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ON SOME FACTOR RINGS AND THEIR CONNECTIONS WITH DERIVATIONS

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Abstract. Our purpose in this paper is to investigate certain central valued identities on a factor ring with respect to a prime ideal P of a ring R, involving a pair of derivations of R. Some well-known results characterizing commutativity of prime (semi-prime) rings have been generalized.

1. Introduction

Throughout this paper R will represent an associative ring with center $Z(R)$. Recall that a proper ideal P of R is said to be prime if for any $x, y \in R$, $xRy \subseteq P$ implies $x \in P$ or $y \in P$. The ring R is prime if and only if (0) is a prime ideal of R. Recall that R is a semi-prime ring if $xRx = (0)$ yields $x = 0$. R is said to be 2-torsion free if $2x = 0$ (where $x \in R$) implies $x = 0$. For any $x, y \in R$ the symbol $[x, y]$ will denote the commutator $xy - yx$; while the symbol $x \circ y$ will stand for the anti-commutator $xy + yx$. An additive subgroup L of R is said to be a Lie ideal of R if $[L, R] \subseteq L$. An additive mapping $d : R \longrightarrow R$ is a *derivation* if $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. In particular, for a fixed $a \in R$, the mapping $d_a : R \longrightarrow R$ given by $d_a(x) = [a, x]$ is a derivation called *inner derivation* defined by a. Recently, many results in literature indicate how the global structure of a ring R is often tightly connected to the behaviour of additive mappings defined on R (for example, see [3], [5], [8] and [16]). Herstein [7] showed that a 2-torsion free prime ring R must be a commutative integral domain if it admits a nonzero derivation d satisfying $[d(x), d(y)] = 0$ for all $x, y \in R$, and if the characteristic of R equals two, the ring R must be commutative

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or an order in a simple algebra which is 4-dimensional over its center. Several authors have proved commutativity theorems for prime rings admitting derivations which are centralizing on R. We begin recalling that a mapping $f : R \longrightarrow R$ is called centralizing on R if $[f(x), x] \in Z(R)$ for all $x \in R$. A well known result of Posner [15] states that if d is a derivation of the prime ring R such that $[d(x), x] \in Z(R)$, for any $x \in R$, then either $d = 0$ or R is commutative. Mayne [13] obtained the analogous result of Posner for non-identity centralizing automorphisms. In [9] Lanski generalizes the result of Posner to a Lie ideal.

More recently several authors considered similar situations in the case the derivation d is replaced by a generalized derivation. More specifically an additive map $F : R \longrightarrow R$ is said to be a generalized derivation if there exists a derivation d of R such that, for all $x, y \in R$, $F(xy) = F(x)y + xd(y)$. Basic examples of generalized derivations are the usual derivations on R and left R-module mappings from R into itself. An important example is a map of the form $F(x) = ax + xb$, for some $a, b \in R$; such generalized derivations are called *inner*. Generalized derivations have been primarily studied on operator algebras. Therefore any investigation from the algebraic point of view might be interesting (see for example [6], [10] and [12]).

During the last two decades, many authors have studied commutativity of prime and semi-prime rings admitting suitably constrained additive mappings acting on appropriate subsets of the rings. Moreover, many of the obtained results extend other ones proven previously just for the action of the considered mapping on the entire ring. In this direction, the recent literature contains numerous results on commutativity in prime and semi-prime rings admitting suitably constrained derivations and generalized derivations (for example, see [6], [10], [11] and [14]).

The present paper is motivated by the previous results and we continue this line of investigation by considering a generalization to any ring rather than prime rings. More precisely, we will establish a relationship between the structure of factor rings R/P and the behavior of its derivations satisfying algebraic identities involving prime ideals.

2. Main results

In what follows, $\bar{x}$ for x in R denotes $x+P$ in R/P. We will use frequently the following Lemma which is very crucial for developing the proofs of our main results.

Lemma 2.1 ([1], Theorem 2.2)**.** *Let R be a ring and P be a prime ideal of R. If d is a derivation of R satisfying $\overline{[d(x),x]} \in Z(R/P)$ for all $x \in R$, then $d(R) \subseteq P$ or R/P is a commutative integral domain.*

A classical result of Posner [15] states that the existence of a derivation d of a prime ring R such that $[d(x),x] \in Z(R)$ for all $x \in R$, implies that either $d=0$ or R is commutative.
Our fundamental aim is to generalize this result in two ways. First of all, we will assume that the above algebraic identities belongs to $Z(R/P)$ where P is any prime ideal rather than the zero ideal. Secondly, we will treat more general differential identities involving two derivations. More precisely we will prove the following result.

Theorem 2.1. *Let R be a ring and P be a prime ideal of R. If d and g are derivations of R satisfying*

$$\overline{d(x)x - xg(x)} \in Z(R/P) \quad \text{for all } x \in R,$$

then ($d(R) \subseteq P$ and $g(R) \subseteq P$) or R/P is a commutative integral domain.

Proof. We are given that

$$\overline{d(x)x - xg(x)} \in Z(R/P) \quad \text{for all } x \in R. \tag{2.1}$$

If $Z(R/P) = \{\overline{0}\}$, then R/P is a non-commutative ring and thus our hypothesis becomes

$$d(x)x - xg(x) \in P \quad \text{for all } x \in R. \tag{2.2}$$

Linearizing this relation, we obtain

$$d(x)y + d(y)x - xg(y) - yg(x) \in P \quad \text{for all } x,y \in R. \tag{2.3}$$

Substituting yx for y in (2.3), we find that

$$d(x)yx + d(y)x^2 + yd(x)x - xg(y)x - xyg(x) - yxg(x) \in P \quad \text{for all } x,y \in R. \tag{2.4}$$

Right multiplying (2.3) by x and then subtracting it from (2.4), we obviously see that

$$yd(x)x - xyg(x) - yxg(x) + yg(x)x \in P \quad \text{for all} \quad x, y \in R. \tag{2.5}$$

Using the fact that $d(x)x - xg(x) \in P$, then the above relation reduces to

$$yg(x)x - xyg(x) \in P \quad \text{for all} \quad x, y \in R.$$

Writing ry instead of y in the above expression and applying it, we thereby obtain

$$[x, r]Rg(x) \subseteq P \quad \text{for all} \quad r, x \in R. \tag{2.6}$$

In light of primeness of P, and since R/P is a non-commutative ring, we get $g(R) \subseteq P$ and so the relation (2.3) yields

$$d(x)y + d(y)x \in P \quad \text{for all} \quad x, y \in R. \tag{2.7}$$

Putting yr instead of y in (2.7) and using (2.7), it is easy to see that

$$d(y)[r, x] + yd(r)x \in P \quad \text{for all} \quad r, x, y \in R.$$

Replacing x by xt in the above relation and applying it, we obviously obtain

$$d(y)R[r, t] \subseteq P \quad \text{for all} \quad r, t, y \in R.$$

Since P is prime and R/P is a non-commutative ring, we conclude that $d(R) \subseteq P$. Now assuming that $Z(R/P) \neq \{\overline{0}\}$, then there exists $z \in R - P$ such that $\overline{z} \in Z(R/P)$ and linearizing our hypothesis, we obviously obtain

$$\overline{d(x)y + d(y)x - xg(y) - yg(x)} \in Z(R/P) \quad \text{for all} \quad x, y \in R. \tag{2.8}$$

Replacing y by yz in the above expression, to get

(2.9)

$$\overline{(d(x)y + d(y)x - xg(y) - yg(x))z + yd(z)x - xyg(z)} \in Z(R/P) \quad \text{for all} \quad x, y \in R.$$

Commuting the last equation with r and using (2.8), it is easy to see that

$$[yd(z)x - xyg(z), r] \in P \quad \text{for all} \quad r, x, y \in R. \tag{2.10}$$

Writing ry for y in this relation and applying it, we get $[ryd(z)x - xryg(z), r] \in P$. Combining the two last relations, it follows that

$$[x, r]yg(z)r - r[x, r]yg(z) \in P \quad \text{for all} \quad r, x, y \in R. \tag{2.11}$$

Substituting tx for x in (2.11) and using (2.10), to get

$$t[x,r]yg(z)r - rt[x,r]yg(z) \in P \quad \text{for all} \;\; r,t,x,y \in R.$$

Combining the last relation with (2.10), we thereby obtain $[t,r][x,r]g(z) \in P$, in such a way that

$$[t,r]R[x,r]Rg(z) \subseteq P \quad \text{for all} \;\; r,t,x \in R.$$

In light of primeness of P, we get either R/P is commutative or $g(z) \in P$. By the later case the relation (2.10) reduces to

$$[yd(z)x,r] \in P \quad \text{for all } r,x,y \in R.$$

Now replacing y by sy in the last relation and using it, we can see that

$$[s,r]yd(z)x \in P \quad \text{for all} \;\; r,s,x,y \in R.$$

As a special case of the above relation when we put $x = xd(z)$ we may write

$$[s,r]Rd(z)Rd(z) \in P \text{ for all } r,s \in R.$$

The primeness of P forces that either R/P is commutative or $d(z) \in P$. By the last case putting $y = z$ in (2.8), we arrive at $\overline{(d(x)-g(x))z} \in Z(R/P)$. Because of $\overline{z} \neq \overline{0}$, we get $\overline{d(x)-g(x)} \in Z(R/P)$, and thus $[D(x),x] \in P$, where $D = d - g$ is a derivation of R. According to Lemma (2.1), we get either $D(R) \subseteq P$ or R/P is commutative. So the relation (2.1) can be rewritten as

$$\overline{D(x)x + [g(x),x]} \in Z(R/P) \text{ for all } x \in R.$$

Accordingly

$$\overline{[g(x),x]} \in Z(R/P) \text{ for all } x \in R.$$

According to Lemma (2.1), either $g(R) \subseteq P$ or R/P is commutative. By the first case, using the fact that $d(R) - g(R) \subseteq P$, we conclude that $d(R) \subseteq P$ and this completes our proof. □

Remark 1. *Using the same techniques as in the preceding proof with necessary variations, it is obvious to see that* $\overline{d(x)x + xg(x)} \in Z(R/P)$ *for all* $x \in R$ *implies that either* ($d(R) \subseteq P$ *and* $g(R) \subseteq P$) *or* R/P *is a commutative integral domain.*

By letting R to be prime, then (0) is a prime ideal of R. Then we obtain the following corollary which is a generalization of the classical result of Posner [[15],Theorem 2].

Corollary 2.1. *Let R be a prime ring. If d and g are derivations of R satisfying the condition $d(x)x \pm xg(x) \in Z(R)$ for all $x \in R$, then R is a commutative integral domain or $d = g = 0$.*

Application of Theorem 2.1 yields the famous result of Posner [[15],Theorem 2].

Corollary 2.2. *Let R be a prime ring. If d is a nonzero derivation of R satisfying the condition $[d(x), x] \in Z(R)$ for all $x \in R$, then R is a commutative integral domain.*

We can easily prove the following result which is an application of Remark 1.

Corollary 2.3. *Let R be a prime ring. If d is a nonzero derivation of R satisfying the condition $d(x) \circ x \in Z(R)$ for all $x \in R$, then R is a commutative integral domain.*

By application of the proof of Theorem 2.1, we deduce the following proposition.

Proposition 2.1. *Let R be a semi-prime ring. If d and g are nonzero derivations of R such that $d(x)x - xg(x) = 0$ for all $x \in R$, then R contains a nonzero central ideal.*

Proof. Assume that $d(x)x - xg(x) = 0$ for all $x \in R$. In view of semi-primeness of R, there exists a family Γ of prime ideals such that $\bigcap_{P \in \Gamma} P = (0)$. Thereby obtaining $d(x)x - xg(x) \in P$ for all $x \in R$ and for all $P \in \Gamma$. Invoking the proof of Theorem 2.1 by relation (2.6), we find that $[x, r]Rg(x) \subseteq P$ for all $x, r \in R$, in such a way that $[g(x), r]R[g(x), r] \subseteq P$ for all $x, r \in R$ and for all $P \in \Gamma$. Therefore

$$[g(x), r]R[g(x), r] = 0 \quad \text{for all} \quad r, x \in R.$$

According to semi-primeness of R we obviously obtain $[g(x), r] = 0$ for all $r, x \in R$. In view of [[4], Theorem 3], we conclude that R contains a nonzero central ideal. □

In [[2], Theorems 4.4], it is proved that if R is a 2-torsion free prime ring and I is a nonzero ideal of R with a derivation d which satisfies $d(x) \circ d(y) - x \circ y = 0$ for all $x, y \in I$, then R must be commutative.

Motivated by the above result we investigate a more general context of differential identities involving a prime ideal by omitting the primeness assumption imposed on the ring. This approach allows us to generalize the preceding result in two directions. First of all, we will assume that $\overline{d(x) \circ d(y) - x \circ y}$ belongs to the center of the ring R/P rather than $d(x) \circ d(y) - x \circ y = 0$. Secondly we will investigate the behavior of the more general expression $\overline{d(x) \circ g(y) - x \circ y}$ involving two derivations d and g instead of the expression $d(x) \circ d(y) - x \circ y$. Our result consist to characterize the structure of the ring R/P. More precisely we will prove the following result.

Theorem 2.2. *Let R be a ring, P be a prime ideal of R, and the ring R/P be 2-torsion free. If d and g are derivations of R, then the following assertions are equivalent:*
(1) $\overline{d(x) \circ g(y) - x \circ y} \in Z(R/P)$ *for all* $x, y \in R$
(2) $\overline{[d(x), g(y)] - x \circ y} \in Z(R/P)$ *for all* $x, y \in R$
(3) R/P *is a commutative integral domain.*

Proof. It is obvious that (3) implies both (1) and (2). So we need to prove (1) $\Longrightarrow$ (3) and (2) $\Longrightarrow$ (3).
(1) $\Longrightarrow$ (3) Assume that

$$\overline{d(x) \circ g(y) - x \circ y} \in Z(R/P) \quad \text{for all } x, y \in R. \tag{2.12}$$

If $Z(R/P) = \{\overline{0}\}$, then R/P is a non-commutative ring. Accordingly the relation (2.12) reduces to

$$d(x) \circ g(y) - x \circ y \in P \quad \text{for all } x, y \in R. \tag{2.13}$$

Writing yr instead of y in the last expression and using it, we can see that

$$y(d(x) \circ g(r)) - g(y)[d(x), r] + [d(x), y]g(r) + y[x, r] \in P \quad \text{for all } r, x, y \in R. \tag{2.14}$$

Replacing y by sy in (2.14), and invoking (2.14), we find that

$$g(s)y[d(x), r] + [d(x), s]yg(r) \in P \quad \text{for all } r, s, x, y \in R. \tag{2.15}$$

As a special case of (2.15) when we put $r = d(x)$, we may write

$$[d(x), s]yg(d(x)) \in P \quad \text{for all } s, x, y \in R. \tag{2.16}$$

In light of primeness of P, we get $[d(x), s] \in P$ or $g(d(x)) \in P$ for all $s, x \in R$, which implies that $R = R_1 \cup R_2$ with $R_1 = \{x \in R \ / \ [d(x), s] \in P$ for all $s \in R\}$ and $R_2 = \{x \in R \ / \ g(d(x)) \in P\}$. Since a group cannot be the union of its subgroups, we have $R = R_1$ in which applying Lemma (2.1), we can see that $d(R) \subseteq P$. So the relation (2.13) reduces to $x \circ y \in P$ for all $x, y \in R$ and R/P is a 2-torsion free. Hence $xy \in P$ for all $x, y \in R$ therefore $R = P$ (which is absurd) or $R = R_2$. In this case $g(d(x)) \in P$ for all $x \in R$ and replacing y by $d(y)$ in (2.13), we obviously get $x \circ d(y) \in P$ for all $x, y \in R$. Substituting yx for x in the above expression and using it, we get $[y, d(y)]x \in P$ for all $x, y \in R$. If we put $x = x[y, d(y)]$ in this relation we find that $[y, d(y)]R[y, d(y)] \subseteq P$. Invoking primeness of P, we conclude that $[y, d(y)] \in P$. Using Lemma (2.1) and since R/P is a non-commutative ring, we obtain $d(R) \subseteq P$. Therefore relation (2.13) becomes $x \circ y \in P$ for all $x, y \in R$ which implies that $R = P$ impossible.

Now suppose that $Z(R/P) \neq \{\overline{0}\}$, then there exists $z \in R - P$ such that $\overline{z} \in Z(R/P)$ and substituting yz for y in our hypothesis and using it, we obviously get

$$\overline{y(d(x) \circ g(z)) + [d(x), y]g(z)} \in Z(R/P) \quad \text{for all } x, y \in R. \tag{2.17}$$

Replacing y by ry in the last expression, we arrive at

$$\overline{ry(d(x) \circ g(z)) + r[d(x), y]g(z) + [d(x), r]yg(z)} \in Z(R/P) \quad \text{for all } r, x, y \in R. \tag{2.18}$$

Since $\overline{y(d(x) \circ g(z)) + [d(x), y]g(z)} \in Z(R/P)$ by relation (2.17), and commuting equation (2.18) with r, one can verify that

$$[[d(x), r]yg(z), r] \in P \quad \text{for all } r, x, y \in R. \tag{2.19}$$

This relation can be rewritten as

$$[d(x), r]y[g(z), r] + [[d(x), r]y, r]g(z) \in P \quad \text{for all } r, x, y \in R. \tag{2.20}$$

Putting $y = yg(z)$ in (2.20), and applying (2.19), it is easy to see that

$$[d(x), r]yg(z)[g(z), r] \in P \quad \text{for all } r, x, y \in R. \tag{2.21}$$

Using primeness of P, we get either $[d(x), r] \in P$ for all $r, x \in R$ or $g(z)[g(z), r] \in P$ for all $r \in R$.

Let us set $J = \{r \in R \mid [d(x), r] \in P \text{ for all } x \in R\}$ and $K = \{r \in R \mid g(z)[g(z), r] \in P\}$. Clearly J and K are additive subgroups of R such that $R = J \cup K$. According to Brauer's trick, it follows that either $R = J$ or $R = K$.

Assume that $R = J$, then $[d(x), r] \in P$. Invoking Lemma (2.1), we get $d(R) \subseteq P$ or R/P is commutative. By the first case relation (2.12) reduces to $\overline{x \circ y} \in Z(R/P)$. Therefore R/P is commutative. Now if $R = K$ then $g(z)[g(z), r] \in P$. Accordingly $[g(z), r]R[g(z), r] \subseteq P$. In view of primeness of P, we thereby obtain $\overline{g(z)} \in Z(R/P)$. On the other hand, putting $y = z$ in (2.17) and as R/P is a 2-torsion free ring, we find that

$$\overline{zd(x)g(z)} \in Z(R/P) \quad \text{for all } x \in R. \tag{2.22}$$

Since $\overline{z} \neq \overline{0}$, we have $\overline{d(x)g(z)} \in Z(R/P)$. Hence $\overline{d(x)} \in Z(R/P)$ or $\overline{g(z)} = \overline{0}$. By the later case replacing y by z in our hypothesis, it follows that $\overline{x} \in Z(R/P)$ for all $x \in R$, then R/P is commutative. Now the first case yields that $[d(x), x] \in P$ which implies that either R/P is commutative or $d(R) \subseteq P$. By the second case the relation (2.12) reduces to $\overline{x \circ y} \in Z(R/P)$ for all $x, y \in R$. Finally we conclude that R/P is commutative.

(2) $\Longrightarrow$ (3) We are given that

$$\overline{[d(x), g(y)] - x \circ y} \in Z(R/P) \quad \text{for all } x, y \in R. \tag{2.23}$$

Assuming that $Z(R/P) = \{\overline{0}\}$, then R/P is a non-commutative ring and therefore our hypothesis yields

$$[d(x), g(y)] - x \circ y \in P \quad \text{for all } x, y \in R. \tag{2.24}$$

Putting yr instead of y in the above expression and using it, we obtain

$$g(y)[d(x), r] + y[d(x), g(r)] + [d(x), y]g(r) + y[x, r] \in P \quad \text{for all } r, x, y \in R. \tag{2.25}$$

Replacing y by sy in (2.24) and using (2.24) imply that

$$g(s)y[d(x), r] + [d(x), s]yg(r) \in P \quad \text{for all } r, s, x, y \in R. \tag{2.26}$$

As a special case of (2.26) when we put $s = d(x)$, we may write

$$g(d(x))R[d(x), r] \subseteq P \quad \text{for all } r, x \in R. \tag{2.27}$$

By view of primeness hypothesis, we get either $g(d(x)) \in P$ or $[d(x), r] \in P$. Assume that $[d(x), r] \in P$ and according to Lemma (2.1), the non-commutativity of R/P forces $d(R) \subseteq P$. Then relation (2.24) becomes $x \circ y \in P$ for all $x, y \in R$. Therefore $R = P$ which is absurd.

Now suppose that $g(d(x)) \in P$ for all $x \in R$. Replacing y by $d(r)$ in (2.24), we can see that $x \circ d(r) \in P$ for all $r, x \in R$. Substituting rx for x in the above expression, we get $[r, d(r)]x \in P$, and thus putting $x = x[r, d(r)]$ in the last relation, we obtain $[r, d(r)]R[r, d(r)] \subseteq P$. Using primeness of P, we can see that $[r, d(r)] \in P$. According to Lemma (2.1), and since R/P is non-commutative we obtain $d(R) \subseteq P$. Hence relation (2.24) becomes $x \circ y \in P$, and so $R = P$ which is absurd.

Accordingly, we need to assume that $Z(R/P) \neq \{\overline{0}\}$ in which case there exists $\overline{z} \in Z(R/P)$ such that $\overline{z} \neq \overline{0}$. Substituting yz for y in our hypothesis and using it, one can see that

$$\overline{y[d(x), g(z)] + [d(x), y]g(z)} \in Z(R/P) \quad \text{for all } x, y \in R. \tag{2.28}$$

Putting ry instead of y in the last expression and applying it, we find that

$$[[d(x), r]yg(z), r] \in P \quad \text{for all } r, x, y \in R. \tag{2.29}$$

Since the relation (2.29) is the same as expression (2.19), reasoning in the same manner as above, we arrive at R/P is commutative or $\overline{g(z)} \in Z(R/P)$. By the second case putting $y = z$ in our hypothesis, we can see that $\overline{2xz} \in Z(R/P)$ and because R/P is a 2-torsion free and $\overline{z} \neq \overline{0}$, we conclude that $\overline{x} \in Z(R/P)$ for all $x \in R$ which implies that R/P is commutative and this completes our proof. □

Remark 2. *Reasoning in the same manner as in the preceding proof, it is easy to see the equivalence between the following assertions:*

(1) $\overline{d(x) \circ g(y) + x \circ y} \in Z(R/P)$ *for all* $x, y \in R$

(2) $\overline{[d(x), g(y)] + x \circ y} \in Z(R/P)$ *for all* $x, y \in R$

(3) R/P *is a commutative integral domain.*

As an application of Theorem (2.2), the following corollary gives an improved version of [[2], Theorems 4.4].

Corollary 2.4. *Let R be a 2-torsion free prime ring. If d and g are derivations of R, then the following assertions are equivalent:*
(1) $d(x) \circ g(y) \pm x \circ y \in Z(R)$ *for all* $x, y \in R$
(2) $[d(x), g(y)] \pm x \circ y \in Z(R)$ *for all* $x, y \in R$
(3) *R is a commutative integral domain.*

Using similar arguments used in Theorem (2.2) with necessary variations we can prove the following result even without the characteristic assumption on the ring R/P.

Theorem 2.3. *Let R be a ring and P be a prime ideal of R. If d and g are derivations of R, then the following assertions are equivalent:*
(1) $\overline{[d(x), g(y)] \pm xy} \in Z(R/P)$ *for all* $x, y \in R$
(2) $\overline{[d(x), g(y)] \pm yx} \in Z(R/P)$ *for all* $x, y \in R$
(3) *R/P is a commutative integral domain.*
Moreover, if R/P is 2-torsion free and either $\overline{d(x) \circ g(y) \pm xy} \in Z(R/P)$ or $\overline{d(x) \circ g(y) \pm yx} \in Z(R/P)$ for all $x, y \in R$, then R/P is a commutative integral domain.

As an application of Theorem (2.3), the following corollary gives a commutativity criteria for semi-prime ring.

Corollary 2.5. *Let R be a semi-prime ring. If d and g are derivations of R, then the following assertions are equivalent:*
(1) $[d(x), g(y)] \pm xy \in Z(R)$ *for all* $x, y \in R$
(2) $[d(x), g(y)] \pm yx \in Z(R)$ *for all* $x, y \in R$
(3) *R is commutative.*

Proof. For the non-trivial implication assume that $[d(x), g(y)] - xy \in Z(R)$ for all $x, y \in R$. By view of the semi-primeness of the ring R, there exists a family Γ of prime ideals such that $\bigcap_{P \in \Gamma} P = (0)$, thereby obtaining $[[d(x), g(y)] - xy, r] \in P$ for all $P \in \Gamma$ and for all $r, x, y \in R$. Invoking Theorem 2.3 we conclude that R/P is commutative for all $P \in \Gamma$ which, because of $\bigcap_{P \in \Gamma} P = (0)$, assures that R is commutative. Similarly, if $[d(x), g(y)] - yx \in Z(R)$ for all $x, y \in R$, then the same reasoning proves that R is commutative. $\square$

The next corollary is an immediate consequence of Theorem (2.3).

Corollary 2.6. *Let R be a 2-torsion free prime ring. If d and g are derivations of R, then the following assertions are equivalent :*

(1) $d(x) \circ g(y) \pm xy \in Z(R)$ *for all* $x, y \in R$

(2) $d(x) \circ g(y) \pm yx \in Z(R)$ *for all* $x, y \in R$

(3) *R is a commutative integral domain.*

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