall the functions $\frac{z}{1-z}$, $-\log(1-z), -\frac{1}{2}\log\frac{1+z}{1-z}$ and so on are in the class Σ. The familiar Koebe function which is univalent is not a member of class Σ.

Motivated by the work of Srivastava et al.[20], many researchers investigated the coefficients bounds for various subclasses of bi-univalent function class Σ (see [1, 2, 7, 8, 11, 18, 19, 21]). Not much is known about the bounds on the general coefficients $|a_n|$ for $n \geq 4$. In the literature, there are only a few works determining the general coefficient bounds on $|a_n|$ for the analytic bi-univalent function class (see [6, 12, 13]). The coefficient estimate problem for each of the following Taylor-Maclaurin coefficients $|a_n| \quad (n \in \mathbb{N} \setminus \{1,2\})$ is presumably still an open problem. Motivated by aforementioned works, we introduce a new subclass of bi-univalent function class Σ by using generating function of generalized Laguerre polynomial as follows.

Definition 1.3. A function $f \in \Sigma$ of the form (1.1) is said to be in the class $\mathcal{T}_\Sigma(\lambda,\gamma)$ if it satisfies the following subordination condition:

$$\frac{1}{2}\left[\left(\frac{z}{f(z)}\right)^{1-\gamma} f'(z) + \left\{\left(\frac{z}{f(z)}\right)^{1-\gamma} f'(z)\right\}^{\frac{1}{\lambda}}\right] \prec F(x,z) \quad (z \in \nabla) \tag{1.8}$$

and

$$\frac{1}{2}\left[\left(\frac{w}{g(w)}\right)^{1-\gamma} g'(w) + \left\{\left(\frac{w}{g(w)}\right)^{1-\gamma} g'(w)\right\}^{\frac{1}{\lambda}}\right] \prec F(x,w) \quad (w \in \nabla), \tag{1.9}$$

where $0 < \lambda \leq 1,\ \gamma \geq 0$ and $g = f^{-1}$ is given by (1.7). All of the powers are chosen in order to 1^α=1.

Remark 1. Taking $\lambda = 1$ in Definition 1.3, it can seen that a function $f \in \Sigma$ is in the class $\mathcal{T}_\Sigma(1,\gamma) = \mathcal{T}_\Sigma(\gamma)$ if

$$\left(\frac{z}{f(z)}\right)^{1-\gamma} f'(z) \prec F(x,z) \quad (z \in \nabla) \tag{1.10}$$

and

$$\left(\frac{w}{g(w)}\right)^{1-\gamma} g'(w) \prec F(x,w) \quad (w \in \nabla), \tag{1.11}$$

where $\gamma \geq 0$ and $g = f^{-1}$ is given by (1.7).

Remark 2. Taking $\gamma = 1$ in Definition 1.3, we get the following: A function $f \in \Sigma$ is in the class $\mathcal{T}_\Sigma(\lambda,1) = \mathcal{T}_\Sigma(\lambda)$ if

$$\frac{1}{2}[f'(z) + (f'(z))^{\frac{1}{\lambda}}] \prec F(x,z) \quad (z \in \nabla) \tag{1.12}$$

and

$$\frac{1}{2}[g'(w) + (g'(w))^{\frac{1}{\lambda}}] \prec F(x,w) \quad (w \in \nabla), \tag{1.13}$$

where $\gamma \geq 0$ and $g = f^{-1}$ is given by (1.7).

Remark 3. Letting $\gamma = 0$ in Remark 1, we get the following function class. A function $f \in \Sigma$ is said to be in the class $\mathcal{T}_\Sigma(0) = \mathcal{T}_\Sigma$ if

$$\frac{zf'(z)}{f(z)} \prec F(x,z) \quad (z \in \nabla) \tag{1.14}$$

and

$$\frac{wg'(w)}{g(w)} \prec F(x,w) \quad (w \in \nabla) \tag{1.15}$$

Remark 4. Putting $\gamma = 1$ in Remark 1 or $\lambda = 1$ in Remark 2 we obtain the result for the following function class. A function $f \in \Sigma$ is said to be in the class $\mathcal{T}_\Sigma(1,1) = \mathcal{T}_{\Sigma_1}$ if

$$f'(z) \prec F(x,z) \quad (z \in \nabla) \tag{1.16}$$

and

$$g'(w) \prec F(x,w) \quad (w \in \nabla). \tag{1.17}$$

The main objective of the present paper is sequentially arrange in the following manner. In Section 2, we obtain the coefficient bounds on the Taylor-Maclaurin coefficients on a_2 and a_3 for the function belongs to $\mathcal{T}_\Sigma(\lambda,\gamma)$. In Section 3, we consider the Fekete-Szegö problem for the above mentioned class. Finally, the paper is ended with concluding remark.

2. Coefficient Bounds

The coefficient bounds for the function $f \in \Sigma$ in the class $\mathcal{T}_\Sigma(\lambda,\gamma)$ is given by the following theorem.

Theorem 2.1. *Let the function $f \in \Sigma$ given by (1.1) be in the class $\mathcal{T}_\Sigma(\lambda,\gamma)$ $(0 < \lambda \leq 1,\ \gamma \geq 0)$. Then*

$$|a_2| \leq \frac{2\lambda|1+\beta-x|\sqrt{|1+\beta-x|}}{\sqrt{\left|A_1(1+\beta-x)^2 - A_2\left(\frac{x^2}{2} - (\beta+2)x + \frac{(\beta+1)(\beta+2)}{2}\right)\right|}} \tag{2.1}$$

and

$$|a_3| \leq \frac{4\lambda^2(1+\beta-x)^2}{(1+\gamma)^2(1+\lambda)^2} + \frac{2\lambda|1+\beta-x|}{(1+\lambda)(\gamma+2)}, \tag{2.2}$$

where

$$A_1 = \lambda(\lambda+1)(1+\gamma)(2+\gamma) + (1-\lambda)(1+\gamma)^2 \tag{2.3}$$

$$A_2 = (1+\lambda)^2(1+\gamma)^2. \tag{2.4}$$

Proof. Let $f \in \mathcal{T}_\Sigma(\lambda, \gamma)$ be given by (1.1) and $g = f^{-1}$. Then, by Definition 1.3, we have

$$\frac{1}{2}\left[\left(\frac{z}{f(z)}\right)^{1-\gamma} f'(z) + \left\{\left(\frac{z}{f(z)}\right)^{1-\gamma} f'(z)\right\}^{\frac{1}{\lambda}}\right] = F(x, \phi(z)), \tag{2.5}$$

and

$$\frac{1}{2}\left[\left(\frac{w}{g(w)}\right)^{1-\gamma} g'(w) + \left\{\left(\frac{w}{g(w)}\right)^{1-\gamma} g'(w)\right\}^{\frac{1}{\lambda}}\right] = F(x, \psi(w)). \tag{2.6}$$

Define the functions $\phi(z)$ and $\psi(w)$ by

$$\phi(z) = d_1 z + d_2 z^2 + \ldots, \tag{2.7}$$

$$\psi(z) = e_1 w + e_2 w^2 + \ldots \tag{2.8}$$

which are analytic in ∇ with $\phi(0) = \psi(0) = 0$ and $|\phi(z)| < 1$, $|\psi(z)| < 1$, $(z, w \in \nabla)$. It is fairly well-known that if

$$|\phi(z)| = |d_1 z + d_2 z^2 + \ldots| < 1 \quad (z \in \nabla)$$

and

$$|\psi(w)| = |e_1 w + e_2 w^2 + \ldots| \quad (w \in \nabla), \tag{2.9}$$

then

$$|d_i| \leq 1 \text{ and } |e_i| \leq 1 \quad (i \in \mathbb{N} := \{1, 2, 3, \ldots\}). \tag{2.10}$$

Making use of (2.7) and (2.8) in (2.5) and (2.6) respectively, we obtain

$$\begin{aligned} \frac{1}{2}\left[\left(\frac{z}{f(z)}\right)^{1-\gamma} f'(z) + \left\{\left(\frac{z}{f(z)}\right)^{1-\gamma} f'(z)\right\}^{\frac{1}{\lambda}}\right] &= \sum_{n=0}^{\infty} L_n^\beta(x)[\phi(z)]^n \\ &= 1 + L_1^\beta(x) d_1 z + [L_1^\beta(x) d_2 + L_2^\beta(x) d_1^2] z^2 + \ldots, \end{aligned} \tag{2.11}$$

and

$$\frac{1}{2}\left[\left(\frac{w}{g(w)}\right)^{1-\gamma} g'(w) + \left\{\left(\frac{w}{g(w)}\right)^{1-\gamma} g'(w)\right\}^{\frac{1}{\lambda}}\right] = \sum_{n=0}^{\infty} L_n^{\beta}(x)[\psi(w)]^n$$

$$= 1 + L_1^{\beta}(x)e_1 w + [L_1^{\beta}(x)e_2 + L_2^{\beta}(x)e_1^2]w^2 + \dots. \tag{2.12}$$

A simple calculation give

$$\frac{1}{2}\left[\left(\frac{z}{f(z)}\right)^{1-\gamma} f'(z) + \left\{\left(\frac{z}{f(z)}\right)^{1-\gamma} f'(z)\right\}^{\frac{1}{\lambda}}\right] = 1 + \frac{(1+\lambda)(1+\gamma)}{2\lambda}a_2 z$$

$$+\left[\frac{(\gamma+2)(1+\lambda)}{4\lambda}(2a_3 + (\gamma-1)a_2^2) + \frac{(1-\lambda)(1+\gamma)^2}{4\lambda^2}a_2^2\right] z^2 + \cdots \tag{2.13}$$

and

$$\frac{1}{2}\left[\left(\frac{w}{g(w)}\right)^{1-\gamma} g'(w) + \left\{\left(\frac{w}{g(w)}\right)^{1-\gamma} g'(w)\right\}^{\frac{1}{\lambda}}\right] = 1 - \frac{(1+\gamma)(1+\lambda)}{2\lambda}a_2 w$$

$$+\left[\frac{(1+\lambda)(\gamma+2)}{4\lambda}\{(3+\gamma)a_2^2 - 2a_3\} + \frac{(1-\lambda)(1+\gamma)^2}{4\lambda^2}a_2^2\right] w^2 + \cdots \tag{2.14}$$

Hence, upon comparing coefficients between (2.11) and (2.13) and (2.12) and (2.14) we obtain

$$\frac{(1+\lambda)(1+\gamma)}{2\lambda}a_2 = L_1^{\beta}(x)d_1, \tag{2.15}$$

$$\frac{(1+\lambda)(\gamma+2)}{4\lambda}[2a_3 + (\gamma-1)a_2^2] + \frac{(1-\lambda)(\gamma+1)^2}{4\lambda^2}a_2^2 = L_1^{\beta}(x)d_2 + L_2^{\beta}(x)d_1^2, \tag{2.16}$$

and

$$-\frac{(1+\lambda)(1+\gamma)}{2\lambda}a_2 = L_1^{\beta}(x)e_1, \tag{2.17}$$

$$\frac{(1+\lambda)(\gamma+2)}{4\lambda}[(3+\gamma)a_2^2 - 2a_3] + \frac{(1-\lambda)(1+\gamma)^2}{4\lambda^2}a_2^2 = L_1^{\beta}(x)e_2 + L_2^{\beta}(x)e_1^2. \tag{2.18}$$

It follows from (2.15) and (2.17) that

$$d_1 = -e_1, \tag{2.19}$$

and

$$\frac{(1+\lambda)^2(1+\gamma)^2}{2\lambda^2}a_2^2 = [L_1^\beta(x)]^2(d_1^2+e_1^2). \tag{2.20}$$

Adding (2.16) and (2.18) and using (2.20) in the resulting equation, we get

(2.21)

$$[\lambda(\lambda+1)(\gamma+1)(\gamma+2)+(1-\lambda)(1+\gamma)^2](L_1^\beta(x))^2-(1+\lambda)^2(1+\gamma)^2L_2^\beta(x)]a_2^2 = 2\lambda^2(L_1^\beta(x))^3(d_2+e_2),$$

which implies

$$a_2^2 = \frac{2\lambda^2(L_1^\beta(x))^3(d_2+e_2)}{A_1(L_1^\beta(x))^2 - A_2L_2^\beta(x)}, \tag{2.22}$$

where A_1 and A_2 are given by (2.3) and (2.4) respectively. Applying (2.10) to the coefficients d_2 and e_2 and using (1.4) and (1.5) in (2.22) we have

$$|a_2| \leq \frac{2\lambda|1+\beta-x|\sqrt{|1+\beta-x|}}{\sqrt{|A_1(1+\beta-x)^2 - A_2\left(\frac{x^2}{2}-(\beta+2)x+\frac{(\beta+1)(\beta+2)}{2}\right)|}}.$$

In order to find the bound on $|a_3|$, subtracting (2.18) from (2.16), we obtain

$$\frac{(1+\lambda)(\gamma+2)}{\lambda}(a_3-a_2^2) = L_1^\beta(x)(d_2-e_2)+L_2^\beta(x)(d_1^2-e_1^2). \tag{2.23}$$

By virtue of (2.19) and (2.20), equation (2.23) reduces to

$$a_3 = \frac{2\lambda^2(L_1^\beta(x))^2}{(1+\lambda)^2(1+\gamma)^2}(d_1^2+e_1^2)+\frac{\lambda}{(1+\lambda)(\gamma+2)}L_1^\beta(x)(d_2-e_2).$$

Substituting the values of $L_1^\beta(x)$ and $L_2^\beta(x)$ from (1.4) and (1.5) and applying (2.10) to the coefficients $d_1,\ d_2, e_1$ and e_2, we get

$$|a_3| \leq \frac{4\lambda^2(1+\beta-x)^2}{(1+\gamma)^2(1+\lambda)^2}+\frac{2\lambda|1+\beta-x|}{(1+\lambda)(\gamma+2)}. \tag{2.24}$$

Thus, the proof of Theorem 2.1 is completed. □

Taking $\lambda = 1$ in Theorem 2.1, we obtain the following result:

Corollary 2.1. *Let $f \in \Sigma$ given by (1.1) be in the class $\mathcal{T}_\Sigma(\gamma)$. Then*

$$|a_2| \leq \frac{2|1+\beta-x|^{\frac{3}{2}}}{\sqrt{|2(1+\gamma)(2+\gamma)(1+\beta-x)^2 - 4(1+\gamma)^2\left(\frac{x^2}{2} - (\beta+2)x + \frac{(\beta+1)(\beta+2)}{2}\right)|}}$$

and

$$|a_3| \leq \frac{(1+\beta-x)^2}{(1+\gamma)^2} + \frac{|1+\beta-x|}{(\gamma+2)}.$$

Putting $\gamma = 1$ in Theorem 2.1 gives the following.

Corollary 2.2. *Let $f \in \Sigma$ given by (1.1) belongs to the class $\mathcal{T}_\Sigma(\lambda)$. Then*

$$|a_2| \leq \sqrt{\frac{2\lambda^2|1+\beta-x|^3}{|(3\lambda^2+\lambda+2)(1+\beta-x)^2 - 2(1+\lambda)^2\{\frac{x^2}{2} - (\beta+2)x + \frac{(\beta+1)(\beta+2)}{2}\}|}}$$

and

$$|a_3| \leq \frac{\lambda^2(1+\beta-x)^2}{(1+\lambda)^2} + \frac{2\lambda|1+\beta-x|}{3(1+\lambda)}.$$

Letting $\gamma = 0$ in Corollary 2.1, we get

Corollary 2.3. *Let $f \in \Sigma$ given by (1.1) be in the class $\mathcal{T}_\Sigma$. Then*

$$|a_2| \leq \frac{|1+\beta-x|^{\frac{3}{2}}}{\sqrt{|(1+\beta-x)^2 - \left(\frac{x^2}{2} - (\beta+2)x + \frac{(\beta+1)(\beta+2)}{2}\right)|}}$$

and

$$|a_3| \leq (1+\beta-x)^2 + \frac{|1+\beta-x|}{2}.$$

Taking $\lambda = 1$ in Corollary 2.2 we obtain the following result.

Corollary 2.4. *Let $f \in \Sigma$ given by (1.1) be in the class $\mathcal{T}_{\Sigma_1}$. Then*

$$|a_2| \leq \frac{|1+\beta-x|^{\frac{3}{2}}}{\sqrt{|3(1+\beta-x)^2 - 4\left(\frac{x^2}{2} - (\beta+2)x + \frac{(\beta+1)(\beta+2)}{2}\right)|}}$$

and

$$|a_3| \leq \frac{(1+\beta-x)^2}{4} + \frac{|1+\beta-x|}{3}.$$

3. The Fekete-Szegö Inequalities for the class $\mathcal{T}_\Sigma(\lambda,\gamma)$

It is well-known (cf.[9]) that for a function $f \in \mathcal{S}$ given by (1.1), the sharp inequality $|a_3 - a_2^2| < 1$ holds. Fekete-Szegö [10] obtained sharp upper bounds for $|a_3 - \mu a_2^2|$ for $f \in \mathcal{S}$ where μ is real. Thus, the determination of sharp upper bounds for the nonlinear functional $|a_3 - \mu a_2^2|$ for any compact family $\mathcal{F}$ of functions in Λ is popularly known as the Fekete-Szego problem for $\mathcal{F}$.

The Fekete-Szegö inequalities for function f in the class $\mathcal{T}_\Sigma(\lambda,\gamma)$ is given by the following theorem.

Theorem 3.1. *Let the function f given by (1.1) be in the class $\mathcal{T}_\Sigma(\lambda,\gamma)$. Then for any real number α, we have*

$$|a_3 - \alpha a_2^2| \leq \begin{cases} \frac{2\lambda}{(1+\lambda)(2+\gamma)}|1+\beta-x| & |\alpha-1| \leq \frac{1}{2\lambda(1+\lambda)(2+\gamma)}M \\ \frac{4\lambda^2|1-\alpha||1+\beta-x|^3}{\left|A_1(1+\beta-x)^2 - A_2\left(\frac{x^2}{2}-(\beta+2)x+\frac{(\beta+1)(\beta+2)}{2}\right)\right|} & |\alpha-1| \geq \frac{1}{2\lambda(1+\lambda)(2+\gamma)}M \end{cases}$$

where A_1 and A_2 are given by (2.3) and (2.4) respectively and

$$M = \left| A_1 - \frac{A_2\left(\frac{x^2}{2} - (\beta+2)x + \frac{(\beta+1)(\beta+2)}{2}\right)}{(1+\beta-x)^2} \right|.$$

Proof. From (2.22) and (2.23) we observe that

$$\begin{aligned} a_3 - \alpha a_2^2 &= a_2^2 + \frac{\lambda}{(1+\lambda)(\gamma+2)} L_1^\beta(x)(d_2 - e_2) - \alpha a_2^2 \\ &= (1-\alpha)a_2^2 + \frac{\lambda}{(1+\lambda)(\gamma+2)} L_1^\beta(x)(d_2 - e_2) \\ &= (1-\alpha)\frac{2\lambda^2(L_1^\beta(x))^3(d_2+e_2)}{A_1(L_1^\beta(x))^2 - A_2L_2^\beta(x)} + \frac{\lambda}{(1+\lambda)(\gamma+2)} L_1^\beta(x)(d_2 - e_2) \\ &= L_1^\beta(x)\left[\left(\chi(\alpha,x) + \frac{\lambda}{(1+\lambda)(\gamma+2)}\right)d_2 + \left(\chi(\alpha,x) - \frac{\lambda}{(1+\lambda)(\gamma+2)}\right)e_2\right], \end{aligned}$$

where

$$\chi(\alpha,x) = \frac{2\lambda^2(1-\alpha)(L_1^\beta(x))^2}{A_1(L_1^\beta(x))^2 - A_2L_2^\beta(x)}.$$

In view of (1.4) and (1.5) we conclude that

$$|a_3 - \alpha a_2^2| \leq \begin{cases} \frac{2\lambda|1+\beta-x|}{(1+\lambda)(2+\gamma)} & \left(0 \leq |\chi(\alpha,x)| \leq \frac{\lambda}{(1+\lambda)(2+\gamma)}\right) \\ 2|1+\beta-x||\chi(\alpha,x)| & \left(|\chi(\alpha,x)| \geq \frac{\lambda}{(1+\lambda)(\gamma+2)}\right). \end{cases}$$

This completes the proof of Theorem 3.1. □

Taking $\lambda = 1$ in Theorem 3.1, we obtain the Fekete-Szegö result for the class $\mathcal{T}_\Sigma(\gamma)$ as follows:

Corollary 3.1. *Let the function $f \in \Sigma$ be in the class $\mathcal{T}_\Sigma(\gamma)$. Then for any complex number α, we have*

$$|a_3 - \alpha a_2^2| \leq \begin{cases} \frac{|1+\beta-x|}{\gamma+2} & \left(|\alpha - 1| \leq \frac{1+\gamma}{2(\gamma+2)}\left|2+\gamma-2(1+\gamma)\left(\frac{\frac{x^2}{2}-(\beta+2)x+\frac{(\beta+1)(\beta+2)}{2}}{(1+\beta-x)^2}\right)\right|\right) \\ \frac{2|1-\alpha||1+\beta-x|^3}{\left|(1+\gamma)(2+\gamma)(1+\beta-x)^2-2(1+\gamma)^2\left(\frac{x^2}{2}-(\beta+2)x+\frac{(\beta+1)(\beta+2)}{2}\right)\right|} & \\ \quad \left(|\alpha - 1| \geq \frac{1+\gamma}{2(\gamma+2)}\left|2+\gamma-2(1+\gamma)\left(\frac{\frac{x^2}{2}-(\beta+2)x+\frac{(\beta+1)(\beta+2)}{2}}{(1+\beta-x)^2}\right)\right|\right). & \end{cases}$$

Putting $\gamma = 1$ in Corollary 3.1 we get the following result for the class $\mathcal{T}_{\Sigma_1}$.

Corollary 3.2. *Let the function $f \in \Sigma$ given by (1.1) be in the class $\mathcal{T}_{\Sigma_1}$. Then for any real number α, we obtain*

$$|a_3 - \alpha a_2^2| \leq \begin{cases} \frac{|1+\beta-x|}{3} & \left(|\alpha - 1| \leq \left|1 - \frac{4\left(\frac{x^2}{2}-(\beta+2)x+\frac{(\beta+1)(\beta+2)}{2}\right)}{3(1+\beta-x)^2}\right|\right) \\ \frac{|1-\alpha||1+\beta-x|^3}{\left|3(1+\beta-x)^2-4\left(\frac{x^2}{2}-(\beta+2)x+\frac{(\beta+1)(\beta+2)}{2}\right)\right|} & \\ \quad \left(|\alpha - 1| \geq \left|1 - \frac{4\left(\frac{x^2}{2}-(\beta+2)x+\frac{(\beta+1)(\beta+2)}{2}\right)}{3(1+\beta-x)^2}\right|\right) & \end{cases}$$

Concluding Remarks

: In the present investigation, we have introduced a new subclass $\mathcal{T}_\Sigma(\lambda,\gamma)$ $(0 < \lambda \leq 1,\ \gamma \geq 1)$ of the class Σ of normalized bi-univalent function in the open unit disk ∇. We have obtained the coefficient estimates and Fekete-Szegö inequalities for the function belonging to this bi-univalent function class. Several corollaries of the main result are derived.

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