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EXACT BAHADUR SLOPE FOR COMBINING INDEPENDENT TESTS IN CASE OF LAPLACE DISTRIBUTION

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Abstract. Combining n independent tests of simple hypothesis, vs one-tailed alternative as n approaches infinity, in case of Laplace distribution $\mathbb{L}(\gamma, 1)$ is proposed. Four free-distribution "nonparametric" combination procedures namely; Fisher, logistic, sum of P-values and inverse normal were studied. Several comparisons among the four procedures using the exact Bahadur's slopes were obtained. Results showed that the sum of p-values procedure is better than all other procedures under the null hypothesis, and the inverse normal procedure is better than the other procedures under the alternative hypothesis.

1. Introduction

The combination of n independent tests of hypothesis is an important statistical practice. If H_0 is a simple hypothesis, Birnbaum [3] showed that, for given any nonparametric combination method with a monotone increasing acceptance region, there exists a problem for which this method is most powerful against some alternative. Littell and Folks [6] studied four methods of combining a finite number of independent tests. They found that the Fisher method is better than the inverse normal, the minimum of p-value method and maximum of p-vales via Bahadur efficiency. Later, Littell and Folks [7] showed under mild conditions that the Fisher's method is optimal

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among all methods for combining a finite number of independent tests. Al-Masri [1] studied six methods of combining independent tests. He showed under conditional shifted Exponential distribution that the inverse normal method is the best among six combination methods. Al-Talib, et. al. [2] considered combining independent tests in case of conditional normal distribution with probability density function $X|\theta \sim N(\gamma\theta)$, $\theta \in [a, \infty], a \geq 0$ when $\theta_1, \theta_2, ...$ have a distribution function (DF) F_θ. They concluded that the inverse normal procedure is better than the other procedures. The paper is organized as follows. The specific problem is given in Section 2. The basic definitions and preliminaries are given in Section 3. Section 4 is derivation of the EBS $\mathbb{L}(\gamma, 1)$.

2. The Specific Problem

Consider n hypotheses of the form: See [8]

$$H_0^{(i)} : \eta_i = \eta_0^i, \ vs \ , H_1^{(i)} : \eta_i \in \Omega_i - \{\eta_0^i\} \tag{2.1}$$

such that $H_0^{(i)}$ is rejected for large values, $i = 1, 2, ..., n$ of some continuous random variable $T^{(i)}$. The n hypotheses are combined into one as

$$H_0^{(i)} : (\eta_1, ..., \eta_n) = (\eta_0^1, ..., \eta_0^n), \ vs \ , H_1^{(i)} : (\eta_1, ..., \eta_n) \in \left\{ \prod_{i=1}^{n} \Omega_i - \{(\eta_0^1, ..., \eta_0^n)\} \right\} \tag{2.2}$$

For $i = 1, 2, \ldots, n$ the p-value of the i-th test is given by

$$P_i(t) = P_{H_0^{(i)}} \left(T^{(i)} > t \right) = 1 - F_{H_0^{(i)}} (t) \tag{2.3}$$

where $F_{H_0^{(i)}} (t)$ is the DF of $T^{(i)}$ under $H_0^{(i)}$. Note that $P_i \sim U(0, 1)$ under $H_0^{(i)}$.
If considering the special case where $\eta_i = \theta$ and $\eta_0^i = \theta_0$ for $i = 1, \ldots, n$, and also assume that $T^{(1)}, \ldots, T^{(n)}$ are independent, then (1) reduces to

$$H_0 : \theta = \theta_0, \ vs \ , H_1 : \theta \in \Omega - \{\theta_0\} \tag{2.4}$$

It follows that the p-values $P_1, \ldots, P_n$ are also independent identically distributed random variables that have a $U(0,1)$ distribution under H_0, and under H_1 have a distribution whose support is a subset of the interval $(0,1)$ and is not a $U(0,1)$ distribution. Therefore, if f is the probability density function (pdf) of P, then (4) is equivalent to

$$H_0 : P \sim U(0,1), \ vs \ , H_1 : P \sim f \tag{2.5}$$

where P has a pdf f with support a subset of the interval $(0,1)$.

This study considers the case: $\eta_i = 0, \ i = 1, \ldots, n$. Also we are assuming that $T^{(1)}, T^{(2)}, \ldots, T^{(n)}$ are independent. Then Eq. (4) reduced to

$$H_0 : \gamma = 0, \ vs \ , H_1 : \gamma > 0 \tag{2.6}$$

Thus, the p-values $P_1, P_2, \ldots, P_n$ are i.i.d. r.v.'s distributed with a uniform distribution $U(0,1)$ under H_0 which is given by (6).

We shall assume that the i-th problem in case of the normal distribution is based on $T_1^{(i)}, \ldots, T_{(n_i)}^{(i)}$ which are independent r.v.'s. By sufficiency we may assume $n_i = 1$ and $T^{(i)} = X_i$ for $i = 1, \ldots, n$. Then we consider the sequence $\left\{T^{(n)}\right\}$ of independent test statistics that is we will take a random sample $X_1, \ldots, X_n$ of size n and let $n \to \infty$ and compare the four non-parametric methods via EBS. Although X_i is not sufficient for θ_i under $H_0^{(i)}$ for the other distributions, but we will assume $n_i = 1$ and $T^{(i)} = X_i$ for $i = 1, \ldots, n$.

The following four combination tests: Fisher, logistic, inverse normal and the sum of P-values, that will be used in this paper:

$$\varphi_{Fisher} = \begin{cases} 1, & -2\sum_{i=1}^{n} \ln(P_i) > c \\ 0, & ow \end{cases}$$

$$\varphi_{logistic} = \begin{cases} 1, & -\sum_{i=1}^{n} \ln\left(\frac{P_i}{1-P_i}\right) > c \\ 0, & ow \end{cases}$$

$$\varphi_{Normal} = \begin{cases} 1, & -\sum_{i=1}^{n} \Phi^{-1}(P_i) > c \\ 0, & ow \end{cases}$$

$$\varphi_{Sum} = \begin{cases} 1, & -\sum_{i=1}^{n} P_i > c \\ 0, & ow, \end{cases}$$

where Φ is the cdf of standard normal distribution.

3. Definitions and Preliminaries

In this section we will state some definitions and preliminaries that will be used

Definition 3.1. *(Bahadur efficiency and exact Bahadur slope (EBS))* Let $X_1, \ldots, X_n$ be i.i.d. from a distribution with a probability density function $f(x, \theta)$, and we want to test $H_0 : \theta = \theta_0$ vs. $H_1 : \theta \in \Theta - \{\theta_0\}$. Let $\left\{T_n^{(1)}\right\}$ and $\left\{T_n^{(2)}\right\}$ be two sequences of test statistics for testing H_0. Let the significance attained by $T_n^{(i)}$ be $L_n^{(i)} = 1 - F_i\left(T_n^{(i)}\right)$, where $F_i\left(T_n^{(i)}\right) = P_{H_0}\left(T_n^{(i)} \leq t_i\right)$, $i = 1, 2$. Then there exists a positive valued function $C_i(\theta)$ called the exact Bahadur slope of the sequence $\{T_n^{(i)}\}$ such that

$$C_i(\theta) = \lim_{\theta \to \infty} -2n^{-1} \ln\left(L_n^i\right)$$

with probability 1 (w.p.1) under θ and the Bahadur efficiency of $\left\{T_n^{(1)}\right\}$ relative to $\left\{T_n^{(2)}\right\}$ is given by $e_B\left(T_1, T_2\right) = C_1(\theta)/C_2(\theta)$. See [8]

Theorem 3.1. (Large deviation theorem) *Let $X_1, X_2, \ldots, X_n$ be i.i.d., with distribution F and put $S_n = \sum_{i=1}^{n} X_i$. Assume existence of the moment generating function (mgf) $M(z) = E_F\left(e^{zX}\right)$, z real, and put $m(t) = \inf_z e^{-z(X-t)} = \inf_z e^{-zt} M(z)$. The behavior of large deviation probabilities $P\left(S_n \geq t_n\right)$, where $t_n \to \infty$ at rates slower than $O(n)$. The case $t_n = tn$, if $-\infty < t \leq EY$, then $P\left(S_n \leq nt\right) \leq [m(t)]^n$, the*

$$-2n^{-1} \ln P_F\left(S_n \geq nt\right) \to -2 \ln m(t) \ \ a.s. \ \ \left(F_\theta\right).$$

See [8]

Theorem 3.2. (Bahadur theorem) *Let $\{T_n\}$ be a sequence of test statistics which satisfies the following:*

(1) *Under $H_1 : \theta \in \Theta - \{\theta_0\}$:*

$$n^{-\frac{1}{2}} T_n \to b(\theta) \;\; a.s. \;\; (F_\theta)\,,$$

where $b(\theta) \in \mathbb{R}$.

(2) *There exists an open interval I containing $\{b(\theta) : \theta \in \Theta - \{\theta_0\}\}$, and a function g continuous on I, such that*

$$\lim_n -2n^{-1} \log \sup_{\theta \in \Theta_0} \left[1 - F_{\theta_n}(n^{\frac{1}{2}} t)\right] = \lim_n -2n^{-1} \log \left[1 - F_{\theta_n}(n^{\frac{1}{2}} t)\right] = g(t), \;\; t \in I.$$

If $\{T_n\}$ satisfied (1)-(2), then for $\theta \in \Theta - \{\theta_0\}$

$$-2n^{-1} \log \sup_{\theta \in \Theta_0} \left[1 - F_{\theta_n}(T_n)\right] \to C(\theta) \;\; a.s. \;\; (F_\theta)\,.$$

See [3]

Theorem 3.3. *Let $X_1, \ldots, X_n$ be i.i.d. with probability density function $f(x, \theta)$, and we want to test $H_0 : \theta = 0$ vs. $H_1 : \theta > 0$. For $j = 1, 2$, let $T_{n,j} = \sum_{i=1}^{n} f_i(x_i)/\sqrt{n}$ be a sequence of statistics such that H_0 will be rejected for large values of $T_{n,j}$ and let φ_j be the test based on $T_{n,j}$. Assume $\mathbb{E}_\theta(f_i(x)) > 0, \forall \theta \in \Theta$, $\mathbb{E}_0(f_i(x)) = 0$, $Var(f_i(x)) > 0$ for $j = 1, 2$. Then*

1. If the derivative $b'_j(0)$ is finite for $j = 1, 2$, then

$$\lim_{\theta \to 0} \frac{C_1(\theta)}{C_2(\theta)} = \frac{Var_{\theta=0}(f_2(x))}{Var_{\theta=0}(f_1(x))} \left[\frac{b'_1(0)}{b'_2(0)}\right]^2,$$

where $b_i(\theta) = \mathbb{E}_\theta(f_j(x))$, and $C_j(\theta)$ is the EBS of test φ_j at θ.

2. If the derivative $b'_j(0)$ is infinite for $j = 1, 2$, then

$$\lim_{\theta \to 0} \frac{C_1(\theta)}{C_2(\theta)} = \frac{Var_{\theta=0}(f_2(x))}{Var_{\theta=0}(f_1(x))} \left[\lim_{\theta \to 0} \frac{b'_1(\theta)}{b'_2(\theta)}\right]^2.$$

See [1]

Theorem 3.4. *If $T_n^{(1)}$ and $T_n^{(2)}$ are two test statistics for testing $H_0 : \theta = 0$ vs. $H_1 : \theta > 0$ with distribution functions $F_0^{(1)}$ and $F_0^{(2)}$ under H_0, respectively, and that $T_n^{(1)}$ is at least as powerful as $T_n^{(2)}$ at θ for any α, then if φ_j is the test based on $T_n^{(j)}$, $j = 1, 2$, then*

$$C_{\varphi_1}^{(1)}(\theta) \geq C_{\varphi_2}^{(2)}(\theta).$$

See [8]

Corollary 3.1. *If T_n is the uniformly most powerful test for all α, then it is the best via EBS. See [8]*

Theorem 3.5.

$$2t \leq m_S(t) \leq et, \ \forall : 0 \leq t \leq 0.5,$$

where

$$m_S(t) = \inf_{z>0} e^{-zt} \frac{e^z - 1}{z}.$$

See [1]

Theorem 3.6. (1) $m_L(t) \geq 2te^{-t}, \ \forall t \geq 0,$

(2) $m_L(t) \leq te^{1-t}, \ \forall t \geq 0.852,$

(3) $m_L(t) \leq t \left(\frac{t^2}{1+t^2}\right)^3 e^{1-t}, \ \forall t \geq 4,$

where $m_L(t) = \inf_{z\in(0,1)} e^{-zt}\pi z \ csc(\pi z)$ and csc is an abbreviation for cosecant function.

See [1]

Theorem 3.7. *For $x > 0$,*

$$\phi(x)\left[\frac{1}{x} - \frac{1}{x^3}\right] \leq 1 - \Phi(x) \leq \frac{\phi(x)}{x}.$$

Where ϕ is the pdf of standard normal distribution. See [1]

Theorem 3.8. *For $x > 0$,*

$$1 - \Phi(x) > \frac{\phi(x)}{x + \sqrt{\frac{\pi}{2}}}.$$

See [1]

Lemma 3.1. (1) $m_L(t) \geq \inf_{0<z<1} e^{-zt} = e^{-t}$

(2) $m_L(t) \leq \dfrac{e^{-t^2/(t+1)} \left(\frac{\pi t}{t+1}\right)}{sin\left(\frac{\pi t}{t+1}\right)}$

(3) $\begin{cases} m_s(t) = \inf_{z>0} \frac{e^{-zt}(1-e^{-z})}{z} \leq \inf_{z>0} \frac{e^{-zt}}{z} \leq -et, & t < 0 \\ m_s(t) \geq -2t, & -\frac{1}{2} \leq t \leq 0. \end{cases}$

(4) $\dfrac{x-1}{x} \leq \ln x \leq x - 1, \ x > 0$

See [1]

Theorem 3.9. *For any integrable function f and any η in the interior of Θ, the integral*

$$\int f(x) e^{\sum \eta_i T_i(x)} h(x) d\mu(x)$$

is continuous and has derivatives of all orders with respect to the η's, and these can be obtained by differentiating under the integral sign. See [5]

4. Derivation of The EBS For $\mathbb{L}(\gamma, 1)$

In this section we will study testing problem (6). We will compare the four methods viz. Fisher, logistic, sum of P-values and the inverse normal method via EBS.
Let $X_1, \ldots, X_n$ be i.i.d. with probability density function $\mathbb{L}(\gamma, 1)$, and we want to test (6). The P-value in this case is given by

$$P_n(X_n) = 1 - F^{H_0}(X_n) = 1 - F_0(x) = \tfrac{1}{2}\left\{1 - sgn(x)\left(1 - e^{-|x|}\right)\right\} \tag{4.1}$$

The next four lemmas give the EBS for Fisher (C_F), logistic (C_L), inverse normal (C_N), and sum of p-values (C_S) methods.

Lemma 4.1. *The exact Bahadur's slope (EBS's) result for the tests, which is given in Section 2, are as follows:*

B1. *Fisher method.* $C_F(\gamma) = b_F(\gamma) - 2\ln(b_F(\gamma)) + 2\ln(2) - 2,$ *where*

$$b_F(\gamma) = 2\cosh(\gamma) + \ln(4)\sinh(\gamma).$$

B2. *Logistic method.* $C_L(\gamma) = -2\ln(m(b_L(\gamma))),$ *where*

$$m_L(t) = \inf_{z\in(0,1)} e^{-zt}\pi z\ csc(\pi z)$$

and

$$b_L(\gamma) = \ln(4)\sinh[\gamma].$$

B3. *Sum of p-values method.* $C_S(\gamma) = -2\ln(m(b_S(\gamma))),$ *where*

$$m_S(t) = \inf_{z>0} e^{-zt}\frac{1-e^{-z}}{z}$$

and

$$b_S(\gamma) = \tfrac{1}{4}\left(\sinh(\gamma) - 2\cosh(\gamma)\right).$$

B4. *Inverse Normal method.* $C_N(\gamma) = -2\ln(m(b_N(\gamma))) = \frac{2}{\pi}\sinh^2(\gamma).$

Proof of B1.

$$T_F = -2\sum_{i=1}^{n}\frac{\ln\left[\frac{1}{2}\left\{1 - sgn(x)\left(1-e^{-|x|}\right)\right\}\right]}{\sqrt{n}}.$$

By the strong law of large number (SLLN)

$$\frac{T_F}{\sqrt{n}} \xrightarrow{\text{w.p.1}} b_F(\gamma) = 2\ln 2 - \mathbb{E}^{H_1}\ln\left\{1 - sgn(x)\left(1-e^{-|x|}\right)\right\}$$

then

$$b_F(\gamma) = 2\ln 2 - 2\int_{\mathbb{R}}\ln\left\{1 - sgn(x)\left(1-e^{-|x|}\right)\right\}\tfrac{1}{2}e^{-|x-\gamma|}\,dx = (1+\ln 2)e^{\gamma} - (\ln 2 - 1)e^{-\gamma} = 2\cosh(\gamma)$$

Now under H_0, then by Theorem 1, we have $m_S(t) = \inf_{z>0} e^{-zt} M_S(z)$, where $M_S(z) = \mathbb{E}_F(e^{zX})$. Under $H_0 : -\frac{1}{2}\left\{1 - sgn(x)\left(1 - e^{-|x|}\right)\right\} \sim U(-1, 0)$, so $M_S(z) = \frac{1-e^{-z}}{z}$, by part (2) of Theorem 2 we complete the proof, that is

$$C_F(\gamma) = -2\ln(m_F(b_F(\gamma))) = -2\ln\left(\frac{b_F(\gamma)}{2} e^{1-\frac{b_F(\gamma)}{2}}\right) = b_F(\gamma) - 2\ln(b_F(\gamma)) + 2\ln(2) - 2.$$

□

Proof of B3.

$$T_S = -\sum_{i=1}^{n} \frac{\frac{1}{2}\left\{1 - sgn(x)\left(1 - e^{-|x|}\right)\right\}}{\sqrt{n}}.$$

By the strong law of large number (SLLN)

$$\frac{T_S}{\sqrt{n}} \xrightarrow{\text{w.p.1}} b_S(\gamma) = -\,\mathbb{E}^{H_1}\left[\tfrac{1}{2}\left\{1 - sgn(x)\left(1 - e^{-|x|}\right)\right\}\right]$$

then

$$b_S(\gamma) = -\tfrac{1}{4}\int_{\mathbb{R}} \left\{1 - sgn(x)\left(1 - e^{-|x|}\right)\right\} e^{-|x-\gamma|}\, dx = -\tfrac{1}{8}\left(3e^{-\gamma} + e^{\gamma}\right) = \tfrac{1}{4}\left(\sinh(\gamma) - 2\cosh(\gamma)\right).$$

Now, by Theorem 1, we have $m_S(t) = \inf_{z>0} e^{-zt} M_S(z)$, where $M_S(z) = \mathbb{E}_F(e^{zX})$. Under $H_0 : -\frac{1}{2}\left\{1 - sgn(x)\left(1 - e^{-|x|}\right)\right\} \sim U(-1, 0)$, so $M_S(z) = \frac{1-e^{-z}}{z}$, by part (2) of Theorem 2 we complete the proof, that is $C_S(\gamma) = -2\ln(m_S(b_S(\gamma)))$. □

Proof of B4.

$$T_N = -\sum_{i=1}^{n} \frac{\Phi^{-1}\left(\frac{1}{2}\left\{1 - sgn(x)\left(1 - e^{-|x|}\right)\right\}\right)}{\sqrt{n}}.$$

By the strong law of large number (SLLN)

$$\frac{T_N}{\sqrt{n}} \xrightarrow{\text{w.p.1}} b_N(\gamma) = -\,\mathbb{E}^{H_1}\,\Phi^{-1}\left(\tfrac{1}{2}\left\{1 - sgn(x)\left(1 - e^{-|x|}\right)\right\}\right)$$

where

$$b_N(\gamma) = \tfrac{1}{\sqrt{2\pi}}\left(e^{\gamma} - e^{-\gamma}\right) = \sqrt{\tfrac{2}{\pi}}\sinh(\gamma).$$

Now, by Theorem 1, we have $m_N(t) = \inf_{z>0} e^{-zt} M_N(z)$, where $M_N(z) = \mathbb{E}_F(e^{zX})$. Under $H_0 : -\Phi^{-1}\left(\frac{1}{2}\left\{1 - sgn(x)\left(1 - e^{-|x|}\right)\right\}\right) \sim N(0,1)$, so $M_N(z) = e^{z^2/2}$, by part (2) of Theorem 2, $C_N(\gamma) = -2\ln(m_N(b_N(\gamma))) = b_N^2(\gamma) = \frac{2}{\pi}\sinh^2(\gamma)$. □

4.1. The Limiting ratio of the EBS for different tests when $\gamma \to 0$.

Corollary 4.1. *The limits of ratios for different tests are as follows:*

A1. $\lim_{\gamma\to 0} \dfrac{C_S(\gamma)}{C_F(\gamma)} = 1.56103$

A2. $\lim_{\gamma\to 0} \dfrac{C_L(\gamma)}{C_F(\gamma)} = 1.21585$

A3. $\lim_{\gamma\to 0} \dfrac{C_N(\gamma)}{C_F(\gamma)} = 1.32504$

A4. $\lim_{\gamma\to 0} \dfrac{C_N(\gamma)}{C_L(\gamma)} = 1.08981$

A5. $\lim_{\gamma\to 0} \dfrac{C_S(\gamma)}{C_N(\gamma)} = 1.1781$

A6. $\lim_{\gamma\to 0} \dfrac{C_S(\gamma)}{C_L(\gamma)} = 1.2839$

Proof of A1.

$$b_F(\gamma) = (1 + \ln 2)e^{\gamma} - (\ln 2 - 1)e^{-\gamma} = 2\cosh(\gamma) + \ln(4)\sinh(\gamma).$$

Therefore

$$b_F'(\gamma) = 2\sinh(\gamma) + \ln(4)\cosh(\gamma),$$

then

$$\lim_{\gamma\to 0} b_F'(\gamma) = \ln(4) < \infty.$$

Also

$$b_S(\gamma) = \tfrac{1}{4}\left(\sinh(\gamma) - 2\cosh(\gamma)\right),$$

then

$$\lim_{\gamma\to 0} b'_S(\gamma) = \lim_{\gamma\to 0} \tfrac{1}{4}\left(\cosh(\gamma) - 2\sinh(\gamma)\right) = \tfrac{1}{4} < \infty.$$

Now under $H_0 : h_F(x) = -2\ln\left[\frac{1}{2}\left\{1 - sgn(x)\left(1 - e^{-|x|}\right)\right\}\right] \sim \chi^2_2$ and $h_S(x) = -\frac{1}{2}\left\{1 - sgn(x)\left(1 - e^{-|x|}\right)\right\} \sim U(-1,0)$, so $Var_{\gamma=0}(h_F(x)) = 4$ and $Var_{\gamma=0}(h_S(x)) = \frac{1}{12}$, also, $\dfrac{b'_S(0)}{b'_F(0)} = \dfrac{1}{4\ln(4)}$. By applying Theorem 3 we can get $\lim\limits_{\gamma\to 0} \dfrac{C_S(\gamma)}{C_F(\gamma)} = 1.56103$. Similarly we can prove the other parts. □

4.2. **The Limiting ratio of the EBS for different tests when $\gamma \to \infty$.**

Corollary 4.2. *The limits of ratios for different tests are as follows:*

D1. $\lim\limits_{\gamma\to\infty} \dfrac{C_L(\gamma)}{C_F(\gamma)} = \dfrac{2\ln 2}{1+\ln 2}$

D2. $\lim\limits_{\gamma\to\infty} \dfrac{C_L(\gamma)}{C_N(\gamma)} = \lim\limits_{\gamma\to\infty} \dfrac{C_F(\gamma)}{C_N(\gamma)} = \lim\limits_{\gamma\to\infty} \dfrac{C_S(\gamma)}{C_F(\gamma)} = \lim\limits_{\gamma\to\infty} \dfrac{C_S(\gamma)}{C_N(\gamma)} = \lim\limits_{\gamma\to\infty} \dfrac{C_L(\gamma)}{C_S(\gamma)} = 0$

Proof of $\lim_{\gamma\to\infty} \frac{C_L(\gamma)}{C_F(\gamma)}$. By Lemma 1 part (1) $C_L(\gamma) \leq 2b_L(\gamma)$. So

$$\lim_{\gamma\to\infty} \frac{C_L(\gamma)}{C_F(\gamma)} \leq \lim_{\gamma\to\infty} \frac{2b_L(\gamma)}{b_F(\gamma) - 2\ln(b_F(\gamma)) + 2\ln(2) - 2}.$$

It is sufficient to obtain the limit of $\lim\limits_{\gamma\to\infty} \dfrac{2b_L(\gamma)}{b_F(\gamma)}$. Then by using L'Hopital's rule, we get

$$\lim_{\gamma\to\infty} \frac{2b_L(\gamma)}{b_F(\gamma)} = \lim_{\gamma\to\infty} \frac{2ln2}{\tanh(\gamma) + \ln 2} = \frac{2ln2}{1+\ln 2}$$

Then

$$\lim_{\gamma\to\infty} \frac{C_L(\gamma)}{C_F(\gamma)} \leq \frac{2\ln 2}{1+\ln 2}.$$

Also, by Theorem 6 part (2), we have $C_L(\gamma) \geq 2b_L(\gamma) - 2\ln(b_L(\gamma)) - 2$. So

$$\lim_{\gamma\to\infty} \frac{C_L(\gamma)}{C_F(\gamma)} \geq \lim_{\gamma\to\infty} \frac{2b_L(\gamma) - 2\ln(b_L(\gamma)) - 2}{b_F(\gamma) - 2\ln(b_F(\gamma)) + 2\ln(2) - 2}.$$

It is sufficient to obtain the limit of $\lim_{\gamma\to\infty} \frac{2b_L(\gamma)}{b_F(\gamma)}$. Then by using L'Hopital's rule, we get

$$\lim_{\gamma\to\infty} \frac{2b_L(\gamma)}{b_F(\gamma)} = \lim_{\gamma\to\infty} \frac{2\ln 2}{\tanh(\gamma) + \ln 2} = \frac{2\ln 2}{1+\ln 2}$$

Then

$$\lim_{\gamma\to\infty} \frac{C_L(\gamma)}{C_F(\gamma)} \geq \frac{2\ln 2}{1+\ln 2}.$$

By pinching theorem, we have $\lim_{\gamma\to\infty} \frac{C_L(\gamma)}{C_F(\gamma)} = \frac{2\ln 2}{1+\ln 2}$. □

Proof of $\lim_{\gamma\to\infty} \frac{C_L(\gamma)}{C_N(\gamma)}$. From B4 we have

$$C_N(\gamma) = \tfrac{2}{\pi}\sinh^2(\gamma).$$

By Lemma 1 part (1) $C_L(\gamma) \leq 2b_L(\gamma)$. So

$$\lim_{\gamma\to\infty} \frac{C_L(\gamma)}{C_N(\gamma)} \leq \lim_{\gamma\to\infty} \frac{2b_L(\gamma)}{\frac{2}{\pi}\sinh^2(\gamma)}$$

$$\lim_{\gamma\to\infty} \frac{C_L(\gamma)}{C_N(\gamma)} \leq \lim_{\gamma\to\infty} \frac{2\ln(4)\sinh(\gamma)}{\frac{2}{\pi}\sinh^2(\gamma)} = \lim_{\gamma\to\infty} \frac{2\ln(4)}{\frac{2}{\pi}\sinh(\gamma)} = 0.$$

So

$$\lim_{\gamma\to\infty} \frac{C_L(\gamma)}{C_N(\gamma)} \leq 0.$$

Then

$$\lim_{\gamma\to\infty} \frac{C_L(\gamma)}{C_N(\gamma)} = 0.$$

□

Proof of $\lim_{\gamma\to\infty}\frac{C_S(\gamma)}{C_F(\gamma)}$. By Lemma 1 part (3) $C_S(\gamma) \le -2\ln(2) - 2\ln(-b_S(\gamma))$. So

$$\lim_{\gamma\to\infty}\frac{C_S(\gamma)}{C_F(\gamma)} \le \lim_{\gamma\to\infty}\frac{-2\ln(2) - 2\ln(-b_S(\gamma))}{b_F(\gamma) - 2\ln(b_F(\gamma)) + 2\ln(2) - 2}.$$

It is sufficient to obtain the limit of $\lim_{\gamma\to\infty}\frac{-2\ln(-b_S(\gamma))}{b_F(\gamma)}$.

Then

$$\lim_{\gamma\to\infty}\frac{-2\ln(-b_S(\gamma))}{b_F(\gamma)} = \lim_{\gamma\to\infty}\frac{-2\ln(2) - 2\gamma + 3\ln(2)}{(1+\ln 2)e^{\gamma}},$$

now, by using L'Hopital's rule, we get

$$\lim_{\gamma\to\infty}\frac{-2\ln(-b_S(\gamma))}{b_F(\gamma)} = 0.$$

So

$$\lim_{\gamma\to\infty}\frac{C_S(\gamma)}{C_F(\gamma)} \le 0.$$

Then

$$\lim_{\gamma\to\infty}\frac{C_S(\gamma)}{C_F(\gamma)} = 0.$$

□

4.3. **Comparison of the EBS for the four combination procedures.** From the relations in section (4.1) we conclude that locally as $\gamma \to 0$, the sum of p-values procedure is better than all other procedures since it has the highest EBS, followed in decreasing order by the inverse normal and the logistic procedure. The worst is the Fisher's procedure, i.e,

$$C_S(\gamma) > C_N(\gamma) > C_L(\gamma) > C_F(\gamma).$$

Whereas, from result of Section (4.2) as $\gamma \to \infty$ the inverse normal procedure is better than the other procedures, followed in decreasing order by the Fisher's procedure and the sum of p-values. The worst is the logistic procedure, i.e,

$$C_N(\gamma) > C_F(\gamma) > C_S(\gamma) > C_L(\gamma).$$

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