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EXISTENCE AND UNIQUENESS RESULTS FOR A CLASS OF NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS WITH NONLOCAL BOUNDARY CONDITIONS

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Abstract. In this paper, we study the existence and uniqueness of solutions for fractional differential equations with fractional integral and Caputo fractional derivatives in boundary conditions. Our analysis relies on the Banach contraction principle, Schauder fixed point theorem and Krasnoselskii's fixed point theorem. Examples are provided to illustrate the main results.

1. Introduction

Fractional calculus is a generalization of ordinary differentiation and integration to arbitrary noninteger order. Applications of the fractional differential equations can be found in various fields of science and engineering. Indeed, there are numerous applications in viscoelasticity, electrochemistry, control, porous media, electromagnetism, etc. (see [14, 15, 16, 18]).

At the present day, many authors have studied the existence of solution to the fractional boundary value problems under various boundary conditions and using different approaches, for instance, see ([3]-[9], [17, 19, 21, 22]) and references therein. We also refer the readers to the papers [1, 2, 10, 11, 12, 13] and references therein

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for more literature on some classes of fractional functional or fractional systems with delay in Banach spaces.

A. Guezane-Lakoud and R. khaldi proved the existence and uniqueness of solutions for the following nonlinear fractional boundary value problem [8]:

$$\begin{cases} {}^cD^q_{0^+}u(t) = f(t, u(t), {}^cD^\sigma_{0^+}u(t)), \quad 0 < t < 1 \\ u(0) = 0, \quad u'(1) = I^\sigma_{0^+}u(1). \end{cases}$$

Where $f : [0,1] \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is a given function. $1 < q < 2$, $0 < \sigma < 1$, and ${}^cD^q_{0^+}$ is the Caputo fractional derivative.

B. Ahmad et al. studied the following boundary value problem for fractional differential equations with nonlocal Riemann-Liouville integral boundary conditions [4]:

$$\begin{cases} {}^cD^q_{0^+}x(t) = f(t, x(t)),\ 1 < q \leq 2 \quad t \in [0,1], \\ x(0) = aI^\beta_{0^+}x(\eta) = \frac{a}{\Gamma(\beta)}\int_0^\eta (\eta - s)^{\beta-1}ds, \\ x(1) = bI^\alpha_{0^+}x(\sigma) = \frac{b}{\Gamma(\alpha)}\int_0^\sigma (\sigma - s)^{\alpha-1}ds. \end{cases}$$

Where ${}^cD^q_{0^+}$ denotes the Caputo fractional derivative of order q, f is is a given continuous function, a, b, η, σ are a real constants with $0 < \eta, \sigma < 1$.

M. Houasand and M. Benbachir considered the existence and uniqueness of solutions for the following problem [9]:

$$\begin{cases} {}^cD^\alpha_{0^+}x(t) = f(x(t), {}^cD^\beta_{0^+}x(t)), \quad t \in J \\ x(0) = x_0, \quad x'(0) = 0, \quad x'(1) = \lambda I^\sigma_{0^+}x(\eta). \end{cases} \tag{1.1}$$

Where $2 < \alpha \leq 3$, $1 < \beta \leq 2; 0 < \eta < 1$, ${}^cD^\alpha_{0^+}$ and ${}^cD^\beta_{0^+}$ are the Caputo fractional derivatives, $J = [0,1], \lambda$ is real constant f is continuous function on $\mathbb{R}^2$.

In 2017 C-M. Su et al studied the existence and uniqueness of solutions for boundary value problem of nonlinear fractional differential equation with fractional integral BCs as well as integer and fractional derivative [21]:

$$\begin{cases} {}^{c}\mathcal{D}^{q}_{0+}u(t) = f(t, u(t)) \quad t \in (0,1) \\ u(0) = u''(0) = 0 \quad {}^{c}\mathcal{D}^{\sigma_1}_{0+}u(1) = \lambda I^{\sigma_2}_{0+}u(1). \end{cases}$$

Here $2 < q < 3, 0 < \sigma_1 \leq 1, \sigma_2 > 0, \lambda \neq \Gamma(2+\sigma_2)/\Gamma(2-\sigma_1)$, ${}^{c}\mathcal{D}^{q}_{0+}, {}^{c}\mathcal{D}^{\sigma_1}_{0+}$ denotes the standard Caputo fractional derivatives and $I^{\sigma_2}_{0+}$ denotes the standard Riemann-Liouville fractional integral, $f : [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous.

Motivated by the above works, in this article we aim to establish the existence and uniqueness of solutions to the boundary-value problem of the fractional differential equations

$$\begin{cases} {}^{c}\mathcal{D}^{\alpha}_{0+}y(t) = f(t, y(t), {}^{c}\mathcal{D}^{\beta}_{0+}y(t)), \ 0 < t < 1 \\ y(0) = y_0, y'(0) = aI^{\sigma_1}_{0+}y(\eta_1), {}^{c}\mathcal{D}^{\beta_1}_{0+}y(1) = bI^{\sigma_2}_{0+}y(\eta_2), \end{cases} \tag{1.2}$$

where ${}^{c}\mathcal{D}^{\nu}_{0+}$ is the Caputo fractional derivative of order $\nu \in \{\alpha, \beta, \beta_1\}$ such that $2 < \alpha \leq 3,\ 0 < \beta, \beta_1 \leq 1, I^{\theta}_{0+}$ is the Riemann–Liouville fractional integral of order $\theta > 0, \theta \in \{\sigma_1, \sigma_2\},\ J := [0,1]$ and $f : J \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a continuous function, a, b are suitably chosen real constants.

Note that, in Eq. (1.2), the nonlinear term involves Caputo fractional derivative; to the best of our knowledge, few results can be found in the literature concerning boundary value problems for Caputo fractional differential equations with fractional integral and Caputo fractional derivatives in boundary conditions and nonlinear term depends on Caputo fractional derivative of the unknown function.

In comparison to problem (1.1), our considered BVP (1.2) is more general than the problem studied in [9], as we consider a problem with Riemann-Liouville fractional integral and Caputo fractional derivatives in boundary conditions, while the authors in [9] investigated a problem with three point boundary conditions. Moreover, in [9] the assumption on f are strong (f uniformly Lipschitz condition or uniformly bounded). In this paper the new existence and uniqueness results will be presented for the boundary value problem (1.2) by virtue of fractional calculus and fixed point method under some weak conditions.

Compared with the results appeared in [9], there are some differences. The most important of them is that the assumptions on f are more general and easy to check.

The rest of the paper is organized as follows: In Section 2, we give some definitions and lemmas which we need to prove the main results. In Section 3, several fixed point theorems are used to give sufficient conditions for the existence (uniqueness) of solutions to (1.2) such as Banach's contraction principle, Krasnoselskii's fixed point theorem, and Schauder's fixed point theorem. We end our work with illustrative examples.

2. Preliminaries

Here, we intend to introduce some basic definitions and properties of fractional calculus theory see [14, 16, 18].

Definition 2.1. A real function $f(t); t > 0$ is said to be in space C_μ, $\mu \in \mathbb{R}$ if there exists a real number $p > \mu$, such that $f(t) = t^p f_1(t)$, where $f_1(t) \in C(0, +\infty)$, and it is said to be in the space C_μ^n if and only if $f^{(n)} \in C_\mu; n \in \mathbb{N}$.

Definition 2.2. The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$ of a function $f \in C_\mu$, $\mu \geq -1$, is defined as

$$I_{0^+}^{\alpha} f(t) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, & \alpha > 0, \\ f(t), & \alpha = 0, \end{cases}$$

where $\Gamma(\cdot)$ is the Euler gamma function.

Definition 2.3. The fractional derivative of $f(t)$ in the Caputo sense is defined as

$$^{c}D_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s) ds,$$

for $n-1 < \alpha \leq n, n \in \mathbb{N}, t > 0, f \in C_{-1}^{n}$.

Lemma 2.1. *Let $\alpha > \beta > 0$, and $f \in C_\mu$, $\mu \geq -1$. Then we have:*

(1) $I_{0^+}^{\alpha} I_{0^+}^{\beta} f(t) = I_{0^+}^{\alpha+\beta} f(t)$,

(2) $^{c}D_{0^+}^{\alpha} I_{0^+}^{\alpha} f(t) = f(t)$,

(3) $^{c}D_{0^+}^{\beta} I_{0^+}^{\alpha} f(t) = I_{0^+}^{\alpha-\beta} f(t)$,

(4) $I_{0^+}^{\alpha}(^{c}D_{0^+}^{\alpha} f(t)) = f(t) + \sum_{j=0}^{m-1} c_j t^j$, *for some* $c_j \in \mathbb{R}, j = 0, 1, 2, \dots, m-1$, *where* $m = [\alpha] + 1$.

Lemma 2.2. *Let $\alpha > 0$, then the differential equation*

$$(^{c}D_{0^+}^{\alpha} f)(t) = 0,$$

has the unique solution

$$f(t) = \sum_{j=0}^{m-1} c_j t^j, \quad c_j \in \mathbb{R}, j = 0 \dots m-1,$$

where $m - 1 < \alpha \leq m$.

Lemma 2.3 ([8, 21])**.** *Let $\alpha > 0, f \in L^1([0,1], \mathbb{R}_+)$. Then for all $t \in [0,1]$ we have*

$$I_{0^+}^{\alpha+1} f(t) \leq \|I_{0^+}^{\alpha} f\|_{L^1}.$$

Lemma 2.4 ([14])**.** *The fractional integral $I_{0^+}^{\alpha}$, $\alpha > 0$ is bounded in $L^1(a,b)$ with*

$$\|I_{0^+}^{\alpha} f\|_{L^1(a,b)} \leq \frac{(b-a)^\alpha}{\Gamma(\alpha+1)} \|f\|_{L^1(a,b)}.$$

Let us now introduce the space $E = \big\{ y : y \in C([0,1], \mathbb{R}) : {}^{c}D_{0^+}^{\beta} y \in C([0,1], \mathbb{R}) \big\}$. equipped with the norm

$$\|y\|_E = \|y\|_\infty + \|^{c}D_{0^+}^{\beta} y\|_\infty = \sup_{t \in J} |y(t)| + \sup_{t \in J} |^{c}D_{0^+}^{\beta} y(t)|.$$

Clearly, $(E, \|.\|_E)$ is a Banach space [20].

Theorem 2.1. (*Krasnselskii's fixed point theorem*). *Let U be a closed convex, bounded and nonempty subset of a Banach space E. Let $\mathcal{A}, \mathcal{B}$ be the opertors such that*

- *$\mathcal{A}x + \mathcal{B}y \in U$ whenever $x, y \in U$;*
- *$\mathcal{A}$ is compact and continuous;*
- *$\mathcal{B}$ is contraction mapping.*

Then there exists $z \in U$ such that $z = \mathcal{A}z + \mathcal{B}z$.

Theorem 2.2. (*Schauder's fixed point theorem*). *Let (E, d) be a complete metric space, let U be a closed convex subset of E, and let $\mathcal{A} : U \longrightarrow U$ be a mapping such that the set $\{\mathcal{A}u : u \in U\}$ is relatively compact in E. Then $\mathcal{A}$ has at least one fixed point.*

3. Main Results

Before starting and proving our main result we introduce the following auxiliary lemma.

Lemma 3.1. *Let $2 < \alpha \leq 3$ and h be continuous function on $J := [0, 1]$. Then the linear problem*

$$ {}^{c}D^{\alpha}_{0^+} y(t) = h(t), \tag{3.1} $$

with boundary conditions

$$ y(0) = y_0, \;\; y'(0) = a I^{\sigma_1}_{0^+} y(\eta_1), \;\; {}^{c}D^{\beta_1}_{0^+} y(1) = b I^{\sigma_2}_{0^+} y(\eta_2), \tag{3.2} $$

is equivalent to the fractional integral equation

$$ \begin{aligned} y(t) = {} & I^{\alpha}_{0^+} h(t) + (v_6 t^2 - v_1 t) I^{\alpha+\sigma_1}_{0^+} h(\eta_1) + (v_2 t - v_5 t^2) I^{\alpha+\sigma_2}_{0^+} h(\eta_2) \\ & + (v_8 t^2 - v_4 t) I^{\alpha-\beta_1}_{0^+} h(1) + (1 - v_3 t - v_7 t^2) y_0. \end{aligned} \tag{3.3} $$

Here

$$(3.4)\quad \begin{aligned}
\Delta &= \left[\left(\frac{a\eta_1^{\sigma_1+1}}{\Gamma(\sigma_1+2)} - 1\right)\left(\frac{b\eta_2^{\sigma_2+2}}{\Gamma(\sigma_2+3)} - \frac{1}{\Gamma(3-\beta_1)}\right)\right. \\
&\quad \left. - \left(\frac{b\eta_2^{\sigma_2+1}}{\Gamma(\sigma_2+2)} - \frac{1}{\Gamma(2-\beta_1)}\right)\left(\frac{a\eta_1^{\sigma_1+2}}{\Gamma(\sigma_1+3)}\right)\right] \neq 0, \\
v_1 &= \frac{a}{\Delta}\left(\frac{b\eta_2^{\sigma_2+2}}{\Gamma(\sigma_2+3)} - \frac{1}{\Gamma(3-\beta_1)}\right),\ v_2 = \frac{ab}{\Delta}\left(\frac{\eta_1^{\sigma_1+2}}{\Gamma(\sigma_1+3)}\right), \\
v_3 &= \frac{\eta_1^{\sigma_1}v_1}{\Gamma(\sigma_1+1)} - \frac{\eta_2^{\sigma_2}v_2}{\Gamma(\sigma_2+1)},\ v_4 = \frac{v_2}{b},\ v_5 = \frac{b}{2\Delta}\left(\frac{a\eta_1^{\sigma_1+1}}{\Gamma(\sigma_1+2)} - 1\right), \\
v_6 &= \frac{a}{2\Delta}\left(\frac{b\eta_2^{\sigma_2+1}}{\Gamma(\sigma_2+2)} - \frac{1}{\Gamma(2-\beta_1)}\right),\ v_7 = \frac{\eta_2^{\sigma_2}v_5}{\Gamma(\sigma_2+1)} - \frac{\eta_1^{\sigma_1}v_6}{\Gamma(\sigma_1+1)},
\end{aligned}$$

and

$$v_8 = \frac{v_5}{b}.$$

Proof. By applying Lemma (2.1), we may reduce (3.1) to an equivalent integral equation

$$(3.5)\qquad y(t) = I_{0^+}^{\alpha}h(t) - c_0 - c_1 t - c_2 t^2, \quad c_0, c_1, c_2 \in \mathbb{R}.$$

Applying the boundary conditions (3.2) in (3.5) we find that

$$c_0 = -y_0,$$

Using the boundary conditions of (3.2) in (3.5), we end up with a system of equations:

$$(P_0)\begin{cases}
\left(\frac{a\eta_1^{\sigma_1+1}}{\Gamma(\sigma_1+2)} - 1\right)c_1 + 2\left(\frac{a\eta_1^{\sigma_1+2}}{\Gamma(\sigma_1+3)}\right)c_2 = aI_{0^+}^{\alpha+\sigma_1}h(\eta_1) + \left(\frac{a\eta_1^{\sigma_1}}{\Gamma(\sigma_1+1)}\right)y_0. \\
\left(\frac{b\eta_2^{\sigma_2+1}}{\Gamma(\sigma_2+2)} - \frac{1}{\Gamma(2-\beta_1)}\right)c_1 + 2\left(\frac{b\eta_2^{\sigma_2+2}}{\Gamma(\sigma_2+3)} - \frac{1}{\Gamma(3-\beta_1)}\right)c_2 = bI_{0^+}^{\alpha+\sigma_2}h(\eta_2) + \left(\frac{b\eta_2^{\sigma_2}}{\Gamma(\sigma_2+1)}\right)y_0 \\
\qquad -I_{0^+}^{\alpha-\beta_1}h(1).
\end{cases}$$

Solving (P_0) together with notations (3.4), we find that

$$\begin{aligned}
c_1 &= v_1 I_{0^+}^{\alpha+\sigma_1}h(\eta_1) - v_2 I_{0^+}^{\alpha+\sigma_2}h(\eta_2) + v_3 y_0 + v_4 I_{0^+}^{\alpha-\beta_1}h(1), \\
c_2 &= v_5 I_{0^+}^{\alpha+\sigma_2}h(\eta_2) - v_6 I_{0^+}^{\alpha+\sigma_1}h(\eta_1) + v_7 y_0 - v_8 I_{0^+}^{\alpha-\beta_1}h(1),
\end{aligned}$$

Substituting the value of c_0, c_1, c_2 in (3.5) we get (3.3). □

In view of Lemma 3.1 we define the integral operator $\mathcal{T} : E \longrightarrow E$ by

$$\begin{aligned}\mathcal{T}y(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s, y(s), {}^cD^{\beta}_{0^+}y(s))ds \\ &+ \frac{v_6t^2 - v_1t}{\Gamma(\alpha+\sigma_1)}\int_0^{\eta_1} (\eta_1-s)^{\alpha+\sigma_1-1} f(s, y(s), {}^cD^{\beta}_{0^+}y(s))ds \\ &+ \frac{v_2t - v_5t^2}{\Gamma(\alpha+\sigma_2)}\int_0^{\eta_2} (\eta_2-s)^{\alpha+\sigma_2-1} f(s, y(s), {}^cD^{\beta}_{0^+}y(s))ds \\ &+ \frac{v_8t^2 - v_4t}{\Gamma(\alpha-\beta_1)}\int_0^{1} (1-s)^{\alpha-\beta_1-1} f(s, y(s), {}^cD^{\beta}_{0^+}y(s))ds \\ &+ (1 - v_3t - v_7t^2)y_0. \end{aligned} \tag{3.6}$$

In this section we shall present and prove our main results. First, consider the following hypotheses:

(H1) the function $f : J \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous

(H2)

$$|f(t, x_1, y_1) - f(t, x_2, y_2)| \leq g_1(t)|x_1 - x_2| + g_2(t)|y_1 - y_2|,$$

for all $x_1,\ x_2,\ y_1,\ y_2 \in \mathbb{R}; g_1, g_2 \in L^1(J, \mathbb{R}_+)$ and $t \in J$.

In order to simplify the computations in the main results we set some notations:

$$M_1 = \frac{1}{\Gamma(\alpha)} + \frac{|v_6| + |v_1|}{\Gamma(\alpha+\sigma_1)} + \frac{|v_2| + |v_5|}{\Gamma(\alpha+\sigma_2)} + \frac{|v_8| + |v_4|}{\Gamma(\alpha-\beta_1)},$$

$$M_2 = \frac{1}{\Gamma(\alpha-\beta)} + \frac{2|v_6| + (2-\beta)|v_1|}{\Gamma(3-\beta)\Gamma(\alpha+\sigma_1)} + \frac{2|v_5| + (2-\beta)|v_2|}{\Gamma(3-\beta)\Gamma(\alpha+\sigma_2)} + \frac{2|v_8| + (2-\beta)|v_4|}{\Gamma(3-\beta)\Gamma(\alpha-\beta_1)},$$

$$N_1 = (1 + |v_3| + |v_7|), \quad N_2 = \frac{2|v_7| + (2-\beta)|v_3|}{\Gamma(3-\beta)}.$$

Now we are ready to establish the main results.

Our first results is based on the Banach contraction principle.

Theorem 3.1. *Assume that $(H_1), (H_2)$ true and the following condition*

$$\gamma_1 = (M_1 + M_2)(\|g_1\|_{L^1} + \|g_2\|_{L^1}) < 1. \tag{3.7}$$

true. Then the problem (1.2) *has a unique solution on* J.

Proof. Transform the problem (1.2) into a fixed point problem. Clearly, the fixed points of the operator $\mathcal{T}$ defined by (3.6) are solutions of the problem (1.2).
Let $x, y \in E$ and $t \in J$, using (3.3) we can write

$$\begin{aligned}\mathcal{T}(x)(t) - \mathcal{T}(y)(t) &= I^{\alpha}_{0^+}(f(t, x(t), {}^cD^{\beta}_{0^+}x(t)) - f(t, y(t), {}^cD^{\beta}_{0^+}y(t)))\\ &+ (v_6t^2 - v_1t)I^{\alpha+\sigma_1}_{0^+}(f(\eta_1, x(\eta_1), {}^cD^{\beta}_{0^+}x(\eta_1)) - f(\eta_1, y(\eta_1), {}^cD^{\beta}_{0^+}y(\eta)))\\ &+ (v_2t - v_5t^2)I^{\alpha+\sigma_2}_{0^+}(f(\eta_2, x(\eta_2), {}^cD^{\beta}_{0^+}x(\eta_2)) - f(\eta_2, y(\eta_2), {}^cD^{\beta}_{0^+}y(\eta_2)))\\ &+ (v_8t^2 - v_4t)I^{\alpha-\beta_1}_{0^+}(f(1, x(1), {}^cD^{\beta}_{0^+}x(1)) - f(1, y(1), {}^cD^{\beta}_{0^+}y(1))).\end{aligned}$$

By (H2), we can find that

$$\begin{aligned}|\mathcal{T}(x)(t) - \mathcal{T}(y)(t)| \leq \|x - y\|_{\infty} &\left\{I^{\alpha}_{0^+}g_1(t) + (|v_6| + |v_1|)I^{\alpha+\sigma_1}_{0^+}g_1(\eta_1)\right.\\ &\left.+(|v_2| + |v_5|)I^{\alpha+\sigma_2}_{0^+}g_1(\eta_2) + (|v_8| + |v_4|)I^{\alpha-\beta_1}_{0^+}g_1(1)\right\}\\ &+ \|{}^cD^{\beta}_{0^+}x - {}^cD^{\beta}_{0^+}y\|_{\infty}\left\{I^{\alpha}_{0^+}g_2(t) + (|v_6| + |v_1|)I^{\alpha+\sigma_1}_{0^+}g_2(\eta_1)\right.\\ &\left.+ (|v_2| + |v_5|)I^{\alpha+\sigma_2}_{0^+}g_2(\eta_2) + (|v_8| + |v_4|)I^{\alpha-\beta_1}_{0^+}g_2(1)\right\}.\end{aligned}$$

According to the lemmas 2.3 and 2.4, we have

$$\begin{aligned}\|\mathcal{T}x - \mathcal{T}y\|_{\infty} \leq \|x - y\|_{\infty}&\left[\frac{1}{\Gamma(\alpha)} + \frac{|v_6| + |v_1|}{\Gamma(\alpha + \sigma_1)} + \frac{|v_2| + |v_5|}{\Gamma(\alpha + \sigma_2)} + \frac{|v_8| + |v_4|}{\Gamma(\alpha - \beta_1)}\right]\|g_1\|_{L^1}\\ + \|{}^cD^{\beta}_{0^+}x - {}^cD^{\beta}_{0^+}y\|_{\infty}&\left[\frac{1}{\Gamma(\alpha)} + \frac{|v_6| + |v_1|}{\Gamma(\alpha + \sigma_1)} + \frac{|v_2| + |v_5|}{\Gamma(\alpha + \sigma_2)} + \frac{|v_8| + |v_4|}{\Gamma(\alpha - \beta_1)}\right]\|g_2\|_{L^1}.\end{aligned}$$

Thus,

$$\|\mathcal{T}(x) - \mathcal{T}(y)\|_{\infty} \leq M_1(\|g_1\|_{L^1} + \|g_2\|_{L^1})\|x - y\|_E. \tag{3.8}$$

And

$$\begin{aligned}
{}^c\mathcal{D}^{\beta}_{0^+}\mathcal{T}y(t) &= I^{\alpha-\beta}_{0^+} f(t, y(t), {}^c\mathcal{D}^{\beta}_{0^+}y(t)) \\
&+ \frac{2v_6 t^{2-\beta} + (2-\beta)v_1 t^{1-\beta}}{\Gamma(3-\beta)} I^{\alpha+\sigma_1}_{0^+} f(\eta_1, y(\eta_1), {}^c\mathcal{D}^{\beta}_{0^+}y(\eta_1)) \\
&+ \frac{2v_5 t^{2-\beta} + (2-\beta)v_2 t^{1-\beta}}{\Gamma(3-\beta)} I^{\alpha+\sigma_2}_{0^+} f(\eta_2, y(\eta_2), {}^c\mathcal{D}^{\beta}_{0^+}y(\eta_2)) \\
&+ \frac{2v_8 t^{2-\beta} + (2-\beta)v_4 t^{1-\beta}}{\Gamma(3-\beta)} I^{\alpha-\beta_1}_{0^+} f(1, y(1), {}^c\mathcal{D}^{\beta}_{0^+}y(1)) \\
&+ \frac{2v_7 t^{2-\beta} + (2-\beta)v_3 t^{1-\beta}}{\Gamma(3-\beta)} y_0.
\end{aligned}$$

Using similar techniques applied to get (3.8), we have

$$\|{}^c\mathcal{D}^{\beta}_{0^+}\mathcal{T}x - {}^c\mathcal{D}^{\beta}_{0^+}\mathcal{T}y\|_\infty \le M_2(\|g_1\|_{L^1} + \|g_2\|_{L^1})\|x-y\|_E. \tag{3.9}$$

From (3.8), (3.9) we obtain

$$\|\mathcal{T}x - \mathcal{T}y\|_E \le (M_1 + M_2)(\|g_1\|_{L^1} + \|g_2\|_{L^1})\|x-y\|_E.$$

So we have

$$\|\mathcal{T}x - \mathcal{T}y\|_E \le \gamma_1\|x-y\|_E.$$

By (3.7), we deduce that $\mathcal{T}$ is a contraction. As a consequence of Banach fixed-point theorem, we deduce that $\mathcal{T}$ has a fixed point which is the unique solution of the problem (1.2). □

Our next result is upon the Schauder's fixed point theorem.

Theorem 3.2. *Assume that (H1) true, and there exists a function $\psi \in L^1(J, \mathbb{R}_+)$ such that*

$$|f(t, x, y)| \le \psi(t) + a_1|x|^{\tau_1} + a_2|y|^{\tau_2}, x, y \in \mathbb{R}. \tag{3.10}$$

where $0 < \tau_i < 1$, *and* $a_i \geq 0$ *for* $i = 1, 2, 3$. *Also assume that there exists a number* $r > 0$ *such that*

$$r \geq \max\Big\{4(M_1 + M_2)\|\psi\|_{L^1},\ (4(M_1 + M_2)a_1)^{\frac{1}{1-\tau_1}},$$
$$(4(M_1 + M_2)a_2)^{\frac{1}{1-\tau_2}},\ 4(N_1 + N_2)|y_0|\Big\}. \tag{3.11}$$

Then the problem (1.2) *has at least one solution on* J.

Proof. In order to use the Schauder fixed-point theorem to prove our main result, we define a subset B_r of E defined by

$$B_r = \{y \in E : \|y\|_E \leq r\},$$

where r satisfies inequality (3.11). Notice that B_r is closed, convex and bounded subset of the Banach space E.
Now we prove that $\mathcal{T} : B_r \longrightarrow B_r$. For any $y \in B_r$, we have

$$\begin{aligned}
|\mathcal{T}y(t)| &\leq I^{\alpha}_{0+}|f(s, y(s), {}^c\mathcal{D}^{\beta}_{0+}y(s))| \\
&\quad + (|v_6| + |v_1|)I^{\alpha+\sigma_1}_{0+}|f(\eta_1, y(\eta_1), {}^c\mathcal{D}^{\beta}_{0+}y(\eta_1))| \\
&\quad + (|v_2| + |v_5|)I^{\alpha+\sigma_2}_{0+}|f(\eta_2, y(\eta_2), {}^c\mathcal{D}^{\beta}_{0+}y(\eta_2))| \\
&\quad + (|v_8| + |v_4|)I^{\alpha-\beta_1}_{0+}|f(1, y(1), {}^c\mathcal{D}^{\beta}_{0+}y(1))| \\
&\quad + (1 + |v_3| + |v_7|)|y_0|.
\end{aligned}$$

Using hypothese (3.10) we get

$$
\begin{aligned}
|\mathcal{T}y(t)| \leq\ & I_{0^+}^{\alpha}\psi(t) + (|v_6| + |v_1|)I_{0^+}^{\alpha+\sigma_1}\psi(\eta_1) \\
& + (|v_2| + |v_5|)I_{0^+}^{\alpha+\sigma_2}\psi(\eta_2) + (|v_8| + |v_4|)I_{0^+}^{\alpha-\beta_1}\psi(1) \\
& + a_1 \left(I_{0^+}^{\alpha}|y(t)|^{\tau_1} + (|v_6| + |v_1|)I_{0^+}^{\alpha+\sigma_1}|y(\eta_1)|^{\tau_1}\right. \\
& \left.+(|v_2| + |v_5|)I_{0^+}^{\alpha+\sigma_2}|y(\eta_2)|^{\tau_1} + (|v_8| + |v_4|)I_{0^+}^{\alpha-\beta_1}|y(1)|^{\tau_1}\right) \\
& + a_2 \left(I_{0^+}^{\alpha}|{}^cD_{0^+}^{\beta}y(t)|^{\tau_2} + (|v_6| + |v_1|)I_{0^+}^{\alpha+\sigma_1}|{}^cD_{0^+}^{\beta}y(\eta_1)|^{\tau_2}\right. \\
& \left.+(|v_2| + |v_5|)I_{0^+}^{\alpha+\sigma_2}|{}^cD_{0^+}^{\beta}y(\eta_2)|^{\tau_2} + (|v_8| + |v_4|)I_{0^+}^{\alpha-\beta_1}|{}^cD_{0^+}^{\beta}y(1)|^{\tau_2}\right) \\
& + (1 + |v_3| + |v_7|)|y_0| \\
& \leq M_1\|\psi\|_{L^1} + (a_1 r^{\tau_1} + a_2 r^{\tau_2})M_1 + N_1.
\end{aligned}
$$

Hence

$$
\|\mathcal{T}y\|_{\infty} \leq M_1\|\psi\|_{L^1} + (a_1 r^{\tau_1} + a_2 r^{\tau_2})M_1 + N_1. \tag{3.12}
$$

And

$$
\|{}^cD_{0^+}^{\beta}\mathcal{T}y\|_{\infty} \leq M_2\|\psi\|_{L^1} + M_2(a_1 r^{\tau_1} + a_2 r^{\tau_2}) + N_2. \tag{3.13}
$$

From (3.12), (3.13) we get

$$
\|\mathcal{T}y\|_E \leq (M_1 + M_2)(\|\psi\|_{L^1} + a_1 r^{\tau_1} + a_2 r^{\tau_2}) + (N_1 + N_2) \leq \frac{r}{4} + \frac{r}{4} + \frac{r}{4} + \frac{r}{4} \leq r,
$$

By the condition (3.11), we deduce that

$$
\|\mathcal{T}y\|_E \leq r,
$$

thus, $\mathcal{T}(B_r) \subset B_r$.

In view of the continuity of f, it is easy to verify that $\mathcal{T}$ is continuous.

Next, we show that the families $\mathcal{T}(B_r)$ and ${}^cD^{\beta}_{0+}\mathcal{T}(B_r)$ are equicontinuous. Since f is continuous, we can assume that $|f(t, y(t), {}^cD^{\beta}_{0+}y(t))| \leq M$ for any $y \in B_r$ and $t \in [0, 1]$.

Now, for $0 \leq t_1 < t_2 \leq 1$, we have

$$
\begin{aligned}
|\mathcal{T}(y)(t_2) - \mathcal{T}(y)(t_1)| \leq{}& \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \left[(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}\right] |f(s, y(s), {}^cD^{\beta}_{0+}y(s))|ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} |f(s, y(s), {}^cD^{\beta}_{0+}y(s))|ds \\
&+ \frac{(|v_6|(t_2^2 - t_1^2) + |v_1|(t_2 - t_1))}{\Gamma(\alpha + \sigma_1)} \int_0^{\eta_1} (\eta_1 - s)^{\alpha+\sigma_1-1} |f(s, y(s), {}^cD^{\beta}_{0+}y(s))|ds \\
&+ \frac{(|v_2|(t_2 - t_1) + |v_5|(t_2^2 - t_1^2))}{\Gamma(\alpha + \sigma_2)} \int_0^{\eta_2} (\eta_2 - s)^{\alpha+\sigma_2-1} |f(s, y(s), {}^cD^{\beta}_{0+}y(s))|ds \\
&+ \frac{(|v_8|(t_2^2 - t_1^2) + |v_4|(t_2 - t_1))}{\Gamma(\alpha - \beta_1)} \int_0^{1} (1 - s)^{\alpha-\beta_1-1} |f(s, y(s), {}^cD^{\beta}_{0+}y(s))|ds \\
&+ [|v_3|(t_2 - t_1)|v_7|(t_2^2 - t_1^2)]|y_0| \\
\leq{}& M \left\{ \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \left[(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}\right] ds + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} ds \right. \\
&+ \frac{(|v_6|(t_2^2 - t_1^2) + |v_1|(t_2 - t_1))}{\Gamma(\alpha + \sigma_1)} \int_0^{\eta_1} (\eta_1 - s)^{\alpha+\sigma_1-1} ds \\
&+ \frac{(|v_2|(t_2 - t_1) + |v_5|(t_2^2 - t_1^2))}{\Gamma(\alpha + \sigma_2)} \int_0^{\eta_2} (\eta_2 - s)^{\alpha+\sigma_2-1} ds \\
&\left. + \frac{(|v_8|(t_2^2 - t_1^2) + |v_4|(t_2 - t_1))}{\Gamma(\alpha - \beta_1)} \int_0^{1} (1 - s)^{\alpha-\beta_1-1} ds \right\} \\
&+ [|v_3|(t_2 - t_1) + |v_7|(t_2^2 - t_1^2)]|y_0|.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
|\mathcal{T}(y)(t_2) - \mathcal{T}(y)(t_1)| \leq{}& M \left[\frac{t_2^{\alpha} - t_1^{\alpha}}{\Gamma(\alpha + 1)} + \frac{|v_6||t_2^2 - t_1^2| + |v_1||t_2 - t_1|}{\Gamma(\alpha + \sigma_1 + 1)} \eta_1^{\alpha+\sigma_1} \right. \\
&\left. \frac{|v_2||t_2^2 - t_1^2| + |v_5||t_2 - t_1|}{\Gamma(\alpha + \sigma_2 + 1)} \eta_2^{\alpha+\sigma_2} + \frac{|v_8||t_2^2 - t_1^2| + |v_4||t_2 - t_1|}{\Gamma(\alpha - \beta_1 + 1)} \right] \\
&+ [|v_3|(t_2 - t_1) + |v_7|(t_2^2 - t_1^2)]|y_0| \qquad (3.14)
\end{aligned}
$$

And

$$|{}^cD^{\beta}_{0^+}\mathcal{T}y(t_2) - {}^cD^{\beta}_{0^+}\mathcal{T}y(t_1)| \leq M\left\{\frac{1}{\Gamma(\alpha-\beta)}\int_0^{t_1}\left[(t_2-s)^{\alpha-\beta-1}-(t_1-s)^{\alpha-\beta-1}\right]ds\right.$$
$$+\frac{1}{\Gamma(\alpha-\beta)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-\beta-1}ds$$
$$+\frac{2|v_6||t_2^{2-\beta}-t_1^{2-\beta}|+(2-\beta)|v_1||t_2^{1-\beta}-t_1^{1-\beta}|}{\Gamma(3-\beta)\Gamma(\alpha+\sigma_1)}\int_0^{\eta_1}(\eta_1-s)^{\alpha+\sigma_1-1}ds$$
$$+\frac{2|v_5||t_2^{2-\beta}-t_1^{2-\beta}|+(2-\beta)|v_2||t_2^{1-\beta}-t_1^{1-\beta}|}{\Gamma(3-\beta)\Gamma(\alpha+\sigma_2)}\int_0^{\eta_2}(\eta_2-s)^{\alpha+\sigma_2-1}ds$$
$$\left.+\frac{2|v_8||t_2^{2-\beta}-t_1^{2-\beta}|+(2-\beta)|v_4||t_2^{1-\beta}-t_1^{1-\beta}|}{\Gamma(3-\beta)\Gamma(\alpha-\beta_1)}\int_0^{1}(1-s)^{\alpha-\beta_1-1}ds\right\}.$$

Thus,

$$|{}^cD^{\beta}_{0^+}Ty(t_2) - {}^cD^{\beta}_{0^+}Ty(t_1)| \leq M\left\{\frac{t_2^{\alpha-\beta}-t_1^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right.$$
$$+\frac{2|v_6||t_2^{2-\beta}-t_1^{2-\beta}|+(2-\beta)|v_1||t_2^{1-\beta}-t_1^{1-\beta}|}{\Gamma(3-\beta)\Gamma(\alpha+\sigma_1+1)}\eta_1^{\alpha+\sigma_1}$$
$$+\frac{2|v_5||t_2^{2-\beta}-t_1^{2-\beta}|+(2-\beta)|v_2||t_2^{1-\beta}-t_1^{1-\beta}|}{\Gamma(3-\beta)\Gamma(\alpha+\sigma_2+1)}\eta_2^{\alpha+\sigma_2}$$
$$\left.+\frac{2|v_8||t_2^{2-\beta}-t_1^{2-\beta}|+(2-\beta)|v_4||t_2^{1-\beta}-t_1^{1-\beta}|}{\Gamma(3-\beta)\Gamma(\alpha-\beta_1+1)}\right\}$$
$$+\frac{2|v_7||t_2^{2-\beta}-t_1^{2-\beta}|+(2-\beta)|v_3||t_2^{1-\beta}-t_1^{1-\beta}|}{\Gamma(3-\beta)} \tag{3.15}$$

The right-hand sides of inequality (3.14) and (3.15) tends to zero when $t_1 \to t_2$ independently of y, so $\mathcal{T}$ is compact as consequence of the Arzelá–Ascoli theorem, and $\mathcal{T}$ is continuous. We claim that $\mathcal{T}$ is completely continuous.
As a consequence of Schauder's fixed-point theorem, we deduce that $\mathcal{T}$ has a fixed point. We claim that the problem (1.2) has at least one solution on J. □

For $a_1 = a_2 = 0$, Theorem 3.2 takes the following form.

Corollary 3.1. *Let (H1) hold. In addition, the function f satisfies the assumptions:*

$$|f(t,x,y)| \leq \psi(t),$$

$\forall (t,x,y) \in J \times \mathbb{R} \times \mathbb{R}$ and $\psi \in L^1(J,\mathbb{R}_+)$. Then the boundary value problem (1.2) *has at least one solution on J.*

For $\tau_1 = \tau_2 = 1$, Theorem 3.2 takes the following form.

Corollary 3.2. *Let (H1) hold. In addition, the function f satisfies the assumptions:*

$$|f(t,x,y)| \leq \psi(t) + a_1|x| + a_2|y|,$$

$\forall (t,x,y) \in J \times \mathbb{R} \times \mathbb{R}$, where $\psi \in L^1(J,\mathbb{R}_+)$. Then the boundary value problem (1.2) *has at least one solution on J.*

Our key tool in the next existence result is based on the Krasnoselskii's fixed point theorem.

Theorem 3.3. *Assume that (H1) and (H2) hold, furthermore there exists non-negative function $\phi \in L^1(J,\mathbb{R}_+)$ such that :*

$$|f(t,x,y)| \leq \phi(t), \tag{3.16}$$

for all $x, y \in \mathbb{R}$ and $t \in J$.

Then the problem (1.2), has at least one solution on J, provided that $\gamma_1 < 1$, where γ_1 given in (3.7).

Proof. Let $r > 0$, such that

$$r \geq (M_1 + M_2)\|\phi\|_{L^1} + (N_1 + N_2)|y_0|, \tag{3.17}$$

and consider the ball

$$B_r = \big\{ y \in E : \|y\|_E \leq r \big\}.$$

We define the operators $\mathcal{A}$ and $\mathcal{B}$ on B_r by:

$$\begin{aligned}\mathcal{A}y(t) &= I_{0^+}^{\alpha} f(t, y(t), {}^c\!D_{0^+}^{\beta} y(t)),\\ \mathcal{B}y(t) &= (v_6 t^2 - v_1 t) I_{0^+}^{\alpha+\sigma_1} f(\eta_1, y(\eta_1), {}^c\!D_{0^+}^{\beta} y(\eta_1))\\ &+ (v_2 t - v_5 t^2) I_{0^+}^{\alpha+\sigma_2} f(\eta_2, y(\eta_2), {}^c\!D_{0^+}^{\beta} y(\eta_2))\\ &+ (v_8 t^2 - v_4 t) I_{0^+}^{\alpha-\beta_1} f(1, y(1), {}^c\!D_{0^+}^{\beta} y(1))\\ &+ (1 - v_3 t - v_7 t^2) y_0.\end{aligned}$$

Then the fractional integral Equation (3.6) can be written as the operator equation:

$$\mathcal{T}y(t) = \mathcal{A}y(t) + \mathcal{A}y(t), \quad y \in B_r$$

We will divide the proof into three steps.

Step 1 : $\mathcal{A}x + \mathcal{B}y \in B_r$, for any $x, y \in B_r$ and $t \in J$. By the same way of the proof of Theorem 3.2, we can easily show that

$$\|\mathcal{A}x + \mathcal{B}y\|_E \leq (M_1 + M_2)\|\phi\|_{L^1} + (N_1 + N_2)y_0.$$

Using (3.17) we conclude that $\mathcal{A}x + \mathcal{B}y \in B_r$.

Step 2 : $\mathcal{A}$ is compact and continuous. We have verified that $\mathcal{A} : B_r \to E$ is continuous and compact under its definition, a detailed proof is given in Appendix.

Step 3 : $\mathcal{B}$ is a contraction mapping on B_r. Let $x, y \in B_r$ and $t \in J$ by using the same arguments in Theorem 3.1 we can show that

$$\begin{aligned}\|\mathcal{B}(x) - \mathcal{B}(y)\|_E &\leq (\|g_1\|_{L^1} + |g_2\|_{L^1}) \left(M_1 - \frac{1}{\Gamma(\alpha)} + M_2 - \frac{1}{\Gamma(\alpha-\beta)} \right) \|x - y\|_E,\\ &\leq (M_1 + M_2)(\|g_1\|_{L^1} + \|g_2\|_{L^1})\|x - y\|_E,\\ &\leq \gamma_1 \|x - y\|_E.\end{aligned}$$

By (3.7) , we deduce that $\mathcal{B}$ is a contraction.

As a consequence of Krasnoselskii's fixed point theorem, we conclude that the operator $\mathcal{T}$ defined by (3.6) has at least one fixed point $u \in B_r$, which is just the solution of the boundary value problem (1.2). This completes the proof of Theorem. 3.3. $\square$

4. Examples

In this section, in order to illustrate our results, we consider two examples.

Example 4.1. *Consider the following fractional differential problem:*

$$(4.1)\qquad \begin{cases} {}^cD_{0^+}^{\frac{11}{4}} y(t) = f(t, y(t), {}^cD_{0^+}^{\frac{1}{8}} y(t)) \quad \forall t \in J = [0,1] \\ y(0) = 1, y'(0) = \frac{1}{100} I_{0^+}^{\frac{3}{2}} y(\frac{1}{2}), {}^cD_{0^+}^{\frac{7}{8}} y(1) = \frac{1}{10} I_{0^+}^{\frac{5}{2}} y(\frac{3}{4}) \end{cases}$$

In this case we take $f(t, y(t), {}^cD_{0^+}^{\frac{1}{8}} y(t)) = \frac{t^4}{4!} y(t) + \frac{(1-t)^2}{10} {}^cD_{0^+}^{\frac{1}{8}} y(t) + e^{-t}$,

$$\alpha = \frac{11}{4}, \beta = \frac{1}{8}, \beta_1 = \frac{7}{8}, y_0 = 1, a = \frac{1}{100}, b = \frac{1}{10}, \sigma_1 = \frac{3}{2}, \sigma_2 = \frac{5}{2}, \eta_1 = \frac{1}{2}, \eta_2 = \frac{3}{4}.$$

Let $t \in J, x, x_1, y, y_1 \in \mathbb{R}$

$$|f(t,x,y) - f(t,x_1,y_1)| = \frac{t^4}{4!}|x - x_1| + \frac{(1-t)^2}{10}|y - y_1|$$

Hence the condition (H2) holds with

$$g_1(t) = \frac{t^4}{4!}, \quad g_2(t) = \frac{(1-t)^2}{10}$$

using the Matlab program, we can find $\gamma_1 = (M_1 + M_2)(\|g_1\|_{L^1} + \|g_2\|_{L^1}) = \frac{83}{799} < 1$ *Thus, the assumptions of (Theorem 3.1) hold so the problem* (4.1) *has a unique solution on J.*

Example 4.2. *Let us consider the following fractional boundary value problem:*

$$(4.2)\qquad \begin{cases} {}^cD_{0^+}^{\frac{11}{5}} y(t) = (t - 0.25)^2 y(t)^{\frac{1}{3}} + e^{-t^2} \sin\left(\frac{({}^cD_{0^+}^{\frac{1}{2}} y(t))^{\frac{1}{6}}}{2}\right) \quad \forall t \in J = [0,1] \\ y(0) = \sqrt{3}, y'(0) = \frac{1}{2} I_{0^+}^{\frac{8}{5}} y(\frac{1}{4}), {}^cD_{0^+}^{\frac{1}{7}}(1) = \frac{1}{4} I_{0^+}^{\frac{7}{3}} y(\frac{3}{4}) \end{cases}$$

Let $t \in J, x, y \in \mathbb{R}$

$$|f(t,x,y)| \leq \frac{9}{16}|x|^{\frac{1}{3}} + \frac{|x|^{\frac{1}{6}}}{2}$$

Where $\psi(t) = 0, a_1 = \frac{9}{16}, a_2 = \frac{1}{2}, \tau_1 = \frac{1}{3}, \tau_2 = \frac{1}{6}$

$$\alpha = \frac{11}{5}, \beta = \frac{1}{2}, \beta_1 = \frac{1}{7}, y_0 = \sqrt{3}, a = \frac{1}{2}, b = \frac{1}{4}, \eta_1 = \frac{1}{4}, \eta_2 = \frac{3}{4}, \sigma_1 = \frac{8}{5}, \sigma_2 = \frac{7}{3},$$

the hypothesis of Theorem (3.2) holds wich implies that the problem (4.2) has a solution

5. Conclusion

We have studied a nonlinear fractional differential equation with nonlinearity depending on the unknown function together with its lower-order fractional derivative, equipped with fractional integral and Caputo fractional derivatives in boundary conditions. Several existence and uniqueness results have been derived by applying different tools of the fixed point theory. We also provide examples to make our results clear. For future research directions, it is possible to consider the existence and uniqueness of solutions for systems of fractional differential equations.

Appendix A. A detailed proof of both the compactness and the continuity of the operator $\mathcal{A}$

The operator $\mathcal{A} : B_r \to E$ defined by

$$\mathcal{A}x(t) = I^{\alpha}_{0^+} f(t, x(t), {}^c\mathcal{D}^{\beta}_{0^+} x(t)).$$

is continuous and compact.

The continuity of $\mathcal{A}$ follows from the continuity of f. Next we prove that $\mathcal{A}$ is uniformly bounded on B_r

Let any $x \in B_r$. Then for each $t \in J$ and by (3.16) we we have:

$$|\mathcal{A}x(t)| \leq I^{\alpha}_{0^+}|f(s, x(s), {}^c\mathcal{D}^{\beta}_{0^+} x(s))| \leq I^{\alpha}_{0^+}\phi(t) \leq \frac{\|\phi\|_{L^1}}{\Gamma(\alpha)}.$$

And

$$|{}^cD^{\beta}_{0^+}\mathcal{A}x(t)| \leq I^{\alpha-\beta}_{0^+}|f(s,x(s),{}^cD^{\beta}_{0^+}x(s))| \leq I^{\alpha-\beta}_{0^+}\phi(t) \leq \frac{\|\phi\|_{L^1}}{\Gamma(\alpha-\beta)}.$$

Hence

$$\|\mathcal{A}x\|_{\infty} \leq \frac{\|\phi\|_{L^1}}{\Gamma(\alpha)},$$

and

$$\|{}^cD^{\beta}_{0^+}\mathcal{A}x\|_{\infty} \leq \frac{\|\phi\|_{L^1}}{\Gamma(\alpha-\beta)}.$$

This implies that

$$\begin{aligned}\|\mathcal{A}x\|_E &= \|\mathcal{A}x\|_{\infty} + \|\mathcal{A}x\|_{\infty} \\ &\leq \frac{\Gamma(\alpha)+\Gamma(\alpha-\beta)}{\Gamma(\alpha)\Gamma(\alpha-\beta)}\|\phi\|_{L^1} < \infty.\end{aligned}$$

This proves that $\mathcal{A}$ is uniform bounded.

Next, we show that the families $\mathcal{A}(B_r)$ and ${}^cD^{\beta}_{0^+}\mathcal{A}(B_r)$ are equicontinuous.

Let $t_1, t_2 \in [0,1],\ t_1 < t_2,\ x \in B_r$, we have

$$\begin{aligned}|\mathcal{A}(x)(t_2) - \mathcal{A}(x)(t_1)| \leq &\frac{1}{\Gamma(\alpha)}\int_0^{t_1}\left[(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\right]|f(s,x(s),{}^CD^{\beta}_{0^+}y(s))|ds \\ &+ \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}|f(s,x(s),{}^CD^{\beta}_{0^+}x(s))|ds, \\ &\leq \frac{1}{\Gamma(\alpha)}\int_0^{t_1}\left[(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\right]\phi(s)ds \\ &+ \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\phi(s)ds,\end{aligned} \tag{A.1}$$

and

$$\begin{aligned}|{}^cD^{\beta}_{0^+}\mathcal{A}x(t_2) - {}^cD^{\beta}_{0^+}\mathcal{A}x(t_1)| \leq &\frac{1}{\Gamma(\alpha-\beta)}\int_0^{t_1}\left[(t_2-s)^{\alpha-\beta-1} - (t_1-s)^{\alpha-\beta-1}\right]\phi(s)ds \\ &+ \frac{1}{\Gamma(\alpha-\beta)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-\beta-1}\phi(s)ds.\end{aligned} \tag{A.2}$$

The right-hand sides of inequality (A.1) and (A.2) tends to zero when $t_1 \to t_2$ independently of $x \in B_r$. So $\mathcal{A}$ is relatively compact on B_r. As consequence of the Arzela-Ascoli theorem, $\mathcal{A}$ is compact on B_r.

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Author contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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