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THE DUAL OF THE NOTIONS n-SUBMODULES AND J-SUBMODULES

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ABSTRACT. Let R be a commutative ring with identity and M be an R-module. A proper submodule N of M is called an *n-submodule* if for $a \in R$, $m \in M$, $am \in N$ with $a \notin \sqrt{Ann_R(M)}$, implies $m \in N$. A proper submodule N of M is called a *J-submodule* of M if for $a \in R$ and $m \in M$, whenever $am \in N$ and $a \notin (J(R)M : M)$, then $m \in N$. The aim of this paper is to introduce and investigate the dual notions of n-submodules and J-submodules of M.

1. Introduction

Throughout this paper, R is a commutative ring with identity and $\mathbb{Z}$ is the ring of integers. Moreover, the set of zero divisors and the Jacobson radical of R are denoted by $Z(R)$ and $J(R)$, respectively. The radical of an ideal I of R is defined by $\sqrt{I} = \{a \in R : a^n \in I \ for \ some \ n \in N\}$. For a submodule N of an R-module M, the annihilator of the R-module M/N is defined as $Ann_R(M/N) = (N :_R M) = \{r \in R : rM \subseteq N\}$.

In [11], the n-ideals of R and the n-submodules of an R-module M are defined. A proper ideal P of R is said to be an *n-ideal* if $ab \in P$ and $a \notin n(R)$ for some $a, b \in R$, then $b \in P$, where $n(R)$ is the set of nilpotent elements of R. A proper submodule N of M is called an *n-submodule* if for $a \in R$, $m \in M$, $am \in N$ with $a \notin \sqrt{Ann_R(M)}$, then $m \in N$.

Khashan and Bani-Ata introduced and studied the concepts of J-ideal and J-submodule of an R-module M in [9]. When $a, b \in R$ with $ab \in I$ and $a \notin J(R)$, then

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$b \in I$, a proper ideal I of R is said to be a *J-ideal* of R. A proper submodule N of M is called J-submodule if $am \in N$ and $a \notin (J(R)M : M)$ for some $a \in R$ and $m \in M$, then $m \in N$.

The main purpose of this paper is to introduce the dual notions of n-submodules and J-submodules of an R-module M. We also look into the initial characteristics of these classes of modules. We say that a non-zero submodule N of an R-module M is a *co-n-submodule of* M if for $a \in R$ and submodule K of M, whenever $aN \subseteq K$ and $a \notin \sqrt{Ann_R(M)}$, then $N \subseteq K$ (Definition 2.1). Also, we say that a non-zero submodule N of an R-module M is a *co-J-submodule of* M if for $a \in R$ and submodule K of M, whenever $aN \subseteq K$ and $a \notin Ann_R((0 :_M J(R)))$, then $N \subseteq K$ (Definition 3.1). For an R-module M, among other results, in Example 2.4 we see that the the concepts of second submodules and *co*-n-submodules are different in general. Moreover, we investigate the behavior of *co*-n-submodules under module homomorphisms (Theorem 2.6). In Theorem 2.11, it is proved that every proper submodule of M is an n-submodule if and only if every non-zero submodule of M is a *co*-n-submodule. Also, it is shown that if $(0 :_M J(R)) \subseteq Soc_R(M)$ (in particular, M is a strong comultiplication R-module), then every *co*-n-submodule of M is a *co*-J-submodule of M (Proposition 3.12). In Corollary 3.6, we characterize *co*-J-submodules when M is a strong comultiplication R-module. Moreover, we investigate the behavior of *co*-J-submodules under localizations and module homomorphisms (Proposition 3.11 and Proposition 3.12).

2. The dual of n-submodules of modules

In this section we introduce and investigate the dual notion of n-submodules (that is *co*-n-submodules) of an R-module M.

Definition 2.1. We say that a non-zero submodule N of an R-module M is a *co-n-submodule of* M if for $a \in R$ and submodule K of M, whenever $aN \subseteq K$ and $a \notin \sqrt{Ann_R(M)}$, then $N \subseteq K$.

Let M be an R-module. A non-zero submodule S of M is said to be *second* if for each $a \in R$, the homomorphism $S \xrightarrow{a} S$ is either surjective or zero [13]. Also, a non-zero submodule S of M is said to be *secondary* if for each $a \in R$ the endomorphism of S given by multiplication by a is either surjective or nilpotent [10].

Lemma 2.2. For a submodule S of an R-module M we have the following.

(a) S is a second submodule of M if and only if $S \neq 0$ and $rS \subseteq K$, where $r \in R$ and K is a submodule of M implies either $rS = 0$ or $S \subseteq K$ [5, Theorem 2.10].

(b) S is a secondary submodule of M if and only if $S \neq 0$ and $rS \subseteq K$, where $r \in R$ and K is a submodule of M implies either $r^n S = 0$ for some positive integer n or $S \subseteq K$ [6, Theorem 2.8].

Now $\sqrt{Ann_R(M)} \subseteq \sqrt{Ann_R(N)}$ implies that if N is a *co*-n-submodule of M, then N is a secondary submodule of M by Lemma 2.2(b). The following example shows that the converse is not true, in general.

Example 2.3. Consider the $\mathbb{Z}$-module $\mathbb{Z}_{p^2q}$, where p, q are positive prime numbers. Then since $p \notin pq\mathbb{Z} = \sqrt{Ann_{\mathbb{Z}}(\mathbb{Z}_{p^2q})}$, $p(q\mathbb{Z}_{p^2q}) \subseteq (p\mathbb{Z}_{p^2q})$, and $(q\mathbb{Z}_{p^2q}) \not\subseteq (p\mathbb{Z}_{p^2q})$, we have $N = q\mathbb{Z}_{p^2q}$ is not a *co*-n-submodule of $\mathbb{Z}_{p^2q}$. But N is a secondary submodule of $\mathbb{Z}_{p^2q}$. Because if $r \in p\mathbb{Z}$, then $r^2N = 0$. If $r \notin p\mathbb{Z}$, then $r\mathbb{Z} + p\mathbb{Z} = \mathbb{Z}$. This implies that $rN + p^2N = N$ and so $rN = N$.

The following example shows that the the concepts of second submodules and *co*-n-submodules are are different in general.

Example 2.4. (a) Consider the $\mathbb{Z}$-module $\mathbb{Z}_{27}$. Note that $\sqrt{Ann_{\mathbb{Z}}(\mathbb{Z}_{27})} = 3\mathbb{Z}$. One can see that $\bar{3}\mathbb{Z}_{27}$ is a *co*-n-submodule of $\mathbb{Z}_{27}$. But the submodule $\bar{3}\mathbb{Z}_{27}$ is not second submodule because $(3)(\bar{3})\mathbb{Z}_{27} \subseteq \bar{9}\mathbb{Z}_{27}$, while, $\bar{3}\mathbb{Z}_{27} \not\subseteq \bar{9}\mathbb{Z}_{27}$ and $(3)(\bar{3})\mathbb{Z}_{27} \neq 0$.

(b) Consider the $\mathbb{Z}$-module $\mathbb{Z}_{p^2q}$, where p, q are positive prime numbers. Then since $q \notin pq\mathbb{Z} = \sqrt{Ann_{\mathbb{Z}}(\mathbb{Z}_{p^2q})}$, $q(p^2\mathbb{Z}_{p^2q}) = 0$, and $(p^2\mathbb{Z}_{p^2q}) \neq 0$, we have $N = p^2\mathbb{Z}_{p^2q}$ is not a *co*-n-submodule of $\mathbb{Z}_{p^2q}$. But N is a second submodule of $\mathbb{Z}_{p^2q}$. Because if $r \in q\mathbb{Z}$, then $rN = 0$. If $r \notin q\mathbb{Z}$, then $r\mathbb{Z} + q\mathbb{Z} = \mathbb{Z}$. This implies that $rN + qN = N$ and so $rN = N$.

Let M be an R-module. M is said to be a *multiplication module* if for every submodule N of M there exists an ideal I of R such that $N = IM$ [8]. M is said to be a *comultiplication module* if for every submodule N of M there exists an ideal I of R such that $N = (0 :_M I)$ [7]. Let N and K be two submodules of M. The *product* (resp. *coproduct*) of N and K is defined by $(N :_R M)(K :_R M)M$ (resp. $(0 :_M Ann_R(N)Ann_R(K)))$ and denoted by NK (resp $C(NK)$) [2]. A submodule N of M is said to be *idempotent* (resp. *coidempotent*) if $N = (N :_R M)^2M$ (resp. $N = (0 :_M Ann^2_R(N)))$ [4].

Proposition 2.5. Let N be a submodule of an R-module M. Then we have the following.

(a) If N is a *co*-n-submodule of M such that $\sqrt{0} = \sqrt{Ann_R(M)}$, then $Ann_R(N)$ is an n-ideal of R.

(b) If M is a comultiplication R-module and $Ann_R(N)$ is an n-ideal of R, then N is a *co*-n-submodule of M.

(c) If N is a *co*-n-submodule of a multiplication R-module M such that $(N :_R M) \not\subseteq \sqrt{Ann_R(M)}$, then N is an idempotent submodule of M.

(d) If N is an n-submodule of a comultiplication R-module M such that $Ann_R(N) \not\subseteq \sqrt{Ann_R(M)}$, then N is a coidempotent submodule of M.

Proof. (a) Let $a, b \in R$, $a \notin \sqrt{0}$ and $ab \in Ann_R(N)$. Then $a \notin \sqrt{0} = \sqrt{Ann_R(M)}$ and $aN \subseteq aN$ imply that $N \subseteq aN$. Thus $bN = 0$, as needed.

(b) Let $a \in R$, K be a submodule of M, $a \notin \sqrt{Ann_R(M)}$, and $aN \subseteq K$. Then we have $Ann_R(K)aN = 0$. So by assumption, $Ann_R(K)N = 0$. Hence, $Ann_R(K) \subseteq Ann_R(N)$. Thus as M is a comultiplication R-module, $N \subseteq K$.

(c) As M is a multiplication R-module, we have

$$(N :_R M)N = (N :_R M)(N :_R M)M = N^2.$$

Let $a \in (N :_R M) \setminus \sqrt{Ann_R(M)}$. Then $aN \subseteq (N :_R M)N = N^2$ implies that $N \subseteq N^2$ since N is a *co*-n-submodule of M. The reverse inclusion is clear.

(d) Since M is a comultiplication R-module, we have

$$(N :_M Ann_R(N)) = (0 :_M Ann^2_R(N)) = C(N^2).$$

Let $a \in Ann_R(N) \setminus \sqrt{Ann_R(M)}$. Then

$$C(N^2) = (0 :_M Ann_R^2(N)) \subseteq (0 :_M aAnn_R(N)) = (N :_M a).$$

Thus $aC(N^2) \subseteq N$. Hence, $C(N^2) \subseteq N$ since N is an n-submodule of M. This complete the proof because the reverse inclusion is clear. □

Theorem 2.6. *Let $f : M \to \acute{M}$ be a monomorphism of R-modules. Then we have the following.*

(a) *If N is a co-n-submodule of M, then $f(N)$ is a co-n-submodule of $\acute{M}$.*

(b) *If $\acute{N}$ is a co-n-submodule of $\acute{M}$ and $\acute{N} \subseteq f(M)$, then $f^{-1}(\acute{N})$ is a co-n-submodule of M.*

Proof. (a) Since $N \neq 0$ and f is a monomorphism, we have $f(N) \neq 0$. Let $a \in R$, $\acute{K}$ be a submodule of $\acute{M}$, and $af(N) \subseteq \acute{K}$. Then $aN \subseteq f^{-1}(\acute{K})$. As N is a *co-n*-submodule, $N \subseteq f^{-1}(\acute{K})$ or $a \in \sqrt{Ann_R(M)}$. Therefore,

$$f(N) \subseteq f(f^{-1}(\acute{K})) = f(M) \cap \acute{K} \subseteq \acute{K}$$

or $a \in \sqrt{Ann_R(f(M))}$, as needed.

(b) If $f^{-1}(\acute{N}) = 0$, then $f(M) \cap \acute{N} = f(f^{-1}(\acute{N})) = f(0) = 0$. Thus $\acute{N} = 0$, which is a contradiction. So, $f^{-1}(\acute{N}) \neq 0$. Now let $a \in R$, K be a submodule of M, and $af^{-1}(\acute{N}) \subseteq K$. Then

$$a\acute{N} = a(f(M) \cap \acute{N}) = af(f^{-1}(\acute{N})) \subseteq f(K).$$

As $\acute{N}$ is a *co-n*-submodule, $\acute{N} \subseteq f(K)$ or $a \in \sqrt{Ann_R(\acute{M})}$. Therefore, $f^{-1}(\acute{N}) \subseteq f^{-1}(f(K)) = K$ or $a \in \sqrt{Ann_R(f^{-1}(\acute{M}))}$ as desired. □

Corollary 2.7. Let M be an R-module and $N \subseteq K$ be two submodules of M. Then N is a *co-n*-submodule of K if and only if N is a *co-n*-submodule of M.

Proof. This follows from Theorem 2.6 by using the natural monomorphism $K \to M$. □

Proposition 2.8. Let M be an R-module. Then we have the following.

(a) M is a *co-n*-submodule of M if M is a secondary R-module.

(b) The sum of an arbitrary non-empty set of *co-n*-submodules of M is a *co-n*-submodule of M.

Proof. (a) This is clear.

(b) Let N_i be a *co-n*-submodule of M for every $i \in I$. Assume that $a\sum_{i\in I} N_i \subseteq K$ with $a \notin \sqrt{Ann_R(M)}$ for $a \in R$ and submodule K of M. This implies that $aN_i \subseteq K$ for every $i \in I$. As N_i is a *co-n*-submodule of M, we conclude that $N_i \subseteq K$ for every $i \in I$. Hence $\sum_{i\in I} N_i \subseteq K$, as needed. □

Lemma 2.9. Let N be non-zero submodule of an R-module. Then N is a *co-n*-submodule of M if and only if whenever I is an ideal of R such that $I \not\subseteq \sqrt{Ann_R(M)})$ and K is a submodule of M with $IN \subseteq K$, then $N \subseteq K$.

Proof. Suppose that N is a *co-n*-submodule and $IN \subseteq K$ for some ideal I of R with $I \cap (R \setminus \sqrt{Ann_R(M)}) \neq \emptyset$ and submodule K of M. Then there exists $a \in I$ such that $a \notin \sqrt{Ann_R(M)}$. Since N is a *co-n*-submodule, $N \subseteq K$. For the converse, let $aN \subseteq K$ and $a \notin \sqrt{Ann_R(M)}$ for $a \in R$, and submodule K of M. We take $I = aR$. Note that $I \cap (R \setminus \sqrt{Ann_R(M)}) \neq \emptyset$. Then by assumption we have $N \subseteq K$, and so N is a *co-n*-submodule of M. □

Theorem 2.10. *Let K_1, K_2, K be submodules of an R-module M and I be an ideal of R with $I \not\subseteq \sqrt{Ann_R(M)})$. Then the following hold.*

(a) *If K_1, K_2 are co-n-submodules of M with $(K_1 :_M I) = (K_2 :_M I)$, then $K_1 = K_2$.*

(b) *If $(K :_M I)$ is a co-n-submodule, then $(K :_M I) = K$. In particular, K is a co-n-submodule.*

Proof. (a) Since $IK_1 \subseteq K_2$ and K_1 is a *co-n*-submodule, we have $K_1 \subseteq K_2$ by Lemma 2.9. Similarly, we have $K_2 \subseteq K_1$, and so $K_1 = K_2$.

(b) As $(K :_M I)$ is a *co-n*-submodule and $I(K :_M I) \subseteq K$, we have $(K :_M I) \subseteq K$ by Lemma 2.9. Hence, $(K :_M I) = K$. □

Theorem 2.11. *Let M be an R-module. Every proper submodule of M is an n-submodule if and only if every non-zero submodule of M is a co-n-submodule.*

Proof. First suppose that every proper submodule of M is an n-submodule. Let N be a non-zero submodule of M and $aN \subseteq K$ for some $a \in R$ and a submodule K of M with $a \notin \sqrt{Ann_R(M)}$. If $K = M$, then we are done. If K is proper, then by assumption, K is an n-submodule. Hence for each $x \in N$, $ax \in K$ implies that $x \in K$. Thus $N \subseteq K$, as needed. Now suppose that every non-zero submodule of M is a *co*-n-submodule. Let N be a proper submodule of M and $am \subseteq N$ for some $a \in R$ and $m \in M$ with $a \notin \sqrt{Ann_R(M)}$. If $m = 0$, then we are done. If Rm is non-zero, then by assumption, Rm is a *co*-n-submodule and so $m \in N$, as requested. □

3. The dual of J-submodules of modules

In this section we introduce the dual notion of J-submodules (that is *co*-J-submodules) of an R-module M. Also, we investigate first properties of this class of modules and obtain some related results.

Definition 3.1. We say that a non-zero submodule N of an R-module M is a *co-J-submodule of* M if for $a \in R$ and submodule K of M, whenever $aN \subseteq K$ and $a \notin Ann_R((0 :_M J(R)))$, then $N \subseteq K$.

Recall that an R-module M is said to be *finitely cogenerated* if for every set $\{M_i\}_{i \in I}$ of submodules of M, $\cap_{i \in I} M_i = 0$ implies $\cap_{i=1}^{n} M_i = 0$ for some positive integer n [1].

Remark 3.2. If $Ann_R((0 :_M J(R))) = R$, then our definition implies that any non-zero submodule of M is a *co*-J-submodule of M. The only finitely cogenerated R-module M such that $Ann_R((0 :_M J(R))) = R$ is 0 by the dual of Nakayama's Lemma [3, Theorem 3.14].

Proposition 3.3. Let M be an R-module and N be a submodule of M. Then the following are equivalent:

(a) N is a *co*-J-submodule of M;

(b) $N \neq 0$ and for each $a \notin Ann_R((0 :_M J(R)))$, we have $aN = N$.

Proof. $(a) \Rightarrow (b)$ This follows from the fact that $aN \subseteq aN$.

$(b) \Rightarrow (a)$ This is clear. □

Theorem 3.4. *Let M be an R-module. Then we have the following results.*

(a) *Any sum of co-J-submodules of M is a co-J-submodule of M.*

(b) *If N is a co-J-submodule of M and $Ann_R((0:_M J(R))) = J(R)$, then $Ann_R(N)$ is a J-ideal of R.*

(c) *If N is a co-J-submodule of M, then IN is also a co-J-submodule of M, where I is an ideal of R.*

(d) *If M is a comultiplication R-module and N is a submodule of M such that $Ann_R(N)$ is a J-ideal of R, then N is a co-J-submodule of M.*

Proof. (a) This is straightforward.

(b) Let $ab \in Ann_R(N)$ such that $a \notin J(R)$. Then we have $abN = 0$ and so $aN \subseteq (0:_M b)$. Since N is a *co*-J-submodule of M and $a \notin J(R) = Ann_R((0:_M J(R)))$, we have $bN = 0$, as needed.

(c) Let $aIN \subseteq K$, where $a \in R$ and K is a submodule of M with $a \notin Ann_R((0:_M J(R)))$. Then we have $abN \subseteq K$ for all $b \in I$. Since N is a *co*-J-submodule of M, we have $bN \subseteq K$ for all $b \in I$. Thus, $IN \subseteq K$, as requested.

(d) Let $a \in R$ and K be a submodule of M such that $aN \subseteq K$ and $a \notin Ann_R((0:_M J(R)))$. Then

$$Ann_R(K) \subseteq Ann_R(aN) \subseteq (Ann_R(N) :_R a).$$

Hence, $aAnn_R(K) \subseteq Ann_R(N)$. As $a \notin Ann_R((0:_M J(R)))$, we have $a \notin J(R)$. Therefore, $Ann_R(K) \subseteq Ann_R(N)$ since $Ann_R(N)$ is a J-ideal of R. Now, M is a comultiplication R-module implies that $N \subseteq K$ by [7, Theorem 5]. Hence, N is a *co*-J-submodule of M. □

The following example shows that if $Ann_R((0:_M J(R))) \nsubseteq J(R)$, then part (b) of Theorem 3.4 need not be true in general.

Example 3.5. Consider the $\mathbb{Z}$-module $M = \mathbb{Z}_2$. Then $2\mathbb{Z} = Ann_{\mathbb{Z}}((0:_{\mathbb{Z}_2} J(\mathbb{Z}))) \nsubseteq J(\mathbb{Z}) = 0$. We have $N = M$ is clearly, a *co*-J-submodule of M but $2\mathbb{Z} = Ann_{\mathbb{Z}}(\mathbb{Z}_2)$ is not a J-ideal of $\mathbb{Z}$.

An R-module M satisfies the *double annihilator conditions* (DAC for short) if for each ideal I of R we have $I = Ann_R((0:_M I))$. An R-module M is said to be a *strong comultiplication module* if M is a comultiplication R-module and satisfies the DAC conditions [7].

Corollary 3.6. Let M be a strong comultiplication R-module and N a non-zero submodule of M. Then the following are equivalent:

(a) N is a *co*-J-submodule of M;

(b) $Ann_R(N)$ is a J-ideal of R;

(c) $N = (0 :_M I)$, where I is a J-ideal of R.

Proof. $(a) \Leftrightarrow (b)$ Follows by Theorem 3.4 and the fact that $Ann_R((0 :_M I)) = I$ for any ideal I of R.

$(b) \Leftrightarrow (c)$ We just choose $I = Ann_R(N)$. □

Recall that *the socle of an R-module M* is defined as the sum of all minimal submodules of M and it is denoted by $Soc_R(M)$.

A non-zero submodule N of an R-module M is said to be *large*, if for any submodule K of M, $N \cap K = 0$ implies $K = 0$.

A proper ideal I of R is said to be *small* if whenever J is an ideal of R with $I + J = R$, then $J = R$.

Theorem 3.7. *Let M be a strong comultiplication R-module. If N is a co-J-submodule of M, then we have the following.*

(a) $Soc_R(M) \subseteq N$.

(b) *N is a large submodule of M.*

Proof. (a) We have $Soc_R(M) = (0 :_M J(R))$ by [12, Corollary 2.7]. Suppose $Soc_R(M) \not\subseteq N$. Then clearly, $Ann_R(N) \not\subseteq Ann_R(Soc_R(M)) = J(R)$. But $Ann_R(N)$ is a J-ideal by Corollary 3.6, which contradicts [9, Proposition 2.2]. Hence, $Soc_R(M) \subseteq N$.

(b) $Ann_R(N)$ is a J-ideal of R by Corollary 3.6. Hence, by [9, Proposition 2.9], $Ann_R(N)$ is a small ideal of R. Now the result follows from [12, Theorem 2.5]. □

Proposition 3.8. Let M be an R-module such that $(0 :_M J(R)) \subseteq Soc_R(M)$ (in particular, M be a strong comultiplication R-module). Then every *co*-n-submodule of M is a *co*-J-submodule of M.

Proof. As $(0 :_M J(R)) \subseteq Soc_R(M)$, we have $Ann_R(Soc_R(M)) \subseteq Ann_R((0 :_M J(R)))$. So it is enough to prove that $\sqrt{Ann_R(M)} \subseteq Ann_R(Soc_R(M))$. Let $a \in \sqrt{Ann_R(M)}$ and S be a minimal submodule of M. Then $Ann_R(S)$ is a maximal ideal of R and so $\sqrt{Ann_R(S)} = Ann_R(S)$. Clearly $Ann_R(M) \subseteq Ann_R(S)$. Thus, $a \in Ann_R(S)$ and so $aS = 0$ for all minimal submodules S of M. It follows that $a \in Ann_R(Soc_R(M))$, as needed. □

Proposition 3.9. If N is a secondary submodule of an R-module M such that $Ann_R(N) \subseteq J(R)$, then N is a *co*-J-submodule of M.

Proof. Suppose that N is a secondary submodule of M and $Ann_R(N) \subseteq J(R)$. Then $\sqrt{Ann_R(N)} \subseteq \sqrt{J(R)} = J(R)$. Let $a \in R$ and K be a submodule of M such that $aN \subseteq K$ and $a \notin Ann_R((0 :_M J(R)))$. Then $a \notin J(R)$ and so $a \notin \sqrt{Ann_R(N)}$. It follows that $N \subseteq K$, as needed □

Corollary 3.10. If N is a secondary submodule of an R-module M such that $Ann_R(N)$ is a J-ideal of R, then N is a *co*-J-submodule of M.

Proof. This follows from [9, Proposition 2.2] and Proposition 3.9. □

Proposition 3.11. Let M be an R-module, S a multiplicatively closed subset of R, and N be a finitely generated *co*-J-submodule of M. If $J(R)$ is a finitely generated ideal of R and $S^{-1}(J(R)) = J(S^{-1}R)$, then $S^{-1}N$ is a *co*-J-submodule of $S^{-1}M$ if $S^{-1}N \neq 0$ (in particular, $S \cap Ann_R(N) = \emptyset$).

Proof. As $J(R)$ is a finitely generated ideal of R and $S^{-1}(J(R)) = J(S^{-1}R)$, we have

$$S^{-1}(Ann_R((0 :_M J(R)))) \subseteq Ann_{S^{-1}R}((0 :_{S^{-1}M} J(S^{-1}R))).$$

Now let $(a/s)(S^{-1}N) \subseteq S^{-1}K$ for some $a/s \in S^{-1}R$ and submodule $S^{-1}K$ of $S^{-1}M$ with $a/s \notin Ann_{S^{-1}R}((0 :_{S^{-1}M} J(S^{-1}R)))$. Then $a/s \notin S^{-1}(Ann_R((0 :_M J(R))))$ and so $a \notin Ann_R((0 :_M J(R)))$. As N is finitely generated and $(a/s)(S^{-1}N) \subseteq S^{-1}K$, there exists $t \in S$ such that $taN \subseteq K$. This implies that $aN \subseteq (K :_M t)$. Now since N is a *co*-J-submodule of M, $tN \subseteq K$. It follows that $S^{-1}N \subseteq S^{-1}K$, as needed. □

Proposition 3.12. Let M and $\acute{M}$ be R-modules, and let $f : M \to \acute{M}$ be an R-monomorphism. If $\acute{N}$ is a *co*-J-submodule of $\acute{M}$ such that $\acute{N} \subseteq Im(f)$, then $f^{-1}(\acute{N})$ is a *co*-J-submodule of M.

Proof. As $\acute{N} \neq 0$ and $\acute{N} \subseteq Im(f)$, we have $f^{-1}(\acute{N}) \neq 0$. We have $Ann_R((0 :_{\acute{M}} J)) \subseteq Ann_R((0 :_M J))$ because if $r \in Ann_R((0 :_{\acute{M}} J))$, then $r(0 :_{f(M)} J) = 0$ and so $rf((0 :_M J)) = 0$. Now as $ker(f) = 0$, we have $r(0 :_M J) = 0$ and thus $r \in Ann_R((0 :_M J))$. Let $r \notin Ann_R((0 :_M J))$ and K be a submodule of M with $rf^{-1}(\acute{N}) \subseteq K$. Then $r \notin Ann_R((0 :_{\acute{M}} J))$ and $r\acute{N} \subseteq f(K)$. Thus as $\acute{N}$ is a *co*-J-submodule of $\acute{M}$, we have $\acute{N} \subseteq f(K)$. This implies that $f^{-1}(\acute{N}) \subseteq K$, as requested. □

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