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A GENERALIZATION OF JACOBSTHAL AND JACOBSTHAL-LUCAS NUMBERS

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Abstract. In this paper, we study a generalization of Jacobsthal and Jacobsthal-Lucas numbers. We describe their distinct properties also we give the related matrix representation and sum of terms of the sequences.

1. Introduction

Fibonacci and Lucas sequences and their generalizations/extensions have many interesting, pretty and amazing properties and applications in many fields of science and arts [1, 10, 11, 12, 13, 15]. The Fibonacci and Lucas numbers are defined by the following two recurrence relations

$$F_n = F_{n-1} + F_{n-2} \text{ and } L_n = L_{n-1} + L_{n-2}$$

where $n \geq 2, F_0 = 0, F_1 = 1, L_0 = 2$ and $L_1 = 1$. These sequences are special cases of the Lucas sequences [18].

$$U_n(P, Q) \text{ and } V_n(P, Q)$$

given by the recurrence relation

$$x_n = Px_{n-1} - Qx_{n-2}$$

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where P and Q are fixed integers and initial values $U_0(P,Q)=0, U_0(P,Q)=1, V_0(P,Q)=2$ and $V_1(P,Q)=P$. There are other famous examples of Lucas sequences such as ([4, 9, 16, 17]).

- Pell and Pell-Lucas numbers:

$$P_n = 2P_{n-1} + P_{n-2} \text{ and } Q_n = 2Q_{n-1} + Q_{n-2}$$

  where $n \geq 2, P_0 = 0, P_1 = 1, Q_0 = Q_1 = 1$.

- Jacobsthal and Jacobsthal-Lucas numbers:

$$J_n = J_{n-1} + 2J_{n-2} \text{ and } j_n = j_{n-1} + 2j_{n-2}$$

  where $n \geq 2, J_0 = 0, J_1 = 1, j_0 = j_1 = 2$.

The sequences mentioned above satisfy many common properties and identities for example Binet formulas, Catalan identities and matrix representation. Recently, these sequences were generalized or extended for example see [2, 3, 5, 6, 7, 8, 19]. In [14] a new one-parameter generalization of Pell and Pell-Lucas numbers numbers is given and its properties and related matrix representation are studied.

In this paper we introduce and study a generalization of the Jacobsthal and Jacobsthal-Lucas numbers. This paper is structured as follows in section 2, we introduce the generalized Jacobsthal and Jacobsthal-Lucas numbers and derive their generating functions and binet formulas. In section 3, we find the matrices of the generalized Jacobsthal and Jacobsthal-Lucas numbers. In section 4, we find many properties of the sequences like the Cassini, Catalan and d'Ocagne's formulas. In section 5, we find the sum of terms formulas of the generalized sequences.

2. Definition, generating functions and Binet formulas

Definition 2.1. *Let $k \geq 2, n \geq 0$ be two integers. The generalized Jacobsthal and Jacobsthal-Lucas numbers respectively are defined by:*

$$J_{k,n} = (k-1)J_{k,n-1} + kJ_{k,n-2} \text{ and } j_{k,n} = (k-1)j_{k,n-1} + kj_{k,n-2}$$

where $J_{k,0} = 0, J_{k,1} = 1$ and $j_{k,0} = j_{k,1} = 2$.

When $k = 2$, we have the classical Jacobsthal and Jacobsthal-Lucas numbers. These sequences can be extended to negative indices as follows, for $n \geq 1$

$$J_{k,-n} = (-1)^{n+1}J_{k,n} \text{ and } j_{k,-n} = (-1)^{n+1}j_{k,n}$$

which satisfy the recurrence relations

$$J_{k,-n} = kJ_{k,2-n} - (k-1)J_{k,1-n} \text{ and } j_{k,-n} = kj_{k,2-n} - (k-1)j_{k,1-n}$$

Example 2.1. *The following two tables present the values of $J_{k,n}$ and $j_{k,n}$ for some selected k and n values.*

Table 1. Generalized Jacobsthal numbers

n	-5	-4	-3	-2	-1	0	1	2	3	4	5
$J_{2,n}$	11	-5	3	-1	1	0	1	1	3	5	11
$J_{3,n}$	161	-20	7	-2	1	0	1	2	7	20	161
$J_{4,n}$	205	-51	13	-3	1	0	1	3	13	51	205

Table 2. Generalized Jacobsthal-Lucas numbers

n	-5	-4	-3	-2	-1	0	1	2	3	4	5
$j_{2,n}$	42	-22	10	-6	2	2	2	6	10	22	42
$j_{3,n}$	242	-82	26	-10	2	2	2	10	26	82	242
$j_{4,n}$	818	-206	50	-14	2	2	2	14	50	206	818

In the following theorem we derive the generating functions for the sequences $J_{k,n}$ and $j_{k,n}$.

Theorem 2.1 (Generating functions). *The generating functions of the sequences $J_{k,n}$ and $j_{k,n}$ respectively are*

(1)

$$J(x) = \frac{x}{1-(k-1)x-kx^2}$$

(2)

$$j(x) = \frac{2(x+2-k)}{1-(k-1)x-kx^2}$$

Proof. Let $J(x)$ and $j(x)$ represents the generating functions of $J_{k,n}$ and $j_{k,n}$ respectively.

Note,

$$\begin{aligned}
J(x) &= \sum_{n=0}^{\infty} J_{k,n}x^n \\
&= J_{k,0} + J_{k,1}x + \sum_{n=2}^{\infty} J_{k,n}x^n \\
&= x + \sum_{n=2}^{\infty}((k-1)J_{k,n-1} + kJ_{k,n-2})x^n \\
&= x + (k-1)x\sum_{n=2}^{\infty} J_{k,n-1}x^{n-1} + kx^2\sum_{n=2}^{\infty} J_{k,n-2}x^{n-2} \\
&= x + (k-1)x\sum_{n=0}^{\infty} J_{k,n}x^n + kx^2\sum_{n=0}^{\infty} J_{k,n}x^n \qquad , J_{k,0} = 0 \\
&= x + (k-1)xJ(x) + kx^2J(x)
\end{aligned}$$

Thus, $x = (1 + (1 - k)x - kx^2)J(x)$ and $J(x) = \frac{x}{1-(k-1)x-kx^2}$.

Similarly,

$$\begin{aligned} j(x) = \sum_{n=0}^{\infty} j_{k,n}x^n &= 2 + 2x + \sum_{n=2}^{\infty}((k-1)j_{k,n-1} + kj_{k,n-2})x^n \\ &= 2 + 2x + (k-1)x\sum_{n=2}^{\infty} j_{k,n-1}x^{n-1} + kx^2\sum_{n=2}^{\infty} j_{k,n-2}x^{n-2} \\ &= 2 - 2kx + 4x + (k-1)x\sum_{n=0}^{\infty} j_{k,n}x^n + kx^2\sum_{n=0}^{\infty} j_{k,n}x^n \end{aligned}$$

by adding and subtracting $2(k-1)x$

Thus, $2 - 2kx + 4x = (1 - (k-1)x - kx^2)j(x)$ and $j(x) = \frac{2(2x-kx+1)}{1-(k-1)x-kx^2}$. □

Theorem 2.2 (Binet formulas)**.** *The $n-th$ terms of the generalized Jacobsthal and Jacobsthal-Lucas sequences are given by*

(2.1) $$J_{k,n} = \frac{k^n - (-1)^n}{k+1},$$

and

(2.2) $$j_{k,n} = \frac{4k^n + 2(k-1)(-1)^n}{k+1}$$

Proof. We will use mathematical induction to prove the formulas above. To prove equation 2.1.

- For $n = 0$: $J_{k,0} = \frac{1-1}{k+1} = 0$.
- Assume that $J_{k,n} = \frac{k^n-(-1)^n}{k+1}$
- Note

$$\begin{aligned} J_{k,n+1} &= (k-1)J_{k,n} + kJ_{k,n-1} \\ &= (k-1)(\frac{k^n - (-1)^n}{k+1}) + k(\frac{k^{n-1} - (-1)^{n-1}}{k+1}) \\ &= \frac{k^{n+1} - k(-1)^n - k^n + (-1)^n}{k+1} + \frac{k^n - k(-1)^{n-1}}{k+1} \end{aligned}$$

$$= \frac{k^{n+1} - (-1)^{n+1}}{k+1}$$

Similarly, we prove equation 2.2.

- For $n = 0$: $j_{k,0} = \frac{4k^0 + (2k-2)(-1)^0}{k+1} = 2$.
- Assume that $j_{k,n} = \frac{4k^n + (2k-2)(-1)^n}{k+1}$
- Note

$$\begin{aligned} j_{k,n+1} &= (k-1)j_{k,n} + kj_{k,n-1} \\ &= (k-1)\left(\frac{4k^n + (2k-2)(-1)^n}{k+1}\right) + k\left(\frac{4k^{n-1} + (2k-2)(-1)^{n-1}}{k+1}\right) \\ &= \frac{4k^{n+1} + (2k-2)(-1)^{n+1}}{k+1} \end{aligned}$$

as desired □

3. Matrix representation

As we know classical Jacobsthal numbers can be derived from the matrix $F = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$ for which $F^n = \begin{bmatrix} J_{n+1} & 2J_n \\ J_n & 2J_{n-1} \end{bmatrix}$ (the Jacobsthal F-matrix). Also the Jacobsthal-Lucas numbers can be derived from the matrix $R = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}$ (the Jacobsthal-Lucas R-matrix) for which we can define $R_n = RF^n = \begin{bmatrix} j_{n+1} & 2j_n \\ j_n & 2j_{n-1} \end{bmatrix}$, these matrices were introduced and studied in [7, 8]. In this section we derive the Jacobsthal F_k and the Jacobsthal-Lucas R_k matrices that generates the generalized Jacobsthal and Jacobsthal-Lucas numbers.

Lemma 3.1 (Matrix of generalized Jacobsthal numbers)**.** *Let* $F_k = \begin{bmatrix} k-1 & k \\ 1 & 0 \end{bmatrix}$. *Then for* $n \geq 2$ *we have*

$$F_k^n = \begin{bmatrix} J_{k,n+1} & kJ_{k,n} \\ J_{k,n} & kJ_{k,n-1} \end{bmatrix}$$

Proof. We will use mathematical induction:

- For $n = 2$:

$$F_k^2 = \begin{bmatrix} (k-1)^2 + k & k(k-1) \\ k-1 & k \end{bmatrix} = \begin{bmatrix} J_{k,2+1} & kJ_{k,2} \\ J_{k,2} & kJ_{k,1} \end{bmatrix}$$

- Assume that $F_k^n = \begin{bmatrix} J_{k,n+1} & kJ_{k,n} \\ J_{k,n} & kJ_{k,n-1} \end{bmatrix}$
- Note

$$\begin{aligned}
F_k^{n+1} &= F_k F_k^n \\
&= F_k \begin{bmatrix} J_{k,n+1} & kJ_{k,n} \\ J_{k,n} & kJ_{k,n-1} \end{bmatrix} && \text{by induction} \\
&= \begin{bmatrix} k-1 & k \\ 1 & 0 \end{bmatrix} \begin{bmatrix} J_{k,n+1} & kJ_{k,n} \\ J_{k,n} & kJ_{k,n-1} \end{bmatrix} \\
&= \begin{bmatrix} (k-1)J_{k,n+1} + kJ_{k,n} & k(k-1)J_{k,n} + k^2 J_{k,n-1} \\ J_{k,n+1} & kJ_{k,n} \end{bmatrix} \\
&= \begin{bmatrix} J_{k,n+2} & kJ_{k,n+1} \\ J_{k,n+1} & kJ_{k,n} \end{bmatrix}
\end{aligned}$$

as desired □

Lemma 3.2. *For* $n \geq 1$, $\begin{bmatrix} j_{k,n+1} \\ j_{k,n} \end{bmatrix} = F_k \begin{bmatrix} j_n \\ j_{n-1} \end{bmatrix}$

Proof. Immediate □

Proposition 3.1. *For $n \geq 1$ we have*

(1) $j_{k,n} = 2(J_{k,n} + kJ_{k,n-1})$

(2) $j_{k,n-1} = 2(J_{k,n} + (2-k)J_{k,n-1})$

Proof. From the Binet formula we have:

(1)

$$\begin{aligned} J_{k,n} + kJ_{k,n-1} &= \frac{k^n - (-1)^n}{k+1} + k\frac{k^{n-1} - (-1)^{n-1}}{k+1} \\ &= \frac{2k^n - k(-1)^{n-1} - (-1)^n}{k+1} \\ &= \frac{2k^n + (k-1)(-1)^n}{k+1} \\ &= \frac{1}{2} j_{k,n} \end{aligned}$$

(2)

$$\begin{aligned} 2(J_{k,n} + (2-k)J_{k,n-1}) &= 2\left(\frac{k^n - (-1)^n}{k+1} + (2-k)\frac{k^{n-1} - (-1)^{n-1}}{k+1}\right) \\ &= 2\left(\frac{2k^{n-1} + k(-1)^{n-1} - (-1)^{n-1}}{k+1}\right) \\ &= 2\left(\frac{2k^{n-1} + (k-1)(-1)^{n-1}}{k+1}\right) \\ &= j_{k,n-1} \end{aligned}$$

□

Lemma 3.3 (Matrix of generalized Jacobsthal-Lucas numbers). *Let $n > 0$ and $R_k = \begin{bmatrix} 1 & k \\ 1 & 2-k \end{bmatrix}$. Then $R_k F_k^n = \frac{1}{2} \begin{bmatrix} j_{k,n+1} & kj_{k,n} \\ j_{k,n} & kj_{k,n-1} \end{bmatrix}$*

Proof.

$$R_k F_k^n = R_k \begin{bmatrix} J_{k,n+1} & kJ_{k,n} \\ J_{k,n} & kJ_{k,n-1} \end{bmatrix} \qquad \text{from Lemma 3.1}$$

$$= \begin{bmatrix} 1 & k \\ 1 & 2-k \end{bmatrix} \begin{bmatrix} J_{k,n+1} & kJ_{k,n} \\ J_{k,n} & kJ_{k,n-1} \end{bmatrix}$$

$$= \begin{bmatrix} J_{k,n+1} + kJ_{k,n} & k^2 J_{k,n-1} + kJ_{k,n} \\ J_{k,n+1} + (2-k)J_{k,n} & kJ_{k,n} + (2-k)kJ_{k,n-1} \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} j_{k,n+1} & kj_{k,n} \\ j_{k,n} & kj_{k,n-1} \end{bmatrix} \qquad \text{from Proposition 3.1}$$

□

Remark 1. $R_k F_k = F_k R_k$.

4. More properties and identities

In this section, we derive some identities for the generalized Jacobsthal and Jacobsthal-Lucas sequences.

Theorem 4.1 (Catalan's Identities)**.**

(1) $J_{k,n+r}J_{k,n-r} - J_{k,n}^2 = (-1)^{n-r}k^{n-r}J_{k,r}^2$

(2) $j_{k,n+r}j_{k,n-r} - j_{k,n}^2 = 8(-1)^{n-r}k^n(k-1)J_{k,r}^2$

Proof.

(1) Note

$$\begin{aligned} J_{k,n+r}J_{k,n-r} - J_{k,n}^2 &= \frac{k^{n+r} + (-1)^{n+r}}{k+1} \cdot \frac{k^{n-r} + (-1)^{n-r}}{k+1} - \left(\frac{k^n + (-1)^n}{k+1}\right)^2 \\ &= \frac{k^{2n} + (-1)^{n+r}k^{n-r} + (-1)^{n-r}k^{n+r} + 1}{(k+1)^2} - \left(\frac{k^n + (-1)^n}{k+1}\right)^2 \\ &= \frac{(-1)^{n+r}k^{n-r} + (-1)^{n-r}k^{n+r} + 2(-1)^n k^n}{(k+1)^2} \\ &= (-1)^n k^n \left(\frac{2 + (-1)^r k^{-r} + (-1)^r k^r}{(k+1)^2}\right) \end{aligned}$$

$$= \frac{(-1)^{n-r}k^n}{k^r}\left(\frac{2(-1)^r(k)^r+1+k^{2r})}{(k+1)^2}\right)$$
$$= (-1)^{n-r}k^{n-r}J_{k,r}^2$$

(2) Note

$$j_{k,n+r}j_{k,n-r} - j_{k,n}^2 = \frac{4k^{n+r}+2(k-1)(-1)^{n+r}}{k+1}\cdot\frac{4k^{n-r}+2(k-1)(-1)^{n-r}}{k+1} - j_{k,n}^2$$
$$= \frac{16k^{2n}+8k^{n-r}(k-1)(-1)^{n+r}+8k^{n+r}(k-1)(-1)^{n-r}+4(k-1)^2}{(k+1)^2} - j_{k,n}^2$$
$$= \frac{8k^{n-r}(k-1)(-1)^{n+r}+8k^{n+r}(k-1)(-1)^{n-r}-16(k-1)k^n(-1)^n}{(k+1)^2}$$
$$= 8(-1)^nk^n(k-1)\cdot\frac{(k^{-r}(-1)^r+k^r(-1)^{-r}-2)}{(k+1)^2}$$
$$= 8(-1)^nk^n(k-1)\cdot\frac{(-2+k^{-r}(-1)^r+k^r(-1)^{-r})}{(k+1)^2}$$
$$= 8(-1)^{n-r}k^n(k-1)\cdot\frac{k^{2r}-2(-1)^rk^r+1}{(k+1)^2}$$
$$= 8(-1)^{n-r}k^n(k-1)J_{k,r}^2$$

□

Theorem 4.2 (Cassini's identities)**.** *We have*

(1) $J_{k,n+1}J_{k,n-1} - J_{k,n}^2 = (-1)^nk^{n-1}$

(2) $j_{k,n+1}j_{k,n-1} - j_{k,n}^2 = 8(-1)^nk^{n-1}(1-k)$

Proof. Immediate from Lemma 3.1 and Lemma 3.3 □

Theorem 4.3 (d'Ocagne's Identity identities)**.** *Let n, m be two integers. Then*

(1) $J_{k,n}J_{k,m+1} - J_{k,n+1}J_{k,m} = (-1)^mk^mJ_{k,n-m}$

(2) $j_{k,n}j_{k,m+1} - j_{k,n+1}j_{k,m} = 8(-1)^m(1-k)k^mJ_{k,n-m}$

Proof. First using equation 2.1 in Theorem 2.2 we have

$$J_{k,n}J_{k,m+1} - J_{k,n+1}J_{k,m} = \frac{k^n-(-1)^n}{k+1}\frac{k^{m+1}-(-1)^{m+1}}{k+1} - \frac{k^m-(-1)^m}{k+1}\frac{k^{n+1}-(-1)^{n+1}}{k+1}$$

$$
\begin{aligned}
&= \frac{k^{n+m+1} - (-1)^n k^{m+1} - (-1)^{m+1} k^n + (-1)^{m+n+1}}{(k+1)^2} \\
&- \frac{k^{m+n+1} - (-1)^{n+1} k^m - (-1)^m k^{n+1} + (-1)^{m+n+1}}{(k+1)^2} \\
&= \frac{(-1)^{n+1} k^m - (-1)^n k^{m+1} + (-1)^m k^{n+1} - (-1)^{m+1} k^n}{(k+1)^2} \\
&= \frac{(-1)^n k^m (-1-k) + (-1)^m k^n (k+1)}{(k+1)^2} \\
&= \frac{(-1)^m k^n - (-1)^n k^m}{(k+1)} \\
&= (-1)^m k^m \left(\frac{k^{n-m} - (-1)^{n-m}}{(k+1)} \right) \\
&= (-1)^m k^m J_{k,n-m}
\end{aligned}
$$

Second using equation 2.2 in Theorem 2.2 we have

$$
\begin{aligned}
j_{k,n} j_{k,m+1} - j_{k,n+1} j_{k,m} &= \frac{8k^n(-1)^m + 8k^{m+2}(-1)^n - 8k^m(-1)^n - 8k^{n+2}(-1)^m}{(k+1)^2} \\
&= 8\frac{(-1)^m k^n (1-k^2) + (-1)^n k^m (k^2-1)}{(k+1)^2} \\
&= 8(1-k^2)\frac{(-1)^m k^n + (-1)^n k^m}{(k+1)^2} \\
&= 8(1-k^2)(-1)^m k^m \frac{k^{n-m} + (-1)^{n-m}}{(k+1)^2} \\
&= 8(-1)^m (1-k) k^m J_{k,n-m}
\end{aligned}
$$

□

Theorem 4.4. *For any two integers $m, n \geq 2$ we have*

(1) $J_{k,m+n} = J_{k,m} J_{k,n+1} + k J_{k,m-1} J_{k,n-1}$

(2) $j_{k,m+n} = j_{k,m} J_{k,n+1} + k j_{k,m-1} J_{k,n}$

Proof. (1)

$$F_k^{m+n} = \begin{bmatrix} J_{k,m+n+1} & kJ_{k,m+n} \\ J_{k,m+n} & kJ_{k,m+n-1} \end{bmatrix} = \begin{bmatrix} J_{k,m+1} & kJ_{k,m} \\ J_{k,m} & kJ_{k,m-1} \end{bmatrix} \begin{bmatrix} J_{k,n+1} & kJ_{k,n} \\ J_{k,n} & kJ_{k,n-1} \end{bmatrix}$$

thus $J_{k,m+n} = J_{k,m}J_{k,n+1} + kJ_{k,m-1}J_{k,n-1}$

(2) Similar to number 1

□

5. Sum of terms

Theorem 5.1. *For all integers $k \geq 2$ and $n \geq 0$ we have*

(1) $\sum_{i=0}^{n} J_{k,i} = \frac{1}{2(k-1)}(kJ_{k,n} + J_{k,n+1} - 1)$

(2) $\sum_{i=0}^{n} j_{k,i} = \frac{1}{2(k-1)}(kj_{k,n} + j_{k,n+1} + 2(k-3))$

Proof. We prove the first formula using mathematical induction

(1) For $n = 0$, the result is trivial.

(2) Assume that $\sum_{i=0}^{n} J_{k,i} = \frac{1}{2(k-1)}(kJ_{k,n} + J_{k,n+1} - 1)$

(3) Consider

$$\begin{aligned} \sum_{i=0}^{n+1} J_{k,i} &= \sum_{i=0}^{n} J_{k,i} + J_{k,n+1} \\ &= \frac{1}{2(k-1)}(kJ_{k,n} + J_{k,n+1} - 1) + J_{k,n+1} \qquad \text{by induction} \\ &= \frac{1}{2(k-1)}(kJ_{k,n} + J_{k,n+1} + 2(k-1)J_{k,n+1} - 1) \\ &= \frac{1}{2(k-1)}(kJ_{k,n+1} + (k-1)J_{k,n+1} + kJ_{k,n} - 1) \\ &= \frac{1}{2(k-1)}(kJ_{k,n+1} + J_{k,n+2} - 1) \end{aligned}$$

The second formula can be proved in a similar way to the first one so we omit its proof. □

Remark 2. *Theorem 5.1 also can be proved directly from the binet formula.*

Theorem 5.2. *For all integers $k \geq 2$ and $n \geq 0$ we have*

(1) $\sum_{i=0}^{n} J_{k,2i} = \frac{1}{(k+1)}(\frac{1}{k-1}J_{k,2n+2} - n - 1)$
(2) $\sum_{i=0}^{n} j_{k,2i+1} = \frac{1}{(k+1)}(\frac{k}{k-1}J_{k,2n+2} + n + 1)$
(3) $\sum_{i=0}^{n} j_{k,2i} = \frac{1}{(k+1)}(\frac{4}{k-1}J_{k,2n+2} + 2(k-1)(n+1))$
(4) $\sum_{i=0}^{n} j_{k,2i+1} = \frac{1}{(k+1)}(\frac{4k}{k-1}J_{k,2n+2} - 2(k-1)(n+1))$

Proof. We will prove the first and third equations. For the first one

$$\begin{aligned}
\sum_{i=0}^{n} J_{k,2i} &= \sum_{i=0}^{n} \frac{k^{2i} - (-1)^{2i}}{k+1} \\
&= \sum_{i=0}^{n} \frac{k^{2i} - 1}{k+1} \\
&= \frac{1}{k+1}\left(\sum_{i=0}^{n} (k^2)^i - (n+1)\right) \\
&= \frac{1}{k+1}\left(\frac{1-(k^2)^{n+1}}{1-k^2} - (n+1)\right) \\
&= \frac{1}{(k+1)}\left(\frac{1}{k-1}J_{k,2n+2} - n - 1\right)
\end{aligned}$$

For the third one

$$\begin{aligned}
\sum_{i=0}^{n} j_{k,2i} &= \sum_{i=0}^{n} \frac{4k^{2i} + 2(k-1)(-1)^{2i}}{k+1} \\
&= \sum_{i=0}^{n} \frac{4k^{2i} + 2(k-1)}{k+1} \\
&= \frac{1}{k+1}\left(4\sum_{i=0}^{n} (k^2)^i + 2(k-1)(n+1)\right) \\
&= \frac{1}{k+1}\left(4\frac{1-(k^2)^{n+1}}{1-k^2} + 2(k-1)(n+1)\right) \\
&= \frac{1}{(k+1)}\left(\frac{4}{k-1}J_{k,2n+2} + 2(k-1)(n+1)\right)
\end{aligned}$$

Proving the second and forth equations is similar. □

Remark 3. *Theorem 5.2 also can be proved by mathematical induction.*

6. Conclusion

We present a generalization for two well-known Lucas sequences, namely the Jacobsthal and Jacobsthal-Lucas sequences. Also, we find the matrix representation of both new sequences, these matrices can be used in coding theory and cryptography to create new codes and ciphers. Also they may be combined with some known codes or ciphers like [11, 12, 13, 15] *in order to improve their security levels and error detection/correction abilities.*

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