Jordan Journal of Mathematics and Statistics (JJMS) 14(2), 2021, pp 287 - 305

COEFFICIENT ESTIMATE FOR CLASS OF MEROMORPHIC BI-BAZILEVIČ TYPE FUNCTIONS ASSOCIATED WITH LINEAR OPERATOR DEFINED BY CONVOLUTION

A. NAIK [(1)], T. PANIGRAHI [(2)] AND G. MURUGUSUNDARAMOORTHY[(3)]

Abstract. In the present paper, we propose to investigate a new subclass $\Sigma_M^{*,p,q}(h,\mu,\lambda,k,\gamma)$ of meromorphic functions associated with linear operator defined by means of convolution in the exterior of the unit disk $\nabla := \{z \in \mathbb{C} : 1 < |z| < \infty\}$. We study the behaviour of initial coefficients $b_0,\ b_1$ and b_2 for the function in this newly constructed class. Some interesting remarks of the results presented here are discussed. Our results generalize and improve some of the previously known results of other researchers.

1. Introduction and Motivation

Let Σ^* be denote the family of all functions of the form

$$f(z) = z + b_0 + \sum_{n=1}^{\infty} \frac{b_n}{z^n}, \tag{1.1}$$

which are meromorphic univalent defined in the exterior of the unit disk $\nabla := \{z \in \mathbb{C} : 1 < |z| < \infty\}$ except for a simple pole at ∞ with residue 1. Since $f \in \Sigma^*$ is univalent, it has an inverse $f^{-1} = g$ that satisfy the condition

$$f^{-1}(f(z)) = z, \quad (z \in \nabla)$$

2000 *Mathematics Subject Classification.* Primary: 30C45, 30C80; Secondary 30C50.

Key words and phrases. Analytic function, meromorphic function, bi-univalent function, bi-Bazilevic function, linear operator, convolution.

Received: Nov. 16, 2019 Accepted: Sept. 3, 2020 .

and

$$f(f^{-1}(w)) = w, \quad (M < |w| < \infty,\ M > 0).$$

The inverse function $g = f^{-1}$ has a series expansion of the form

$$\begin{aligned} g(w) &= w + \sum_{n=0}^{\infty} \frac{B_n}{w^n} \\ &= w - b_0 - \frac{b_1}{w} - \frac{b_2 + b_0 b_1}{w^2} \\ &\quad - \frac{b_3 + 2b_0 b_2 + b_0^2 b_1 + b_1^2}{w^3} - \cdots . \quad (M < |w| < \infty) \end{aligned} \tag{1.2}$$

For a function $f \in \Sigma^*$ given by (1.1) and the function $h \in \Sigma^*$ defined by

$$h(z) = z + d_0 + \sum_{n=1}^{\infty} \frac{d_n}{z^n}, \quad (d_n > 0) \tag{1.3}$$

we define the Hadamard product (or convolution) of f and h , written as $f * h$ as

$$(f * h)(z) = z + b_0 d_0 + \sum_{n=1}^{\infty} \frac{b_n d_n}{z^n} \quad (z \in \nabla). \tag{1.4}$$

For the function $f \in \Sigma^*$ given by (1.1), we define the linear operator $\mathcal{H}_\gamma^k : \Sigma^* \longrightarrow \Sigma^*$ defined as follows:

$$\begin{aligned} \mathcal{H}_\gamma^0 f(z) &= f(z) \\ \mathcal{H}_\gamma^1 f(z) &= \mathcal{H}_\gamma f(z) = (1-\gamma) f(z) + \gamma z f'(z) \\ &= z + \sum_{n=0}^{\infty} [1 - (n+1)\gamma] \frac{b_n}{z^n} \quad (0 \le \gamma < \frac{1}{n+1}) \\ \mathcal{H}_\gamma^2 f(z) &= \mathcal{H}_\gamma[\mathcal{H}_\gamma^1 f(z)] = z + \sum_{n=0}^{\infty} [1 - (n+1)\gamma]^2 \frac{b_n}{z^n}. \end{aligned}$$

In general, for $k \in \mathbb{N}_0 := \{1, 2, 3, \cdots\}$

$$\begin{aligned} \mathcal{H}_\gamma^k f(z) &= \mathcal{H}_\gamma[\mathcal{H}_\gamma^{k-1} f(z)] \\ &= z + \sum_{n=0}^{\infty} [1 - (n+1)\gamma]^k \frac{b_n}{z^n} \quad (0 \le \gamma < \frac{1}{n+1}; k \in \mathbb{N}_0). \end{aligned} \tag{1.5}$$

We say that the function $f \in \Sigma^*$ is bi-univalent in ∇ if $f^{-1}(w)$ has univalent analytic continuation to ∇. The class of all meromorphic bi-univalent functions in ∇ given by (1.1) is denoted by Σ_M^*. Estimates on the coefficient of classes of meromorphic univalent functions were widely investigated in the literature. For instance, Schiffer [16] obtained the sharp estimate $|b_2| \le \frac{2}{3}$ for meromorphic univalent function $f \in \Sigma^*$ with $b_0 = 0$. Duren [2, 3] gave an elementary proof of the inequality $|b_n| \le \frac{2}{2n+1}$ for $f \in \Sigma^*$ with $b_k = 0$ for $1 \le k \le \frac{n}{2}$. For the coefficient of inverse of meromorphic univalent function, Springer [18] prove that $|B_3| \le 1$, $|B_3 + \frac{1}{2}B_1^2| \le \frac{1}{2}$ and conjectured that

$$|B_{2n-1}| \le \frac{(2n-2)!}{n!(n-1)!}, \ n = 3, 4, 5, \cdots .$$

Kubota [12] has proved that Springer conjecture is true for $n = 3, 4, 5$ by an elementary application of Grunsky's inequalities and subsequently Schober [17] obtained a sharp bounds for the coefficient B_{2n-1}, $1 \le n \le 7$. In 2007, Kapoor and Mishra [11] considered the inverse function $g = f^{-1}$ where $g \in B(\alpha; 0)$ and obtained the bounds $\frac{2(1-\alpha)}{n+1}$, if $\frac{n-1}{n} \le \alpha < 1$. Hamidi et al. [7] (also see, [6]) improved the coefficient estimate given by Kapoor and Mishra [11]. Recently, Orhan et al. [13] introduced the following two subclasses of the meromorphic bi-univalent function class Σ_M^* and find estimates on the coefficient $|b_0|$ and $|b_1|$ for the function in each of the subclasses.

Definition 1.1. (see [13]) For $\mu \ge 0$, $\lambda \ge 1$, $\lambda > \mu$; $0 \le \alpha < 1$ a function $f \in \Sigma_M^*$, given by (1.1) is said to be in the class $\Sigma_M^*(\alpha, \mu, \lambda)$, if the following conditions are satisfied:

$$\Re\left[(1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu} + \lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}\right] > \alpha,$$

and

$$\Re\left[(1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu} + \lambda g'(w)\left(\frac{g(w)}{w}\right)^{\mu-1}\right] > \alpha,$$

for some where $z, w \in \nabla$ and g is given by (1.2).

Theorem 1.1. *(see [13], Theorem 2.1) Let the function f(z) given by (1.1) be in the class* $\Sigma_M^*(\alpha,\mu,\lambda)$ *Then*

$$|b_0| \le \frac{2(1-\alpha)}{\lambda-\mu}, \ \textit{and}\ |b_1| \le 2(1-\alpha)\sqrt{\frac{(1-\mu)^2(1-\alpha)^2}{(\lambda-\mu)^4}+\frac{1}{(2\lambda-\mu)^2}}.$$

where $0 \le \alpha < 1,\ \lambda \ge 1,\ \mu \ge 0,\ \lambda > \mu.$

Definition 1.2. (see [13]) For $\mu \ge 0,\ \lambda \ge 1,\ \lambda > \mu;$ a function $f \in \Sigma_M^*$, given by (1.1) is said to be in the class $\widetilde{\Sigma}_M^*(\alpha,\mu,\lambda)$ if the following conditions are satisfied:

$$\left|arg\left[(1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu}+\lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}\right]\right| < \frac{\alpha\pi}{2}$$

and

$$\left|arg\left[(1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu}+\lambda g' w\left(\frac{g(w)}{w}\right)^{\mu-1}\right]\right| < \frac{\alpha\pi}{2}$$

where $z, w \in \nabla$ and g is given by (1.2).

Theorem 1.2. *(see [13], Theorem 2.2) Let the function* $f(z) \in \widetilde{\Sigma}_M^*(\alpha,\mu,\lambda)$ *given by (1.1). Then*

$$|b_0| \le \frac{2\alpha}{\lambda-\mu} \quad \textit{and} \quad |b_1| \le 2\alpha^2\sqrt{\frac{1}{(2\lambda-\mu)^2}+\frac{(1-\mu)^2}{(\lambda-\mu)^4}}.$$

The unexpected and unusual behaviour for finding bounds for the coefficients of f and its inverse map $g = f^{-1}$ makes the task challenging.Various researchers (see [4, 5, 6, 7, 10, 14, 15, 19]) introduced and investigated the coefficient bounds for different subclasses of meromorphic bi-univalent function. Recall from [9] that a meromorphic function is said to be bi-Bazilevič in a given domain ∇ if both the functions and its inverse map are Bazilevič.

Motivated by the aforecited works, in this paper we introduce certain subclass of meromorphic bi-Bazilevič type function and obtain estimates on the initial coefficients $|b_0|$, $|b_1|$ and $|b_2|$ of function in the newly introduced subclass. Our results generalize and improve some recent works of Orhan et al. [13], Hajiparvaneh and Zireh [4] and Halim et al. [5].

2. The class $\Sigma_M^{*,p,q}(h,\mu,\lambda,k,\gamma)$

In this section, we define the generalize class of meromorphic bi-Bazilevič type functions which includes the two classes of bi-univalent function introduced in [4, 5, 13].

Definition 2.1. Let the functions $p, q : \nabla \to \mathbb{C}$ be analytic and

$$p(z) = 1 + \frac{p_1}{z} + \frac{p_2}{z^2} + \cdots,$$

$$q(z) = 1 + \frac{q_1}{z} + \frac{q_2}{z^2} + \cdots$$

such that

$$min\left\{\Re(p(z)), \Re(q(z))\right\} > 0 \quad (z \in \nabla).$$

For $0 \leq \gamma < \frac{1}{n+1}; k \in \mathbb{N}_0$ a function $f \in \Sigma_M^*$ given by (1.1) is in the class $\Sigma_M^{*,p,q}(h,\mu,\lambda,k,\gamma)$ if the conditions

$$\left[(1-\lambda)\left(\frac{(\mathcal{H}_\gamma^k f * h)(z)}{z}\right)^\mu + \lambda(\mathcal{H}_\gamma^k f * h)'(z)\left(\frac{(\mathcal{H}_\gamma^k f * h)(z)}{z}\right)^{\mu-1}\right] \in p(\nabla), \tag{2.1}$$

and

$$(1-\lambda)\left[\left(\frac{(\mathcal{H}_\gamma^k g * h)(w)}{w}\right)^\mu + \lambda(\mathcal{H}_\gamma^k g * h)'(w)\left(\frac{(\mathcal{H}_\gamma^k g * h)(w)}{w}\right)^{\mu-1}\right] \in q(\nabla) \tag{2.2}$$

are satisfied where the functions g and h are defined in (1.2) and (1.3) respectively.

For suitable choices of functions $p(z),\ q(z),\ h(z)$ and parameters λ, μ, γ and k, the class of meromorphic bi-univalent function $\Sigma_M^{*,p,q}(h,\mu,\lambda,k,\gamma)$ leads to certain well-known class of meromorphic bi-univalent function studied by earlier researchers in literature.

Example 2.1. *If we take*

$$p(z) = q(z) = \frac{1 + \frac{1-2\beta}{z}}{1 - \frac{1}{z}} = 1 + \frac{2(1-\beta)}{z} + \frac{2(1-\beta)}{z^2} + \cdots \quad (0 \leq \beta < 1; z \in \nabla),$$

then the conditions of Definition 2.1 are satisfied for both functions p(z) and q(z). Now,for $0 \leq \beta < 1,\ \lambda \geq 1,\ \lambda > \mu,\ \mu \geq 0, \gamma < \frac{1}{n+1}, k \in \mathbb{N}_0$ *a function* $f \in \Sigma_M^*$ *given by* (1.1) *is in the class* $f \in \Sigma_M^{*,p,q}(h, \mu, \lambda, k, \gamma)$ *if*

$$\Re\left[(1-\lambda)\left(\frac{(\mathcal{H}_\gamma^k f * h)(z)}{z}\right)^\mu + \lambda(\mathcal{H}_\gamma^k f * h)'(z)\left(\frac{(\mathcal{H}_\gamma^k f * h)(z)}{z}\right)^{\mu-1}\right] > \beta$$

and

$$\Re\left[(1-\lambda)\left(\frac{(\mathcal{H}_\gamma^k g * h)(w)}{w}\right)^\mu + \lambda(\mathcal{H}_\gamma^k g * h)'(w)\left(\frac{(\mathcal{H}_\gamma^k g * h)(w)}{w}\right)^{\mu-1}\right] > \beta$$

$z, w \in \nabla$. *We denote the above class as* $\Sigma_M^*(h, \beta, \mu, \lambda, k, \gamma)$.

Example 2.2. *Taking* $k = \gamma = 0$,

$$h(z) = \frac{z}{1 - \frac{1}{z}} = z + 1 + \frac{1}{z} + \frac{1}{z^2} + \cdots$$

in class $\Sigma_M^*(h, \beta, \mu, \lambda, k, \gamma)$, *it reduce to class* $\Sigma_M^*(\beta, \mu, \lambda,)$ *as discussed in Definition 1.1 (also see* [4, 8]*).*

Example 2.3. *Letting λ=1 and μ=0 in the class* $\Sigma_M^*(\beta, \mu, \lambda)$, *we obtain the class* $\Sigma_M^*(\beta)$, *the class of all meromorphic bi-univalent starlike function of order* β *studied by Halim et al.* [5]*(also see,* [6]*).*

Example 2.4. *Taking* $\mu = 1$ *in the class* $\Sigma_M^*(\beta, \mu, \lambda)$, *we get* $B_\Sigma(\beta; \lambda)$, *the class of meromorphic bi-univalent function introduced by Hamidi et al. (*[7]*, p.350).*

Example 2.5. *Putting λ=1 in* $\Sigma_M^*(\beta, \mu, \lambda)$, *we get* $B_\Sigma(\beta, \mu)$, *the class of bi-Bazilevič function of order* β *and type* μ, *studied by Jahangiri and Hamidi* [9] *(also see,* [1]*).*

Remark 1. *If we set*

$$p(z) = q(z) = \left(\frac{1+\frac{1}{z}}{1-\frac{1}{z}}\right)^{\alpha} = 1 + \frac{2\alpha}{z} + \frac{2\alpha^2}{z^2} + \cdots \quad (0 < \alpha \leq 1,\ z \in \nabla),$$

it is easy to verify that the functions $p(z)$ and $q(z)$ satisfies the condition of Definition 2.1 .

For $\mu \geq 0,\ \lambda \geq 1,\ 0 < \alpha \leq 1, \gamma < \frac{1}{n+1}, k \in \mathbb{N}_0$ and $f \in \Sigma_M^*$ then, $f \in \Sigma_M^{*,p,q}(h,\mu,\lambda,k,\gamma)$,if

$$\left|arg\left[(1-\lambda)\left(\frac{(\mathcal{H}_\gamma^k f * h)(z)}{z}\right)^{\mu} + \lambda(\mathcal{H}_\gamma^k f * h)'(z)\left(\frac{(\mathcal{H}_\gamma^k f * h)(z)}{z}\right)^{\mu-1}\right]\right| < \frac{\alpha\pi}{2}$$

and

$$\left|arg\left[(1-\lambda)\left(\frac{(\mathcal{H}_\gamma^k g * h)(w)}{w}\right)^{\mu} + \lambda(\mathcal{H}_\gamma^k g * h)'(w)\left(\frac{(\mathcal{H}_\gamma^k g * h)(w)}{w}\right)^{\mu-1}\right]\right| < \frac{\alpha\pi}{2}$$

where $z, w \in \nabla$ and g and h are defined in (1.2) and (1.3) respectively. We denote the above class as $\widetilde{\Sigma}_M^*(h,\alpha,\mu,\lambda,k,\gamma)$.

Example 2.6. *Taking $k = \gamma = 0,\ h(z) = \frac{z}{1-\frac{1}{z}}$ in $\widetilde{\Sigma}_M^*(h,\alpha,\mu,\lambda,k,\gamma)$, it reduce to class $\widetilde{\sum}_M^*(\alpha,\mu,\lambda)$ as discussed in Definition 1.2(also, see [4]).*

Example 2.7. *Putting λ=1, μ=0 in class $\widetilde{\Sigma}_M^*(\alpha,\mu,\lambda)$ it reduce to $\widetilde{\Sigma}_M^*(\alpha)$, the class of bi-univalent strongly starlike meromorphic function of order α studied by Hamil et al.[5].*

Example 2.8. *Letting λ=1 in $\widetilde{\Sigma}_M^*(\alpha,\mu,\lambda)$ we get $\Sigma_M^B(\mu,\alpha)$, the class of meromorphic strongly Bazilevič bi-univalent functions of type μ and order α (see [5]).*

3. Main Results

The initial coefficient bounds $|b_0|, |b_1|$ and $|b_2|$ for the class $\Sigma_M^{*,p,q}(h,\mu,\lambda,k,\gamma)$ is given by the following theorem.

Theorem 3.1. *If the function f given by (1.1) is in the class $\Sigma_M^{*,p,q}(h,\mu,\lambda,k,\gamma)$, then the coefficients b_0, b_1 and b_2 satisfy the inequalities*

$$|b_0| \leq \frac{1}{(1-\gamma)^k|d_0|} min\left\{\sqrt{\frac{|p_1|^2+|q_1|^2}{2(\lambda-\mu)^2}}, \sqrt{\frac{|p_2|+|q_2|}{(2\lambda-\mu)|1-\mu|}}\right\}, \tag{3.1}$$

$$|b_1| \leq \frac{1}{(1-2\gamma)^k|d_1|} \times min\left\{\frac{|p_2|+|q_2|}{2(2\lambda-\mu)}, \sqrt{\frac{|p_2|^2+|q_2|^2}{2(2\lambda-\mu)^2}+\frac{(1-\mu)^2(|p_1|^2+|q_1|^2)^2}{16(\lambda-\mu)^4}}\right\}, \tag{3.2}$$

and

$$|b_2| \leq \frac{1}{(3\lambda-\mu)(1-3\gamma)^k|d_2|} \times \left[\frac{|(\mu-1)(1-\gamma)^k(1-2\gamma)^k d_0 d_1-(1-3\gamma)^k d_2||p_3|+|(\mu-1)(1-\gamma)^k(1-2\gamma)^k d_0 d_1||q_3|}{|2(\mu-1)(1-\gamma)^k(1-2\gamma)^k d_0 d_1-(1-3\gamma)^k d_2|} \frac{|(1-\mu)(2-\mu)|(3\lambda-\mu)}{6(\lambda-\mu)^3}|p_1|^3\right]. \tag{3.3}$$

Proof. Let the function f given by (1.1) be in the class $\Sigma_M^{*,p,q}(h,\mu,\lambda,k,\gamma)$. Then there exists two functions $p,q:\nabla\to C$ satisfies the condition of Definition 2.1 such that

$$(1-\lambda)\left(\frac{(\mathcal{H}_\gamma^k f*h)(z)}{z}\right)^\mu+\lambda(\mathcal{H}_\gamma^k f*h)'(z)\left(\frac{(\mathcal{H}_\gamma^k f*h)(z)}{z}\right)^{\mu-1}=p(z), \quad (z\in\nabla) \tag{3.4}$$

and

$$(1-\lambda)\left[\frac{(\mathcal{H}_\gamma^k g*h)(w)}{w}\right]^\mu+\lambda(\mathcal{H}_\gamma^k g*h)'(w)\left(\frac{(\mathcal{H}_\gamma^k g*h)(w)}{w}\right)^{\mu-1}=q(w), \quad (w\in\nabla). \tag{3.5}$$

Further, the functions $p(z)$ and $q(w)$ have the following form:

$$p(z)=1+\frac{p_1}{z}+\frac{p_2}{z^2}+\cdots \tag{3.6}$$

and

$$q(w)=1+\frac{q_1}{w}+\frac{q_2}{w^2}+\cdots. \tag{3.7}$$

It follows from the relations (1.2), (1.3) and (1.5) that

$$
\begin{aligned}
& (1-\lambda)\left(\frac{(\mathcal{H}_\gamma^k f*h)(z)}{z}\right)^{\mu}+\lambda(\mathcal{H}_\gamma^k f*h)'(z)\left(\frac{(\mathcal{H}_\gamma^k f*h)(z)}{z}\right)^{\mu-1}\\
= & \ 1+\frac{(\mu-\lambda)(1-\gamma)^k}{z}b_0d_0\\
+ & \ \frac{2(\mu-2\lambda)(1-2\gamma)^k b_1d_1+(\mu-1)(\mu-2\lambda)(1-\gamma)^{2k}b_0^2d_0^2}{2z^2}\\
+ & \ \frac{6(\mu-3\lambda)(1-3\gamma)^k b_2d_2+6(\mu-1)(\mu-3\lambda)(1-\gamma)^k(1-2\gamma)^k b_0b_1d_0d_1}{6z^3}\\
+ & \ \frac{(\mu-1)(\mu-2)(\mu-3\lambda)(1-\gamma)^{3k}b_0^3d_0^3}{6z^3}+\cdots
\end{aligned}
\tag{3.8}
$$

and

$$
\begin{aligned}
& (1-\lambda)\left(\frac{(\mathcal{H}_\gamma^k g*h)(w)}{w}\right)^{\mu}+\lambda(\mathcal{H}_\gamma^k g*h)^{'}(w)\left(\frac{(\mathcal{H}_\gamma^k g*h)(w)}{w}\right)^{\mu-1}\\
= & \ 1+\frac{(\lambda-\mu)(1-\gamma)^k}{w}b_0d_0+\frac{2(2\lambda-\mu)(1-2\gamma)^k b_1d_1+(\mu-2\lambda)(\mu-1)(1-\gamma)^{2k}b_0^2d_0^2}{2w^2}\\
+ & \ \frac{6(\mu-1)(\mu-3\lambda)(1-\gamma)^k(1-2\gamma)^k b_0d_0b_1d_1+6(3\lambda-\mu)(1-3\gamma)^k b_2d_2}{6w^3}\\
- & \ \frac{(\mu-1)(\mu-2)(\mu-3\lambda)(1-\gamma)^{3k}b_0^3d_0^3+6(\mu-3\lambda)(1-3\gamma)^k b_0b_1d_2}{6w^3}+\cdots.
\end{aligned}
\tag{3.9}
$$

Making use of (3.8), (3.6) in (3.4) and (3.9),(3.7) in (3.5) and comparing the initial coefficients, we obtain following relations:

$$(\mu-\lambda)(1-\gamma)^k b_0d_0 = p_1 \tag{3.10}$$

$$\frac{2(\mu-2\lambda)(1-2\gamma)^k b_1d_1+(\mu-1)(\mu-2\lambda)(1-\gamma)^{2k}b_0^2d_0^2}{2} = p_2 \tag{3.11}$$

$$\frac{6(\mu-3\lambda)(1-3\gamma)^k b_2 d_2 + 6(\mu-1)(\mu-3\lambda)(1-\gamma)^k(1-2\gamma)^k b_0 d_0 b_1 d_1}{6}$$
$$+\frac{(\mu-1)(\mu-2)(\mu-3\lambda)(1-\gamma)^{3k} b_0^3 d_0^3}{6} = p_3 \tag{3.12}$$

and

$$-(\mu-\lambda)(1-\gamma)^k b_0 d_0 = q_1 \tag{3.13}$$
$$\frac{-2(\mu-2\lambda)(1-2\gamma)^k b_1 d_1 + (\mu-2\lambda)(\mu-1)(1-\gamma)^{2k} b_0^2 d_0^2}{2} = q_2 \tag{3.14}$$

$$\frac{6(\mu-1)(\mu-3\lambda)(1-\gamma)^k(1-2\gamma)^k b_0 d_0 b_1 d_1 + 6(3\lambda-\mu)(1-3\gamma)^k b_2 d_2}{6}$$
$$-\frac{(\mu-1)(\mu-2)(\mu-3\lambda)(1-\gamma)^{3k} b_0^3 d_0^3 + 6(\mu-3\lambda)(1-3\gamma)^k b_0 b_1 d_2}{6} = q_3. \tag{3.15}$$

From (3.10) and (3.13), we have

$$p_1 = -q_1 \tag{3.16}$$
$$2(\lambda-\mu)^2(1-\gamma)^{2k} b_0^2 d_0^2 = p_1^2 + q_1^2 \tag{3.17}$$

which implies

$$b_0^2 = \frac{p_1^2 + q_1^2}{2(\lambda-\mu)^2(1-\gamma)^{2k} d_0^2}. \tag{3.18}$$

Adding (3.11) and (3.14) we get

$$(\mu-1)(\mu-2\lambda)(1-\gamma)^{2k} b_0^2 d_0^2 = p_2 + q_2$$

which implies

$$b_0^2 = \frac{p_2 + q_2}{(2\lambda-\mu)(1-\mu)(1-\gamma)^{2k} d_0^2}. \tag{3.19}$$

By virtue of triangle inequality the estimate (3.1) follows from (3.18) and (3.19).

In order to determine the coefficient bound for $|b_1|$, we subtract (3.14) from (3.11) which gives

$$2(\mu-2\lambda)(1-2\gamma)^k b_1 d_1 = p_2 - q_2$$

$$b_1 = \frac{p_2-q_2}{2(\mu-2\lambda)(1-2\gamma)^k d_1}. \tag{3.20}$$

By squaring and adding (3.11) and (3.14) one may get

$$b_1^2 = \frac{1}{d_1^2}\left[\frac{p_2^2+q_2^2}{2(2\lambda-\mu)^2(1-2\gamma)^{2k}} - \frac{(1-\mu)^2(1-\gamma)^{4k} b_0^4 d_0^4}{4(1-2\gamma)^{2k}}\right]. \tag{3.21}$$

Upon substituting the values of b_0^2 from (3.18) in (3.21), it follows that

$$b_1^2 = \frac{1}{(1-2\gamma)^{2k} d_1^2}\left[\frac{p_2^2+q_2^2}{2(2\lambda-\mu)^2} - \frac{(1-\mu)^2(p_1^2+q_1^2)^2}{16(\lambda-\mu)^4}\right]. \tag{3.22}$$

Applying triangle inequality to equations (3.20) and (3.22) we respectively obtain

$$|b_1| \leq \frac{|p_2|+|q_2|}{2(2\lambda-\mu)(1-2\gamma)^k |d_1|} \tag{3.23}$$

and

$$|b_1| \leq \frac{1}{(1-2\gamma)^k |d_1|}\sqrt{\frac{|p_2|^2+|q_2|^2}{2(2\lambda-\mu)^2} + \frac{(1-\mu)^2(|p_1|^2+|q_2|^2)^2}{16(\lambda-\mu)^4}}. \tag{3.24}$$

The assertion (3.2) follow from (3.23) and (3.24). Next, we have to find the bound for the coefficient $|b_2|$. Adding (3.12) and (3.15) after simplifying we get

$$b_0 b_1 = \frac{p_3+q_3}{2(\mu-1)(\mu-3\lambda)(1-\gamma)^k(1-2\gamma)^k d_0 d_1 - (\mu-3\lambda)(1-3\gamma)^k d_2}. \tag{3.25}$$

Subtracting (3.15) from (3.12) we have

$$\begin{aligned} & 2(\mu-3\lambda)(1-3\gamma)^k b_2 d_2 + (\mu-3\lambda)(1-3\gamma)^k b_0 b_1 d_2 \\ + & \frac{1}{3}(\mu-1)(\mu-2)(\mu-3\lambda)(1-\gamma)^{3k} b_0^3 d_0^3 = p_3 - q_3, \end{aligned}$$

which implies

$$\begin{aligned} 2(\mu-3\lambda)(1-3\gamma)^k b_2 d_2 &= p_3 - q_3 - (\mu-3\lambda)(1-3\gamma)^k b_0 b_1 d_2 \\ &\quad - \frac{1}{3}(\mu-1)(\mu-2)(\mu-3\lambda)(1-\gamma)^{3k} b_0^3 d_0^3. \end{aligned} \tag{3.26}$$

Substituting the relation (3.10) and (3.25) into (3.26) and after simplification we obtain

$$
\begin{aligned}
b_2 &= \frac{1}{(\mu-3\lambda)(1-3\gamma)^k d_2} \\
&\times \left[\frac{[(\mu-1)(1-\gamma)^k(1-2\gamma)^k d_0 d_1 - (1-3\gamma)^k d_2]p_3 - (\mu-1)(1-\gamma)^k(1-2\gamma)^k d_0 d_1 q_3}{2(\mu-1)(1-\gamma)^k(1-2\gamma)^k d_0 d_1 - (1-3\gamma)^k d_2}\right] \\
&- \frac{(\mu-1)(\mu-2)}{6(\mu-\lambda)^3(1-3\gamma)^k d_2} p_1^3.
\end{aligned}
\tag{3.27}
$$

Applying triangle inequality to equation (3.27) we obtain desire estimate (3.3). This complete the proof of Theorem 3.1. □

Remark 2. *Putting $\gamma = k = 0$ in Theorem 3.1 we obtain the bounds for $|b_0|$ and $|b_1|$ due to Hajiparvaneh and Zireh (see [4], Theorem 2.3).*

Taking

$$
p(z) = q(z) = \frac{1 + \frac{1-2\beta}{z}}{1 - \frac{1}{z}} = 1 + \frac{2(1-\beta)}{z} + \frac{2(1-\beta)}{z^2} + \cdots \quad (0 \leq \beta < 1;\ z \in \nabla),
$$

in Theorem 3.1, we get the following corollary:

Corollary 3.1. *Let the function $f(z)$ given by (1.1) be in the class $\Sigma_M^*(h,\beta,\mu,\lambda,k,\gamma)$. Then*

$$
\begin{aligned}
|b_0| &\leq \frac{1}{(1-\gamma)^k|d_0|} min\left\{\frac{2(1-\beta)}{\lambda-\mu}, 2\sqrt{\frac{(1-\beta)}{(2\lambda-\mu)|(1-\mu)|}}\right\}, \\
|b_1| &\leq \frac{1}{(1-2\gamma)^k|d_1|} min\left\{\frac{2(1-\beta)}{2\lambda-\mu}, 2(1-\beta)\sqrt{\frac{1}{(2\lambda-\mu)^2} + \frac{(1-\mu)^2(1-\beta)^2}{(\lambda-\mu)^4}}\right\}
\end{aligned}
$$

and

$$\begin{aligned}|b_2| &\leq \frac{2(1-\beta)}{(3\lambda-\mu)(1-3\gamma)^k|d_2|}\\ &\times \left[\frac{|(\mu-1)(1-\gamma)^k(1-2\gamma)^k d_0 d_1-(1-3\gamma)^k d_2|+|(\mu-1)(1-\gamma)^k(1-2\gamma)^k d_0 d_1|}{|2(\mu-1)(1-\gamma)^k(1-2\gamma)^k d_0 d_1-(1-3\gamma)^k d_2|}\right.\\ &+ \left.\frac{2|(\mu-1)(\mu-2)|(3\lambda-\mu)}{3(\lambda-\mu)^3}(1-\beta)^2\right].\end{aligned}$$

Remark 3. *Taking $\gamma = k = 0$ and $h(z) = \frac{z}{1-\frac{1}{z}}$ in Corollary 3.1 we obtain the coefficient bounds for $|b_0|$ and $|b_1|$ due to Hajiparvaneh and Zireh (see [4], Corollary 3.4).*

Taking $h(z) = \frac{z}{1-\frac{1}{z}}$ in Corollary 3.1 we obtain the following results

Corollary 3.2. *Let the function $f(z)$ given by (1.1) be in the class $\Sigma_M^*(\beta,\mu,\lambda,k,\gamma)$. Then*

$$|b_0| \leq \frac{1}{(1-\gamma)^k} \leq \begin{cases} \sqrt{\frac{4(1-\beta)}{(2\lambda-\mu)|(\mu-1)|}} & \text{if} \quad 0 \leq \beta < 1-\frac{(\lambda-\mu)^2}{|1-\mu|(2\lambda-\mu)} \\ \frac{2(1-\beta)}{\lambda-\mu} & \text{if} \quad 1-\frac{(\lambda-\mu)^2}{|1-\mu|(2\lambda-\mu)} \leq \beta < 1 \end{cases}$$

$$|b_1| \leq \frac{2(1-\beta)}{(1-2\gamma)^k(2\lambda-\mu)}$$

and

$$\begin{aligned}|b_2| &\leq \frac{2(1-\beta)}{(3\lambda-\mu)(1-3\gamma)^k}\\ &\times \left[\frac{|(\mu-1)(1-\gamma)^k(1-2\gamma)^k-(1-3\gamma)^k|+|\mu-1|(1-\gamma)^k(1-2\gamma)^k}{|2(\mu-1)(1-\gamma)^k(1-2\gamma)^k-(1-3\gamma)^k|}\right.\\ &+ \left.\frac{2|(\mu-1)(\mu-2)|(3\lambda-\mu)}{3(\lambda-\mu)^3}(1-\beta)^2\right].\end{aligned}$$

Remark 4. *For $\gamma = k = 0$, Corollary 3.2 is an improvement of estimates obtained in Theorem 1.2 (also see [10], Corollary 3.5) because*

$$\frac{2(1-\beta)}{2\lambda-\mu} \leq \frac{2(1-\beta)}{(2\lambda-\mu)}\sqrt{1+\left[\frac{(2\lambda-\mu)(1-\mu)(1-\beta)}{(\lambda-\mu)^2}\right]^2}.$$

Remark 5. *Our results in Corollary 3.2 with $\gamma = k = 0$ is coincident with the results of Hamidi et al. ([8], Theorem 3.2, p.281) but the estimation was derived by making use of Faber polynomial.*

Taking $\gamma = k = \mu = 0$ and $\lambda = 1$ in Corollary 3.2 we obtain the following.

Corollary 3.3. *Let $f(z) \in \Sigma^*_M(\beta)$ $(0 \leq \beta < 1)$. Then*

$$|b_0| \leq \begin{cases} \sqrt{2(1-\beta)} & \text{if } \; 0 \leq \beta \leq \frac{1}{2} \\ 2(1-\beta) & \text{if } \; \frac{1}{2} \leq \beta < 1 \end{cases}$$

$$|b_1| \leq (1-\beta)$$

and

$$|b_2| \leq \frac{2(1-\beta)}{3}\left(1+4(1-\beta)^2\right).$$

Remark 6. *Corollary 3.3 is an improvement of estimate obtained by Halim et al. (see [5], Theorem 1) because $(1-\beta) < (1-\beta)\sqrt{4\beta^2 - 8\beta + 5}$.*

It may be noted that the estimate given in Corollary 3.3 is an improvement to the bound given by Hamidi et al. (see [6], Theorem 2(i)).

Taking $\lambda = 1,\ k = \gamma = 0$ and $0 \leq \mu < 1$ Corollary 3.2 we obtain the following results.

Corollary 3.4. *(see [9], Theorem 2) Let $f \in B(\beta,\mu)$ be bi-univalent in ∇. Then*

$$|b_0| \leq \begin{cases} \sqrt{\frac{4(1-\beta)}{(2-\mu)(1-\mu)}} & if \quad \left(0 \leq \beta < \frac{1}{2-\mu}\right), \\ \frac{2(1-\beta)}{1-\mu} & if \quad \left(\frac{1}{2-\mu} \leq \beta < 1\right) \end{cases}$$

$$|b_1| \leq \frac{2(1-\beta)}{2-\mu}$$

$$|b_2| \leq \frac{2(1-\beta)}{(3-\mu)}\left[1+\frac{2(2-\mu)(3-\mu)(1-\beta)^2}{3(1-\mu)^2}\right].$$

where $0 \leq \beta < 1,\ 0 \leq \mu < 1.$

Taking $\gamma = k = 0$ and $\mu = 1$ in Corollary 3.2, our results coincidence with the results obtain by Hamidi et al.[6] as follows:

Corollary 3.5. *Let the function f given by (1.1) be in the class $B_\Sigma(\beta,\lambda)$. Then*

$$|b_0| \leq \frac{2(1-\beta)}{\lambda-1} \qquad |b_1| \leq \frac{2(1-\beta)}{2\lambda-1} \qquad and \quad |b_2| \leq \frac{2(1-\beta)}{3\lambda-1}.$$

By setting $p(z) = q(z) = \left(\frac{1+\frac{1}{z}}{1-\frac{1}{z}}\right)^\alpha = 1+\frac{2\alpha}{z}+\frac{2\alpha^2}{z^2}+\frac{2\alpha^3}{z^3}+\cdots (0<\alpha\leq 1)$ in Theorem 3.1 we conclude the following result.

Corollary 3.6. *Let the function f(z) given by (1.1) be in the class $\widetilde{\Sigma}_M(h,\alpha,\mu,\lambda,k,\gamma)$. Then*

$$|b_0| \leq \frac{1}{(1-\gamma)^k|d_0|} min\left\{\frac{2\alpha}{\lambda-\mu}, \frac{2\alpha}{\sqrt{|(2\lambda-\mu)(1-\mu)|}}\right\},$$

$$|b_1| \leq \frac{1}{(1-2\gamma)^k|d_1|} min\left\{\frac{2\alpha^2}{2\lambda-\mu}, \sqrt{\frac{4\alpha^4}{(2\lambda-\mu)^2}+\frac{4\alpha^4(1-\mu)^2}{(\lambda-\mu)^4}}\right\}$$

$$= \frac{2\alpha^2}{(1-2\gamma)^k|d_1|} min\left\{\frac{1}{2\lambda-\mu}, \sqrt{\frac{1}{(2\lambda-\mu)^2}+\frac{(1-\mu)^2}{(\lambda-\mu)^4}}\right\}$$

and

$$\begin{aligned} |b_2| &\leq \frac{2\alpha^3}{(3\lambda-\mu)(1-3\gamma)^k|d_2|} \\ &\times \left[\frac{|(\mu-1)(1-\gamma)^k(1-2\gamma)^k d_0 d_1-(1-3\gamma)^k d_2|+|(\mu-1)(1-\gamma)^k(1-2\gamma)^k d_0 d_1|}{|2(1-\mu)(1-\gamma)^k(1-2\gamma)^k d_0 d_1-(1-3\gamma)^k d_2|} \right. \\ &+ \left. \frac{2|(1-\mu)(2-\mu)|(3\lambda-\mu)}{3(\lambda-\mu)^3}\right]. \end{aligned}$$

Taking $\gamma = k = 0,\ h(z) = \frac{z}{1-\frac{1}{z}}$ in Corollary 3.6, we obtain the bounds for $|b_0|$ and $|b_1|$ due to Hajiparvaneh and Zireh [4] as follows:

Corollary 3.7. *Let the function f be in the class $\widetilde{\Sigma}_M(\alpha,\mu,\lambda)$. Then*

$$\begin{aligned} |b_0| &\leq min\left\{\frac{2\alpha}{\lambda-\mu}, \frac{2\alpha}{\sqrt{(2\lambda-\mu)|1-\mu|}}\right\}, \\ |b_1| &\leq 2\alpha^2 min\left\{\frac{1}{2\lambda-\mu}, \sqrt{\frac{1}{(2\lambda-\mu)^2}+\frac{(1-\mu)^2}{(\lambda-\mu)^4}}\right\} \qquad and \\ |b_2| &\leq \frac{2\alpha^3}{(3\lambda-\mu)}\left[\frac{|\mu-1|+|\mu-2|}{|2\mu-3|}+\frac{2|(1-\mu)(2-\mu)|(3\lambda-\mu)}{3(\lambda-\mu)^3}\right]. \end{aligned}$$

Remark 7. *Corollary 3.7 is an improvement of estimate obtained in Theorem 1.4. (see [4], Corollary 3.1).*

Taking λ=1 in Corollary 3.7 we get the following result.

Corollary 3.8. *Let the function f given by (1.1) be in the class $\widetilde{\Sigma}_M(\alpha,\mu)$. Then for $0 < \alpha \leq 1$ we have*

$$|b_0| \leq \begin{cases} \frac{2\alpha}{\sqrt{(2-\mu)(1-\mu)}} & 0 \leq \mu < 1 \\ \frac{2\alpha}{\mu-1} & \mu > 1, \end{cases}$$

$$
\begin{aligned}
|b_1| &\leq 2\alpha^2 min\left\{\frac{1}{|2-\mu|}, \sqrt{\frac{1}{(2-\mu)^2}+\frac{1}{(1-\mu)^2}}\right\} \\
&= \frac{2\alpha^2}{|(1-\mu)(2-\mu)|} min\left\{|1-\mu|, \sqrt{1+\mu^2+4+\mu^2-2\mu-4\mu}\right\} \\
&= \frac{2\alpha^2}{|(1-\mu)(2-\mu)|} min\left\{|1-\mu|, \sqrt{2\mu^2+6\mu+5}\right\} \\
&= \frac{2\alpha^2}{|(1-\mu)(2-\mu)|} min\left\{|(1-\mu)|, \sqrt{2|(1-\mu)(2-\mu)|+1}\right\} \\
&= \frac{2\alpha^2}{|2-\mu|} \qquad \textit{and} \\
|b_2| &\leq \frac{2\alpha^3}{|3-\mu|}\left[\frac{|\mu-1|+|\mu-2|}{|2\mu-3|}+\frac{2|(2-\mu)(3-\mu)|}{3(1-\mu)^2}\right].
\end{aligned}
$$

The last line in $|b_1|$ follows from the well-known fact that if $a,\ b > 0$ such that $a^2 < b^2 \Longrightarrow a < b$. Note that

$$\left[(1-\mu)^2 - \left(\sqrt{2(1-\mu)(2-\mu)+1}\right)^2\right] < 0.$$

Taking $\mu = 0$ in Corollary 3.8 we get the following deduction:

Corollary 3.9. *Let $f \in \Sigma^*_M(\alpha)$. Then for $0 < \alpha \leq 1$, we have*

$$|b_0| \leq \alpha; \qquad |b_1| \leq \alpha^2 \qquad \textit{and} \qquad |b_2| \leq \frac{10}{3}\alpha^3.$$

Remark 8. *Corollary 3.9 is an improvement of the result obtained by Halim et al. ([5], Theorem 2) because*

$$\alpha < 2\alpha \qquad \textit{and} \qquad \alpha^2 < \sqrt{5}\alpha^2.$$

Acknowledgement

The authors would like to thank to the editor and anonymous referees for reading the manuscript carefully and give their comments and suggestions which improve the contents of the manuscript. The present investigation of the second-named author is supported by CSIR research project scheme no: 25(0278)/17/EMR-II, New Delhi, India.

References

[1] I. E. Bazilevic, On a case of integrability in quadratures of the Loewner-Kufarev equation, *Matematicheskii Sbornik*, **37**(79)(1955), 471-476.

[2] P. L. Duren, Coefficients of meromorphic schlict functions, *Proc. Amer. Math. Soc.*, **28**(1971), 169-172.

[3] P. L. Duren, *Univalent Function of Grundlehren der Mathematischen Wissenschaffen*, **259**, Springer, New York, NY, USA, 1983.

[4] S. Hajiparvaneh and A. Zireh, Coefficient estimates for certain subclass of meromorphic and bi-univalent functions, *Commun. Fac. Sci. Univ. Ank. Ser. A I Math. Stat.*, **68**(1)(2019), 654-662.

[5] S. A. Halim, S. G. Hamidi and V. Ravichandran, Coefficient estimates for meromorphic bi-univalent functions, *arXiv:1108.4089vI [Math.CV]* (2011), 1-9.

[6] S. G. Hamidi, S. A. Halim and J. M. Jahangiri, Faber polynomial coefficient estimates for meromorphic bi-starlike functions, *Int. J. Math. Math. Sci*, 2013; Art. ID: 498159, 4pages.

[7] S. G. Hamidi, S. A .Halim and J. M. Jahangiri, Coefficient estimates for a class of meromorphic bi-univalent functions, *C. R. Acad. Sci. Paris, Ser.I*, **351**(2013), 349-352.

[8] S. G. Hamidi, T. Janani, G. Murugusundaramoorthy and J. M. Jahangiri, Coefficient estimates for certain classes of meromorphic bi-univalent functions , *C. R. Acad. Sci. Paris, Ser.I*, **352** (2014), 277-282.

[9] J. M. Jahangiri and S. G. Hamidi, Coefficients of meromorphic bi-Bazilevic functions, *J. Complex. Anal.*(2014), Art. ID: 263917, 4 pages, http://dx.doi.org/10.1155/2014/263917.

[10] T. Janani and G. Murugusundaramoorthy, Coefficient estimates of meromorphic bi-starlike functions of complex order, *Int. J. Anal. Appl.*, **4**(1)(2014), 68-77.

[11] G. P. Kapoor and A. K. Mishra, Coefficient estimates for inverses of starlike functions of positive order, *J. Math. Anal. Appl.*, **329** (2)(2007), 922-934.

[12] Y. Kubota, Coefficients of meromorphic univalent functions, *Kodai Math. Sem. Rep.*, **28** (2-3)(1977), 253-261.

[13] H. Orhan, N. Magesh and V. K. Balaji, Initial coefficient bounds for certain classes of meromorphic bi-univalent functions, *Asian-Europian J. Math*, **7** (1)(2014); doi: 10.1142/S1793557114500053.

[14] T. Panigrahi, Coefficient bounds for certain subclasses of meromorphic and bi-univalent functions, *Bull. Korean Math. Soc.*, **50**(5)(2013), 1531-1538.

[15] S. Salehian and H. Zireh, Coefficient estimates for certain subclass of meromorphic and bi-univalent functions, *Commun. Korean Math. Soc.*, **32**(2)(2017), 389-397.

[16] M. Schiffer, Faber polynomials in the theory of univalent functions, *Bull. Amer. Math. Soc.*, **54**(1948), 503-517.

[17] G. Schober, Coefficients of inverses of meromorphic univalent functions, *Proc. Amer. Math. Soc.*, **67** (1)(1977), 111-116.

[18] G. Springer, The coefficient problem for schlicht mapping of the exterior of the unit circle, *Trans. Amer. Math. Soc.*, **70**(1951), 421-450.

[19] H. M. Srivastava, A. K. Mishra and S. N. Kund, Coefficient estimates for the inverse of starlike functions represented by symmetric gap series, *PanAmer. Math. J.*, **21**(4) (2011), 105-123.

(1) Department of Mathematics, School of Applied Sciences, KIIT Deemed to be University,Bhubaneswar-751024, Odisha, India

Email address: avayanaik@gmail.com

(2) Institue of Mathematics and Applications, Andharua, Bhubaneswar-751029, Odisha, India.

Email address: trailokyap6@gmail.com

(3) School of Advanced Sciences, Vellore Institute of Technology Deemed to be University,Vellore-632014, Tamilnadu,India

Email address: (3)gmsmoorthy@yahoo.com; gms@vit.ac.in