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APPROXIMATE SOLUTION OF FRACTIONAL ALLEN-CAHN EQUATION BY THE MITTAG-LEFFLER TYPE KERNELS

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Abstract. This study presents the approximate analytic solution of the fractional Allen-–Cahn equation involving fractional-order derivatives with the Mittag-Leffler type kernels. The fractional derivative contains three parameters that can adjust the model. We utilize the homotopy analysis method (HAM) to generate the solution of the fractional differential equations. The effect of the fractional parameters on the solution behaviors is studied, and the approximate analytical solution of the fractional Allen-–Cahn equation has been acquired successfully. Numerical results are given through graphs and tables. Since the exact solution of the obtained differential equation is unknown, we calculate the residual error to demonstrate the algorithm's efficiency.

1. Introduction

Fractional calculus is an extension of ordinary calculus and was established more than three hundred years. Recently, fractional calculus has become an interesting mathematical tool for researchers in different areas such as finance, physics, chemistry, biology, and engineering [1]. Extensive efforts have been made to find a solution to fractional differential equations by suggesting several numerical and analytical methods that are strong and stable. These methods are based on Laplace and Fourier transforms, finite difference and finite element methods, the convergent series generated by differential transform method (DTM)[2], Adomian decomposition method

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(ADM)[3], variational iteration method (VIM)[4], iterative natural transform method [5], collocation method [6], homotopy methods [7, 8], and its modifications.

In 1992, Shijun Liao proposed an approximate analytic solution for linear and nonlinear differential equations called the homotopy analysis method (HAM). The method is widely used for solving strongly nonlinear ODEs and PDEs problems in several sciences and engineering, especially those without small/large physical parameters. One of the most important features of using HAM is that the solution via HAM contains a convergent control parameter that can adjust the convergence region of the series solution.

One of the most effective approaches for solving fractional differential equations is the HAM and its modifications. HAM generates a series solution whose convergence is determined by a convergent control parameter, and the series can be expressed using a variety of basis functions [8].

Fractional calculus with nonsingular kernels become one of the most used derivatives because the singularities are believed to be troublesome especially when these operators are applied to model some physical phenomena. Several applications of the nonsingular fractional derivative have been investigated such as Caputo–Fabrizio [9, 10], Yang-Abdel-Aty-Cattani [11], and Atangana and Baleanu [12, 13, 14].

Fractional derivative with Mittage-Leffler function kernel was introduced by Atangana and Baleanu [15]. One of the limitations of this kind of derivative is that easy fractional differential equations $D^{\alpha}y(t) + y(t) = 0, y(0) = 0$ have only a trivial solution. In 2018 Abdeljawad and Baleanu [16] extended the definition by replacing the kernel with the Mittage-Leffler function with three parameters that idea can overcome some of the limitations of the previous definition. Several applications have been adapted via this generalization such as Srivastava et. al. [17] applying the definition for some dynamical models, and Alomari et. al. solving the fractional Parabolic equation [18] using the generalized definition.

The Allen-Cahn equation is a nonlinear parabolic partial differential equation representing some natural physical phenomenon. This equation has been extensively used to study various physical problems, such as crystal growth, image segmentation, and motion by mean curvature flows. Moreover, it is a mathematical model to study the phase separation process in binary alloys and emerged as a convection-diffusion equation in fluid dynamics or reaction-diffusion equation in material sciences. In particular, it has become a basic model equation for the diffuse interface approach developed to study phase transitions and interfacial dynamics in material science [19].

This paper aims to provide an accurate approximate analytic solution of the time-fractional Allen-Cahn equation in which the fractional derivative contains three parameters, and the kernel is the Mittage-Leffler function which is formulated as

$$ {}_0^{ABC}D^{\alpha,\mu,\gamma}u(x,t) = \varepsilon u_{xx}(x,t) + u(x,t) - u^3(x,t), \qquad x \in [-1,1], t > 0, \tag{1.1} $$

where ${}_0^{ABC}D^{\alpha,\mu,\gamma}$ is the generalized ABC fractional derivative with three parameters.

2. Generalized AB fractional derivative

The standard AB fractional integral and derivative are based on Mittag-Leffler Kernel with one parameter, while its generalization depends on three parameters. Some definitions and properties of this generalization will be presented in this section.

Abdeljawad and Baleanu [16] define the fractional integrals operator of two parameters

$$ (a^{AB}I^{\alpha,\mu}u)(t) = \frac{1-\alpha}{M(\alpha)}({}_aI^{1-\mu}u)(t) + \frac{\alpha}{M(\alpha)}({}_aI^{1-\mu+\alpha}u)(t) \tag{2.1} $$

for the left of the interval $[a,b]$, and for the right as

$$ ({}^{AB}I_b^{\alpha,\mu}u)(t) = \frac{1-\alpha}{M(\alpha)}(I_b^{1-\mu}u)(t) + \frac{\alpha}{M(\alpha)}(I_b^{1-\mu+\alpha}u)(t), \tag{2.2} $$

where $\alpha > 0, \mu \leq 1$, $({}_aI^{\alpha}u)(t)$ is the left Riemann fractional integrals and $(I_b^{\alpha}u)(t)$ is the right one.

If $\gamma = 1, 2, 3, \cdots$, the AB fractional integrals of order $\alpha > 0$, $\mu \leq 1$ can be written as

$$ (a^{AB}I^{\alpha,\mu,\gamma}u)(t) = \sum_{i=0}^{\gamma} \binom{\gamma}{i} \frac{\alpha^i}{M(\alpha)(1-\alpha)^{i-1}}({}_aI^{\alpha i+1-\mu}u)(t) \tag{2.3} $$

and

$$({}^{AB}I_b^{\alpha,\mu,\gamma}u)(t) = \sum_{i=0}^{\gamma}\binom{\gamma}{i}\frac{\alpha^i}{M(\alpha)(1-\alpha)^{i-1}}(I_b^{\alpha i+1-\mu}u)(t). \tag{2.4}$$

We noted that

$$(a^{AB}I^{\alpha,1}u)(t) = (a^{AB}I^{\alpha}u)(t)$$

and

$$({}^{AB}I_b^{\alpha,1}u)(t) = ({}^{AB}I_b^{\alpha}u)(t).$$

Definition 2.1. The fractional derivative with three parameters of the kernel $E^{\gamma}_{\alpha,\mu}(\lambda,t)$ from the left of the interval $[a,b]$ is defined by

$$\begin{aligned}(a^{ABC}D^{\alpha,\mu,\gamma}f)(x) &= \frac{M(\alpha)}{1-\alpha}\int_a^x E^{\gamma}_{\alpha,\mu}(\lambda,x-t)f'(t)\mathrm{d}t,\\ &= \frac{M(\alpha)}{1-\alpha}E^{\gamma}_{\alpha,\mu}(\lambda,x-a)*f'(x).\end{aligned} \tag{2.5}$$

The right one by

$$({}^{ABC}D_b^{\alpha,\mu,\gamma}f)(x) = \frac{-M(\alpha)}{1-\alpha}\int_x^b E^{\gamma}_{\alpha,\mu}(\lambda,x-t)f'(t)\mathrm{d}t, \tag{2.6}$$

where $\alpha \in (0,1)$, $\mu > 0$, γ is real number, $\lambda = \frac{-\alpha}{1-\alpha}$, and $E^{\gamma}_{\alpha,\beta}(z)$ is the generalized Mittag–Leffler function of three parameters

$$E^{\gamma}_{\alpha,\beta}(z) = \sum_{k=0}^{\infty}(\gamma)_k\frac{z^k}{k!\Gamma(\alpha k+\beta)}, \tag{2.7}$$

where $(\gamma)_k$ is the Pochhammer symbol, defined by

$$\begin{aligned}(\gamma)_k &= \frac{\Gamma(\gamma+k)}{\Gamma(\gamma)}\\ &= \begin{cases} 1 & ,(k=0, 0\neq\gamma\in\mathbb{C})\\ \gamma(\gamma+1)\cdots(\gamma+k-1) & ,(k\in\mathbb{N},\gamma\in\mathbb{C}).\end{cases}\end{aligned} \tag{2.8}$$

Theorem 2.1. *If $\alpha \in (0,1)$, $\mu > 0$, $\gamma \in \mathbb{N}$, then*

$$\begin{aligned}(a^{AB}I^{\alpha,\mu,\gamma}a^{ABC}D^{\alpha,\mu,\gamma}f)(x) &= f(x) - f(a)\sum_{k=0}^{\gamma}(-1)^k\lambda^k E^{\gamma}_{\alpha,\alpha k+1}(\lambda,x-a)\\ &= f(x)-f(a).\end{aligned} \tag{2.9}$$

and

$$
\begin{aligned}
({}^{AB}I_b^{\alpha,\mu,\gamma})({}^{ABC}D_b^{\alpha,\mu,\gamma}f)(x) &= f(x) - f(b)\sum_{k=0}^{\gamma}(-1)^k\lambda^k E^{\gamma}_{\alpha,\alpha k+1}(\lambda, b-x) \\
&= f(x) - f(b). \qquad (2.10)
\end{aligned}
$$

Definition 2.2. If $u(x,t)$ is approximate analytic solution for the fractional differential equation $0^{ABC}D_t^{\alpha,\mu,\gamma}u(x,t) - N[u(x,t)] = 0$, then its residual error can be defined as

$$
Res(x,t) = 0^{ABC}D_t^{\alpha,\mu,\gamma}u(x,t) - N[u(x,t)], \qquad (2.11)
$$

and the average residual error function

$$
\zeta(\hbar) = \frac{1}{(N_1+1)(N_2+1)}\sum_{i=0}^{N_1}\sum_{j=0}^{N_2} Res^2(x_i, t_j), \qquad (2.12)
$$

where $x_i = \frac{iL_1}{N_1}$, $t_j = \frac{jK}{N_2}$, L_1 is the endpoint of space along x and K is the endpoint of time.

3. Solution by HAM

In this section, we give a general frame-work for solving time-fractional partial differential equations using HAM. For that, Let $0 < \alpha < 1$, $\mu > 0$ $\gamma = 1, 2, 3\ldots$, for the following differential equation

$$
0^{ABC}D_t^{\alpha,\mu,\gamma}u(x,t) = N[u(x,t)], \qquad (3.1)
$$

with initial condition:

$$
u(x,0) = f(x),
$$

where $u(x,t)$ is an unknown function with independent parameters x and t. N is a linear or non-linear operator.

Firstly, a homotopy map can be defined as:

$$
(1-q)L[\phi(u(x,t);q) - u_0(x,t)] = \hbar q(0^{ABC}D_t^{\alpha,\mu,\gamma}\phi(u(x,t);q) - N[\phi(u(x,t);q)]), \qquad (3.2)
$$

where $q \in [0,1]$ is an embedding parameter, $\hbar$ is a nonzero convergent control parameter, L is an auxiliary linear operator, $u_0(x,t)$ is an initial solution approximation, and $\phi(u(x,t);q)$ is an unknown function. When $q=0$ and $q=1$, we have

$$\phi(u(x,t);0) = u_0(x,t), \qquad \phi(u(x,t);1) = u(x,t). \tag{3.3}$$

So that, when q increases from 0 to 1, $\phi(u(x,t);q)$ varies from the initial guess $\phi(u(x,t);0)$ to the exact solution $\phi(u(x,t);1)$. For succinctness, equation (3.2) is called the zero-order deformation equation.

When we use HAM, we have the freedom to choose the auxiliary linear operator L, the initial approximation $u_0(x,t)$, and the convergent control parameter $\hbar$. If all of them are properly chosen, then the solution $\phi(u(x,t);q)$ of the zero-order deformation equation (3.2) exists for $0 \le q \le 1$. Define the i-th-order derivative of $\phi(u(x,t);q)$ with respect to the embedding parameter q at $q=0$ as

$$u^{[i]}(x,t) = \frac{\partial^i \phi(u(x,t);q)}{\partial q^i} \mid_{q=0}, \tag{3.4}$$

where $i \in \mathbb{N}$ and, $u^{[i]}(x,t)$ is called the ith-order deformation derivative. Define

$$u_i(x,t) = \frac{u^{[i]}(x,t)}{i!} = \frac{1}{i!}\frac{\partial^i \phi(u(x,t);q)}{\partial q^i} \mid_{q=0} . \tag{3.5}$$

Expanding, $\phi(u(x,t);q)$ in Taylor's series with respect to q, we have

$$\phi(u(x,t);q) = \phi(u(x,t);0) + \sum_{i=1}^{\infty} \frac{1}{i!}\frac{\partial^i \phi(u(x,t);q))}{\partial q^i} \mid_{q=0} q^i. \tag{3.6}$$

Form equation (3.3) and (3.5), the above power series can be written as:

$$\phi(u(x,t);q) = u_0(x,t) + \sum_{i=1}^{\infty} u_i(x,t) q^i. \tag{3.7}$$

Substitute the value of $\phi(u(x,t);q)$ into equation (3.2), we get

$$(1-q)L[\sum_{i=1}^{\infty} u_i q^i] = \hbar(0^{ABC}D_t^{\alpha,\mu,\gamma} \sum_{i=0}^{\infty} u_i q^{i+1} - qN[\sum_{i=0}^{\infty} u_i q^i]). \tag{3.8}$$

By equating like powers of q from both sides in Eq.(3.8), we get

$$\begin{aligned}
q^1 : L[u_1(x,t) - 0] &= \hbar(0^{ABC}D_t^{\alpha,\mu,\gamma}u_0(x,t) - R_1), \\
q^2 : L[u_2(x,t) - u_1(x,t)] &= \hbar(0^{ABC}D_t^{\alpha,\mu,\gamma}u_1(x,t) - R_2), \\
&\vdots \\
q^n : L[u_n(x,t) - u_{n-1}(x,t)] &= \hbar(0^{ABC}D_t^{\alpha,\mu,\gamma}u_{n-1}(x,t) - R_n), \qquad (3.9)
\end{aligned}$$

where

$$R_n = \frac{1}{(n-1)!}\frac{\partial^{n-1}N[\Phi(u(x,t),q)]}{\partial q^{n-1}}|_{q=0}. \qquad (3.10)$$

Suppose that the linear operator is $L = 0^{ABC}D_t^{\alpha,\mu,1}$. Applying the integral operator $0^{AB}I^{\alpha,\mu,\gamma}$ with $\gamma = 1$ on the above equations, with the helps of equations (2.9) and choosing $u_0(x,t) = u(x,0)$, we get

$$\begin{aligned}
q^1 : u_1(x,t) &= u_1(x,0) + \hbar[u_0(x,t) - u_0(x,0) - 0^{AB}I^{\alpha,\mu,\gamma}[R_1]], \\
q^2 : u_2(x,t) &= (1+\hbar)u_1(x,t) - \hbar 0^{AB}I^{\alpha,\mu,\gamma}[R_2], \\
&\vdots \\
q^n : u_n(x,t) &= (1+\hbar)u_{n-1}(x,t) - \hbar 0^{AB}I^{\alpha,\mu,\gamma}[R_n]. \qquad (3.11)
\end{aligned}$$

The initial conditions define as $\phi(u(x,0);q) = u_0(x,0) + \sum_{i=1}^{\infty} u_i(x,0)q^i = f(x)$. Thus $u_0(x,0) = f(x)$ and $u_i(x,0) = 0$, where $i = 1, 2, 3, \cdots$. Assume that the auxiliary linear operator L, the initial guess $u_0(x,t)$, and the auxiliary parameter $\hbar$ are selected such that the series (3.7) is convergent at $q = 1$, then due to (3.3) we have

$$u(x,t) = u_0(x,t) + \sum_{i=1}^{\infty} u_i(x,t).$$

In addition, the convergence of the series of $u(x,t)$ and the rate of approximation for the solution strongly depends on the values of the convergent control parameter $\hbar$. A proper value of $\hbar$ can be chosen to ensure that the solution series is convergent. To discover the valid region of $\hbar$, we need to determine at what region the solution does not depend on $\hbar$. For that, we can plot the $\hbar$-curve. The line segments nearly parallel to the horizontal axis will be the valid region. If the $\hbar$ is properly, thus it can be greatly enlarged the convergence region of the series. To do that, we proceed with the following steps, 1) fixed $\alpha = \mu$, 2) calculate the average residual error function $\zeta(\hbar)$, Therefore, we can use Eq. (2.12) to find the optimal value $\hbar$. Note that $\zeta(\hbar)$ contains unknown convergence-control parameter $\hbar$. As $\zeta(\hbar)$ decreases to zero, the corresponding homotopy-series solution will rapidly converge. Now, By finding the minimum of $\zeta(\hbar)$, corresponding to a set of the nonlinear algebraic equation, we get the optimal value of $\hbar$. 3) we solve the nonlinear algebraic equation

$$\frac{\partial\zeta(\hbar)}{\partial\hbar} = 0.$$

For the Allen-Cahn equation 1.1, we define the nonlinear term

$$\begin{aligned} N[u(x,t)] &= \varepsilon u_{xx}(x,t) + u(x,t) - u^3(x,t) \\ &= \varepsilon \frac{\partial^2}{\partial x^2}\sum_{i=0}^{\infty} u_i q^i + \sum_{i=0}^{\infty} u_i q^i - \left(\sum_{i=0}^{\infty} u_i q^i\right)^3 \\ &= \varepsilon \frac{\partial^2}{\partial x^2}\sum_{i=0}^{\infty} u_i q^i + \sum_{i=0}^{\infty} u_i q^i - \sum_{i=0}^{\infty} q^i \sum_{s=0}^{i} u_{i-s} \sum_{j=0}^{s} u_j u_{s-j}. \end{aligned} \tag{3.12}$$

Now, the coefficients of

$$\begin{aligned} q^0 : &\quad \varepsilon \frac{\partial^2}{\partial x^2} u_0 + u_0 - u_0^3 \\ q^1 : &\quad \varepsilon \frac{\partial^2}{\partial x^2} u_1 + u_1 - (3u_0^2 u_1) \\ &\quad \vdots \\ q^n : &\quad \varepsilon \frac{\partial^2}{\partial x^2} u_n + u_n - \sum_{s=0}^{n} u_{n-s} \sum_{j=0}^{s} u_j u_{s-j}. \end{aligned} \tag{3.13}$$

Using equations (3.13) and (3.11), the $n-$th order formula can be written as

$$\begin{aligned} u_n(x,t) &= (\chi_n + h)u_{n-1}(x,t) - (\chi_n + h)u_{n-1}(x,0) + h\Big[{}_0^{AB}I^{\alpha,\mu,1}\Big[-\varepsilon u_{n-1}(x,t) \\ &\quad -u_{n-1}(x,t) + \sum_{s=0}^{n-1} u_{n-1-s}(x,t) \sum_{j=0}^{s} u_{s-j}(x,t)u_j(x,t)\Big]\Big]. \end{aligned} \tag{3.14}$$

In summary, the solution can be generated via the following steps

(1) Choose $u_0(x,t) = f(x)$.
(2) For $n = 1, 2, 3, \ldots M$ apply equation (3.14).
(3) Define $u(x,t) = \sum_{i=0}^{M} u_i(x,t)$.

(4) Get $res(x,t)$ in (2.11).

(5) Get $\xi(\hbar)$ in (2.12).

(6) Solve $\frac{\partial\xi(\hbar)}{\hbar}$ with respect to $\hbar$.

4. Examples.

Example 4.1. *Consider the time-fractional Allen-Cahn equation of the form*

$$ {}_0^{ABC}D^{\alpha,\mu,1}u(x,t) = \varepsilon u_{xx}(x,t) + u(x,t) - u^3(x,t), x \in [-1,1], t > 0, \tag{4.1}$$

where $0 < \alpha < 1$, *subject to the initial condition,*

$$ u(x,0) = 0.53x + 0.47\sin(-1.5\pi x), \tag{4.2}$$

For $\alpha = \mu = \gamma = 1$, *the solution presented in* [19]. *The homotopy expression for* (4.1) *will be,*

$$\begin{aligned} {}_0^{ABC}D^{\alpha,\mu,1}[u_n(x,t) - \chi_n u_{n-1}(x,t)] &= h[{}_0^{ABC}D^{\alpha,\mu,1}u_{n-1}(x,t) - \varepsilon(u_{n-1})_{xx}(x,t) \\ &\quad -u_{n-1}(x,t) + \sum_{s=0}^{n-1} u_{n-1-s}(x,t)\sum_{j=0}^{s} u_{s-j}(x,t)u_j(x,t)], \end{aligned} \tag{4.3}$$

for $n = 1, 2, 3, \cdots$, *we choose the initial guess* $u_0(x,t) = 0.53x + 0.47\sin(-1.5\pi x)$. *By applying* $0^{AB}I^{\alpha,\mu,1}$ *on equation* (4.3), *we get equation* (3.14).

For $n = 1$ *we have:*

$$\begin{aligned} u_1(x,t) \;=\; & -\frac{0.148877\alpha hx^3t^{1-\mu}}{\Gamma(2-\mu)} + \frac{0.148877\alpha hx^3t^{\alpha-\mu+1}}{\Gamma(\alpha-\mu+2)} + \frac{0.148877hx^3t^{1-\mu}}{\Gamma(2-\mu)} \\ & +\frac{0.396069\alpha hx^2t^{1-\mu}\sin(4.71239x)}{\Gamma(2-\mu)} - \frac{0.396069\alpha hx^2\sin(4.71239x)t^{\alpha-\mu+1}}{\Gamma(\alpha-\mu+2)} \\ & -\frac{0.396069hx^2t^{1-\mu}\sin(4.71239x)}{\Gamma(2-\mu)} + \frac{0.53\alpha hxt^{1-\mu}}{\Gamma(2-\mu)} - \frac{0.53\alpha hxt^{\alpha-\mu+1}}{\Gamma(\alpha-\mu+2)} \\ & +\frac{0.103823\alpha ht^{1-\mu}\sin^3(4.71239x)}{\Gamma(2-\mu)} - \frac{0.103823\alpha h\sin^3(4.71239x)t^{\alpha-\mu+1}}{\Gamma(\alpha-\mu+2)} \\ & -\frac{0.351231\alpha hxt^{1-\mu}\sin^2(4.71239x)}{\Gamma(2-\mu)} + \frac{0.351231\alpha hx\sin^2(4.71239x)t^{\alpha-\mu+1}}{\Gamma(\alpha-\mu+2)} \\ & -\frac{0.47\alpha ht^{1-\mu}\sin(4.71239x)}{\Gamma(2-\mu)} + \frac{0.47\alpha h\sin(4.71239x)t^{\alpha-\mu+1}}{\Gamma(\alpha-\mu+2)} \\ & +\frac{10.4371\alpha h\epsilon t^{1-\mu}\sin(4.71239x)}{\Gamma(2-\mu)} - \frac{10.4371\alpha h\epsilon\sin(4.71239x)t^{\alpha-\mu+1}}{\Gamma(\alpha-\mu+2)} \end{aligned}$$

$$-\frac{0.53hxt^{1-\mu}}{\Gamma(2-\mu)}-\frac{0.103823ht^{1-\mu}\sin^3(4.71239x)}{\Gamma(2-\mu)}$$
$$+\frac{0.351231hxt^{1-\mu}\sin^2(4.71239x)}{\Gamma(2-\mu)}+\frac{0.47ht^{1-\mu}\sin(4.71239x)}{\Gamma(2-\mu)}$$
$$-\frac{10.4371h\epsilon t^{1-\mu}\sin(4.71239x)}{\Gamma(2-\mu)}. \tag{4.4}$$

Using this manner, we find $u_i(x,t)$ *for* $i = 2,3,\dots M$. *The HAM solution using* M*-terms of the series become as* $u(x,t) = \sum_{i=0}^{M} u_i(x,t)$ *which is depends on the convergent control parameter* $\hbar$. *To find it, we fixed* $\varepsilon = 0.001$, $\alpha = \mu$, *and use the least square method. For that, we define the residual error*

$$Res(x,t) = {}_0^{ABC} D^{\alpha,\mu,1}u(x,t) - \varepsilon u_{xx}(x,t) - u(x,t) + u^3(x,t), \tag{4.5}$$

and the average residual error (ARE) function

$$\zeta(\hbar) = \frac{1}{(N1+1)(N2+1)}\sum_{i=0}^{N1}\sum_{j=0}^{N2} Res^2(x_i,t_j). \tag{4.6}$$

Now, The $\hbar$*-curve is plotted in figure 1. The average residual error for* α = *0.9 is plotted in figure 2. Table 1 gives the convergent control parameter* $\hbar$ *and its ARE for several values of* α *using 6-order of approximation. The solution* $u(x,t)$ *and its residual error with the value of* α=*0.9 is given in figure 3 a) and b) respectively. We observed that, if* $\mu = 1$ *and vary* $0 < \alpha < 1$, *then the initial condition does not satisfied; which means the problem may have no solution when* $\mu = 1$.

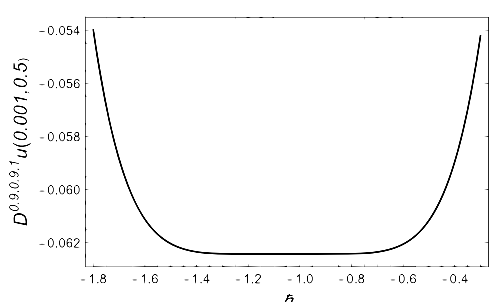


FIGURE 1. The $\hbar$ -curve for example 4.1 with $\alpha = \mu = 0.9$ using 6-order of approximation.

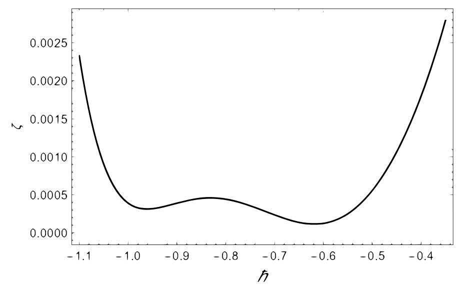


FIGURE 2. The ARE along $\hbar$ for $\alpha = \mu = 0.9$.

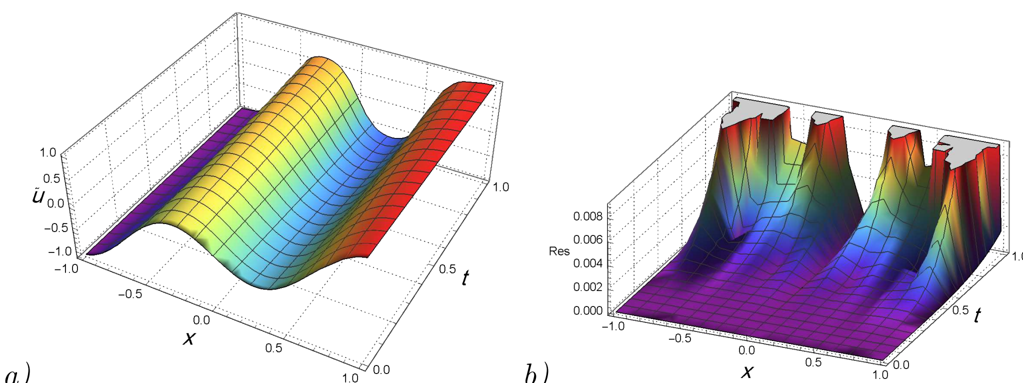


FIGURE 3. a) The approximate solution, b) the residual error of example 4.1 for $\alpha = \mu = 0.9$ and the optimal value of $\hbar = -0.923497$.

TABLE 1. ARE and it's optimal $\hbar$ for example 4.1 at $\mu = 0.5$ and vary α.

α	*ARE*	$\hbar$
0.1	*0.00110832*	*-0.53341*
0.3	*0.000934675*	*-0.556101*
0.5	*0.000602546*	*-0.614374*
0.7	*0.000202705*	*-0.923076*
0.9	*0.0000229112*	*-1.00508*

Example 4.2. *Consider the time-fractional Allen-Cahn equation.*

$$ {}_{0}^{ABC}D^{\alpha,\mu,1}u(x,t) = \varepsilon u_{xx}(x,t) + u(x,t) - u^3(x,t), x \in [0,1], t > 0, \tag{4.7}$$

with the initial condition,

$$u(x,0) = 0.25\sin(x), \tag{4.8}$$

The homotopy expression for (4.7) *will be*

$$
\begin{aligned}
0^{ABC}D^{\alpha,\mu,1}[u_n(x,t)-\chi_n u_{n-1}(x,t)] &= \hbar([0^{ABC}D^{\alpha,\mu,1}u_{n-1}(x,t) \\
&\quad -\varepsilon(u_{n-1}(x,t))_{xx}-u_{n-1}(x,t) \\
&\quad -\sum_{s=0}^{n-1}u_{n-1-s}(x,t)\sum_{j=0}^{s}u_j(x,t)u_{s-j}(x,t)]), \qquad (4.9)
\end{aligned}
$$

for $n=1,2,3,\cdots$, *we choose the initial guess* $u_0(x,t)=0.53x+0.47\sin(-1.5\pi x)$. *By applying* $0^{ABC}I^{\alpha,\mu,1}$ *on equation* (4.9), *we get equation* (3.14).

Now, for $n=1$ *we have:*

$$
\begin{aligned}
u_1(x,t) &= -\frac{0.015625\alpha\hbar t^{1-\mu}\sin^3(x)}{\Gamma(2-\mu)}+\frac{0.015625\alpha\hbar\sin^3(x)t^{\alpha-\mu+1}}{\Gamma(\alpha-\mu+2)}+\frac{0.25\alpha\hbar t^{1-\mu}\sin(x)}{\Gamma(2-\mu)} \\
&\quad -\frac{0.25\alpha\hbar\sin(x)t^{\alpha-\mu+1}}{\Gamma(\alpha-\mu+2)}-\frac{0.25\alpha\hbar\epsilon t^{1-\mu}\sin(x)}{\Gamma(2-\mu)}+\frac{0.25\alpha\hbar\epsilon\sin(x)t^{\alpha-\mu+1}}{\Gamma(\alpha-\mu+2)} \\
&\quad +\frac{0.015625\hbar t^{1-\mu}\sin^3(x)}{\Gamma(2-\mu)}-\frac{0.25\hbar t^{1-\mu}\sin(x)}{\Gamma(2-\mu)}+\frac{0.25\hbar\epsilon t^{1-\mu}\sin(x)}{\Gamma(2-\mu)}. \qquad (4.10)
\end{aligned}
$$

Similarly, we find $u_i(x,t)$ *for* $i=2,3,\ldots,M$. *Now, fixed* ε *= 0.001,* μ*=*α *and we take several values of* α *between 0 to 1. Figure 4 presents the* $\hbar$*-curve using 6-order of approximation while figure 5 gives the average residual error for* $\alpha=0.9$. *The HAM solution* $u(x,t)$ *and its residual error with the value of* $\alpha=0.9$ *are given in figure 6 a) and b). In Table 2 we give the optimal values of the convergent control parameter* $\hbar$ *and its ARE for several values of* α *using 6-order of approximation. The solution when* $\mu=1=\gamma, x=1, t=0$ *is given as a function of* $\hbar$*;* $u(1,0;\hbar)=9.5036*10^{-8}\hbar^6+2.313*10^{-7}\hbar^5+5.6265*10^{-6}\hbar^4-0.000125\hbar^3+0.00173977\hbar^2-0.020\hbar+0.210368$. *The solution of the equations* $u(1,0,\hbar)=0.25Sin(1)=0$ *is* $\hbar=0$, *this is a contradiction of HAM assumption;* $\hbar\neq 0$. *When* μ*= 1 =* γ *we have ABC fractional problem for example 2 which means this problem may have no solution in this case. .*

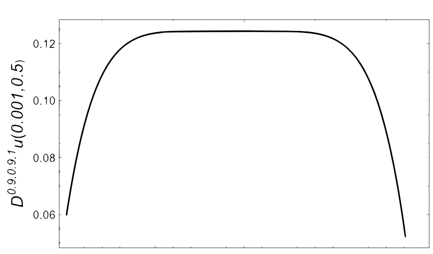


FIGURE 4. The $\hbar$ -curve for (4.2) using 6-order of approximation.

TABLE 2. ARE for example 4.2 with $\mu = 0.5$ and vary α.

α	ARE	$\hbar$
0.1	0.000405026	-1.57095
0.3	0.000175436	-1.51145
0.5	0.0000338132	-1.41423
0.7	$2.75591178 \times 10^{-6}$	-1.29294
0.9	$7.882607651 \times 10^{-8}$	-1.16123

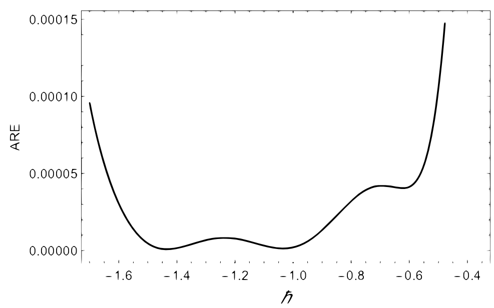


FIGURE 5. The ARE with optimal $\hbar$ for α= 0.9.

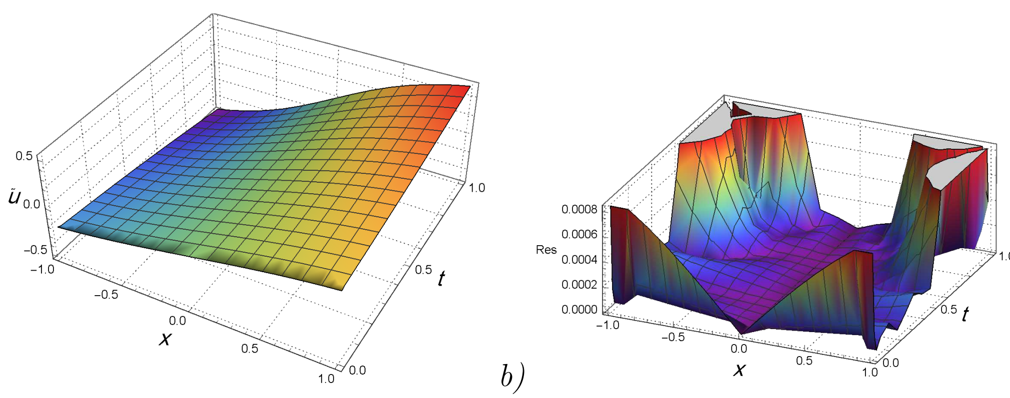
a)  b)

FIGURE 6. a) The HAM solution, b) the residual error for example (4.2) with $\alpha = \mu = 0.9$ and the optimal value of $\hbar = -1.43369$.

5. CONCLUSION

This paper has successfully implemented the HAM to obtain an approximate analytical solution to the time-fractional Allen-Cahn equation. We investigated the approximate solutions of the time-fractional order Allen-Cahn equation, and the computed results are illustrated graphically. We satisfied the accuracy of the approximate solutions by computing the average residual error. We also numerically discuss the existence of the solution. This is the first work that obtained the solutions of Allen-Cahn using GABC definition. The method can apply to more models in physics and engineering problems with easy algorithm.

References

[1] N. A. Khan, M. Jamil, A. Ara, Approximate solutions to time-fractional Schrödinger equation via homotopy analysis method, *International Scholarly Research Notices*, **2012** (2012).

[2] A. R. Kanth, K. Aruna, Two-dimensional differential transform method for solving linear and non-linear Schrodinger equations, *Chaos, Solitons & Fractals*, **41(5)** (2009), 2277-2281.

[3] S. A. Khuri, A new approach to the cubic Schrodinger equation: an application of the decomposition technique, *Applied Mathematics and Computation*, **97(2-3)** (1998), 251-254.

[4] A. M. Wazwaz, A study on linear and nonlinear Schrodinger equations by the variational iteration method, *Chaos, Solitons & Fractals*, **37(4)** (2008), 1136-1142.

[5] A. Khalouta, A. Kadem, A new iterative natural transform method for solving nonlinear Caputo time-fractional partial differential equations, *Jordan Journal of Mathematics and Statistics*, **13(3)** (2020), 459-476.

[6] E. Hengamian Asl, J. Saberi-Nadjafi, M. Gachpazan, Numerical solution of fractional-order population growth model using fractional-order Muntz-Legendre collocation method and Pade-approximants, *Jordan Journal of Mathematics and Statistics*, **15(2)** (2022), 157–175.

[7] A. Sadighi, D. D. Ganji, Analytic treatment of linear and nonlinear Schrodinger equations: a study with homotopy-perturbation and Adomian decomposition methods, *Physics Letters A*, **372(4)** (2008), 465-469.

[8] A. K. Alomari, Homotopy-Sumudu transforms for solving system of fractional partial differential equations, *Advances in Difference Equations*, **2020(1)** (2020), 1-16.

[9] H. Mohammadi, S. Kumar, S. Rezapour, S. Etemad, A theoretical study of the Caputo–Fabrizio fractional modeling for hearing loss due to Mumps virus with optimal control, *Chaos, Solitons & Fractals*, **144** (2021), 110668.

[10] S. Aljhani, M. S. M. Noorani, K. M. Saad, A. K. Alomari, Numerical solutions of certain new models of the time-fractional Gray-Scott, J*Journal of Function Spaces*, **2021**(2021).

[11] S. Kumar, S. Ghosh, B. Samet, E. F. D. Goufo, An analysis for heat equations arises in diffusion process using new Yang-Abdel-Aty-Cattani fractional operator, *Mathematical Methods in the Applied Sciences*, **43(9)** (2020), 6062-6080.

[12] S. Kumar, R. P. Chauhan, S. Momani, S. Hadid, Numerical investigations on COVID-19 model through singular and non-singular fractional operators, *Numerical Methods for Partial Differential Equations*, (2020), https://doi.org/10.1002/num.22707.

[13] P. Veeresha, D. G. Prakasha, S. Kumar, A fractional model for propagation of classical optical solitons by using nonsingular derivative, *Mathematical Methods in the Applied Sciences* 2020, https://doi.org/10.1002/mma.6335Citations

[14] S. Aljhani, M. S. M. Noorani, A. K. Alomari, Numerical solution of fractional-order HIV model using homotopy method, *Discrete Dynamics in Nature and Society*, **2020** (2020).

[15] A. Atangana, D. Baleanu, New fractional derivative with non-local and non-singular kernel, *Therm. Sci.*, **20(2)** (2016), 757–763.

[16] T. Abdeljawad, D. Baleanu, On fractional derivatives with generalized Mittag-Leffler kernels, *Advances in Difference Equations*, **2018(1)** (2018), 1-15.

[17] H. M. Srivastava, A. K. Alomari, K. M. Saad, W. M. Hamanah, Some dynamical models involving fractional-order derivatives with the Mittag-Leffler type kernels and their applications based upon the Legendre spectral collocation method, *Fractal and Fractional*, **5(3)** (2021), 131.

[18] A. K. Alomari, T. Abdeljawad, D. Baleanu, K. M. Saad, Q. M. Al-Mdallal, Numerical solutions of fractional parabolic equations with generalized Mittag–Leffler kernels. *Numerical methods for partial differential equations*, doi.org/10.1002/num.22699.

[19] H. Song, L. Jiang, Q. Li, A reduced order method for Allen–Cahn equations, *Journal of Computational and Applied Mathematics*, **292** (2016), 213-229.

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